Strongly disordered metals and disorder-driven metal-insulator transitions in holography

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Recently, much progress has been made on understanding transport properties of strongly coupled quantum field theories by employing gauge-gravity duality. However, a theory of transport at finite density and temperature is still lacking for strongly disordered systems. We reduce the computation of direct current electrical conductivity, for a wide variety of strongly disordered holographic systems with no background charge density, to the solution of a linear differential equation dependent only on data on the black hole horizon of the bulk theory. Some strongly coupled theories in two spatial dimensions have a universal conductivity, independent of disorder strength. We realize a disorder-driven holographic metal-insulator transitions through the percolation of poorly-conducting regions across the black hole horizon. We compare results from our exact realizations of holographic disorder with simpler approaches to the problem, such as massive gravity.

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1. Conductivity

Let us consider a static, asymptotically anti-de Sitter space with a black hole horizon. Without loss of generality, we use diffeomorphism invariance to choose the metric

$$\mathrm{d}s^2 = L^2 \left[P\mathrm{d}r^2 - Q\mathrm{d}t^2 + G_{ij}\mathrm{d}x^i\mathrm{d}x^j \right]. \tag{1}$$

i, j indices represent the spatial boundary directions, while M, N represent all dimensions, and L is the AdS radius. All functions in the metric are functions of r and $\mathbf{x}$. We further choose the bulk coordinate $0 < r < \infty$, with $r = 0$ the black hole horizon, and $r = \infty$ the AdS boundary. We do not need knowledge of what uncharged matter is required to set up this geometry, but do assume that all energy conditions are obeyed.

We add a U(1) gauge field to the bulk, so the action of our theory is

$$S = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(\mathcal{L}_{\text{uncharged}} - \frac{Z}{4}F^2\right). \tag{2}$$

The two-point functions of this gauge field correspond to calculations of current-current correlation functions in the boundary theory, such as the direct current electrical conductivity matrix σ^{ij}. The conductivity may be related, via the membrane paradigm [1], to data on the horizon of the black hole alone. In particular, the expected value of the boundary current is given by

$$J^i = \sigma^{ij}E_j = \mathbb{E}\left[Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right], \tag{3}$$

where E_j is the applied electric field, $\mathbb{E}[\cdots]$ denotes a uniform spatial average, $\gamma_{ij} = G_{ij}(r = 0)$ is the induced metric on the horizon, and α obeys the equation

$$0 = \partial_i\left(Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right). \tag{4}$$

A proof is given in Appendix A. The membrane paradigm was used in holographic systems in [2], and similar computations appear in [3, 4] for black holes with translational symmetry broken only in one direction. These results are special cases of this general formula. This formula may break down if the black hole horizon fragments and becomes disconnected, as was considered in [5, 6].

In Appendix B we derive remarkable results for theories in $d = 2$. In particular, if $\sigma[Z; \gamma_{ij}]$ is the conductivity matrix with given Z and γ_{ij}:

$$\det\left(\sigma[Z;\gamma_{ij}]\right)\det\left(\sigma\left[\frac{1}{Z};\gamma_{ij}\right]\right) = \frac{1}{e^8}. \tag{5}$$

This identity can be used to analytically compute σ_{ij} for a large disordered sample, with (without loss of generality)

$$Z(\varPhi) = \mathrm{e}^{-\varPhi} \tag{6}$$

and with

$$\varPhi = \varPhi_0 + \hat{\varPhi}(\mathbf{x}), \tag{7}$$

with $\hat{\varPhi}(\mathbf{x})$ an arbitrary random variable whose distribution is symmetric about $\hat{\varPhi} = 0$. In this case we conclude as $Z \to 1/Z$ simply changes $\varPhi_0 \to -\varPhi_0$:

$$\det\left(\sigma[Z;\gamma_{ij}]\right) = \frac{\mathrm{e}^{-2\varPhi_0}}{e^4}. \tag{8}$$

This result holds for a thermodynamically large, self-averaging sample – for any finite size sample there will be fluctuations.

If we set $Z = 1$, (5) gives

$$\det(\sigma) = \frac{1}{e^4}. \tag{9}$$

If we expect that on average for a disordered sample, the conductivity matrix is isotropic ($\sigma^{ij} = \sigma\delta^{ij}$), that fixes the conductivity to be $\sigma = 1/e^2$, exactly the clean result! A simple way to understand this result is as follows: suppose that in local coordinates, the metric is given by

$$\gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j \approx \left(\frac{l_x}{l_0}\right)^2 \mathrm{d}x^2 + \left(\frac{l_y}{l_0}\right)^2 \mathrm{d}y^2. \tag{10}$$

Then we expect "locally" $\sigma_{xx} \sim l_y/l_x$ and $\sigma_{yy} \sim l_x/l_y$ [7]: note (9) is obeyed "locally". On average l_y and l_x should have identical distributions, and thus local fluctuations in the metric wash out.

The robustness of σ in strongly disordered two-dimensional models is remarkable, and deserves further comments. In models where momentum dissipation is introduced through massive gravity [8] or "Q-lattice" axions [**?**], one finds the hydrodynamic result [9]

$$\sigma = \sigma_{\mathrm{Q}} + \frac{\mathcal{Q}^2\tau}{\epsilon + P}, \tag{11}$$

where $\mathcal{Q}$ is the charge density, ϵ the energy density, P the pressure, σ_{Q} the dissipative "quantum critical" conductivity without disorder, and τ a "momentum relaxation time", inversely related to the graviton mass. Before now, it was unclear whether the fact that (11) holds beyond the hydrodynamic limit was an unrealistic feature of massive gravity or similar theories. Our work confirms this is a sensible prediction of massive gravity for many systems at $\mathcal{Q} = 0$. (11) further implies another mechanism, $\tau \to 0$, by which the conductivity can reach its lower bound, σ_{Q}. This is expected to occur at strong disorder. Confirmation that strongly-disordered charged holographic models have a conductivity no smaller than $1/e^2$ in $d = 2$ would be a further non-trivial test of predictions of massive gravity.

Another instructive simplification is to assume that

$$\gamma_{ij} = \varSigma(\mathbf{x})^2\delta_{ij}. \tag{12}$$

Here it becomes simple to employ insight gained from the equivalence between Markov chains on lattices and the resistance of a resistor lattice [10]: see Appendix C. For arbitrary Z, we leverage this analogy to postulate (rather weak) lower and upper bounds to σ, for a self-averaging disordered sample:

$$\frac{1}{e^2\mathbb{E}[1/Z]} \le \sigma \le \frac{\mathbb{E}[Z]}{e^2}. \tag{13}$$

Let us briefly comment on the $d > 2$ case. Here, a conformal rescaling of the metric:

$$\gamma_{ij} \to \gamma_{ij}Z^{-2/(d-2)}, \tag{14}$$

removes the dilaton coupling entirely, so we may assume $Z = 1$ henceforth in $d > 2$. Analogous to (13), one finds, for a disordered black hole in the thermodynamic limit, with metric (12) (equivalently, choosing $Z = \varSigma^{d-2}$ and $\gamma_{ij} = \delta_{ij}$)

$$\frac{L^{d-2}}{e^2}\frac{1}{\mathbb{E}[\varSigma^{2-d}]} \le \sigma \le \frac{L^{d-2}}{e^2}\mathbb{E}\left[\varSigma^{d-2}\right] \tag{15}$$

We check these results by numerically solving (4) for various disorder realizations. To solve these equations and minimize finite size effects, large domains are necessary, and spectral methods are employed, along with domain decomposition [11]. Good agreement with our exact analytic results and consistency with our bounds is obtained.

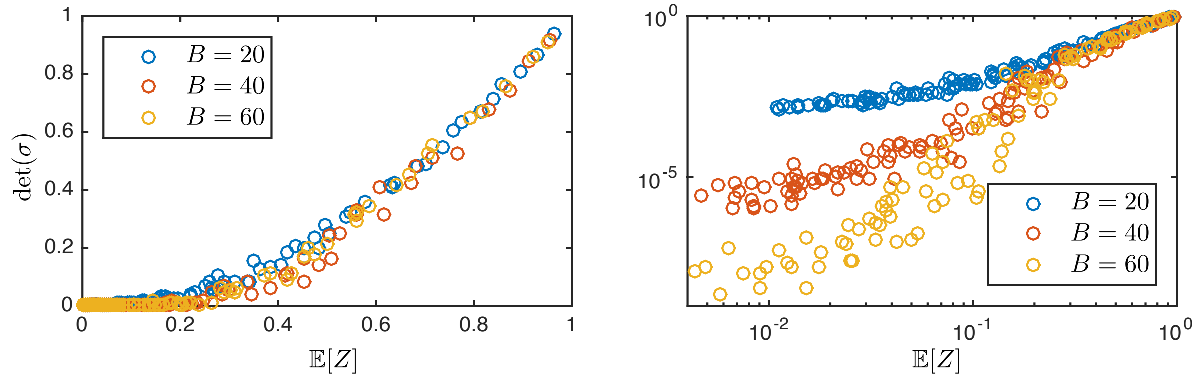

FIGURE 1: $\det(\sigma)$ from a black hole horizon for a theory in $d = 2$; we set $e = 1$, and use periodic boundary conditions with $|x|, |y| \le \pi$, with a discretized spatial grid of 701^2 points. We take $\gamma_{ij} = \delta_{ij}$ and $Z = \exp[-B\mathcal{Z}/(1 + 2\mathcal{Z})]$, where $\mathcal{Z} = \sum_{j=1}^{N} \exp(-(\sin^2(\phi_{jx} + x/2) + \sin^2(\phi_{jy} + y/2))/2\xi^2)$, with ϕ_{jx} and ϕ_{jy} independent random phases, and $B > 0$ is a random constant. We took various values of B and fixed $\xi = 20\pi/701$. When $\mathbb{E}[Z] \gtrsim 0.28 \equiv Z^*$, curves at different B approximately collapse, implying that current avoids the non-conducting bubbles; when $\mathbb{E}[Z] \lesssim Z^*$, the value of conductivity is sensitive to B. In the limit $B \to \infty$ and $\xi \to 0$, a metal-insulator transition appears at Z^*.

2. Metal-Insulator Transition

(13) and (15) constrain σ to deviate from the clean result by the strength of fluctuations. In non-interacting quantum field theory, the metal-insulator transition occurs at a finite disorder strength [12] in $d > 2$, and at arbitrarily small disorder in $d \le 2$. This transition relates to the destructive interference of matter waves scattering off of the disorder. Apparently, bulk fluctuations of the gauge field in holographic theories do not suffer from such interference.

Realizing the holographic metal-insulator transition takes more care. A "helical lattice" approach has generated such a transition in [13, 14], but there is no satisfying physical interpretation. Henceforth, we focus on the case $d = 2$, though our discussion readily generalizes. We will also assume a probe limit where the geometry is described by AdS-Schwarzchild, though we expect our qualitative approach to generalize.

(13) implies that to obtain an insulator with vanishing conductivity, we need $\mathbb{E}[1/Z]$ to be parametrically large. A substantial fraction of the horizon must have $Z \to 0$, as in these regions charge cannot effectively be transported. In fact, the $Z \to 0$ "bubbles" must percolate across the black hole horizon – this is because otherwise, electrical current can simply flow around these bubbles. When regions of space where Z is finite become disconnected from each other, charge transport is no longer possible. The classical percolation transition of these bubbles is a disorder-driven holographic metal-insulator transition. Mathematical support for this argument can be found in Appendix C. "Metal-insulator" transitions, similarly driven by the percolation of regions where charge cannot propagate, in a simple random resistor lattice are well-known [15].

A simple test of this proposal is to simply write down an ansatz for Z where "bubbles" where $Z \to 0$ percolate across the horizon, and to numerically compute the conductivity. Our numerics support this picture: see Figures 1 and 2.

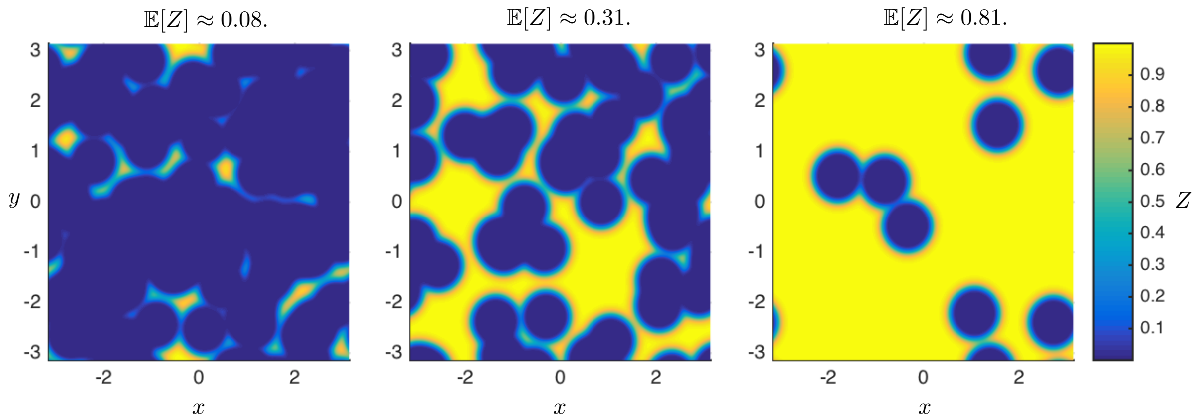

FIGURE 2: Surface plots of $Z(x, y)$ for various bubble densities. Depending on whether regions of high or low Z percolate across the horizon determines whether we are in the metallic or insulating phase, as is clear upon comparing with Figure 1.

2.1. Holographic Realizations

We now ask whether percolation mechanism proposed above for a disorder-driven metal-insulator transition can occur in a "realistic" holographic model: the Einstein-Maxwell-Dilaton theory with action

$$S = \int \mathrm{d}^{d+2}x\sqrt{-g}\left(\frac{R-2\Lambda}{16\pi G} - \mathcal{M}^d\left[\frac{1}{2}(\partial\Phi)^2 - \frac{V(\Phi)}{L^2}\right] - \frac{Z(\Phi)}{4e^2}F^2\right). \tag{16}$$

Here $\mathcal{M}$ is a mass scale, whose precise value is unimportant – we choose it so that Φ is strictly dimensionless, for simplicity, and

$$\Lambda = -\frac{d(d+1)}{2L^2}. \tag{17}$$

We are in a probe limit, so $G \to 0$. The equations of motion of the dilaton are

$$\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}g^{MN}\partial_N\Phi\right) = \frac{\partial V}{\partial\Phi}. \tag{18}$$

Let us begin by sourcing the dilaton with δ-like sources on the AdS boundary – analogous to point-like impurities in the dual theory. More carefully, if the density of the impurities is n, and the width of the impurity on the boundary is ξ, we need $n\xi^2 \ll 1$. A single impurity will induce an expanding dilaton bubble, as it propagates into the bulk.

Suppose for simplicity that V is quadratic. The width of the impurity scales as $\max(\xi, 1/T)$ and the percolation transition thus occurs when $n \sim T^2$ – the conductivity will transition from $\sigma \sim 1/e^2$ to $\sigma \sim \min(Z)/e^2$. If Φ is large in each bubble, $\min(Z)$ is small and we find an insulating phase.

In a typical supergravity truncation, however, we find that higher order corrections to Z and V tend to become non-neligible at the same scale in Φ. Thus for Z to be parametrically small in bubbles, nonlinear terms in V cannot be neglected.

We argue it's also possible to obtain a metal-insulator transition from sourcing with plane wave disorder. Suppose we pick an even $V(\Phi)$ with a local maximum at $\Phi = 0$, $V(\Phi) \approx -\Phi^2 + \cdots$ near $\Phi = 0$, and two global minima at $\Phi = \pm\Phi_0$, and then we pick

$$Z(\Phi) = \left(1 + \frac{\Phi}{\Phi_0}\right)^2. \tag{19}$$

Similar choices of V and Z were made in [16] to study translation-invariant insulators. Suppose that regions where $\Phi \to \pm\Phi_0$ persist all the way to the horizon as $T \to 0$. Then we associate a conductivity of 0 with regions where $\Phi \to -\Phi_0$, and $4L^{d-2}/e^2$ with regions where $\Phi \to +\Phi_0$. When the $+\Phi_0$ regions become disconnected, we obtain an insulator; otherwise, we have a metal. The precise nature of disorder on the boundary tunes the transition between the two phases. We give supporting arguments in Appendix D.

An alternative mechanism is to choose

$$Z(\Phi) = \left(1 - \left(\frac{\Phi}{\Phi_0}\right)^2\right)^2. \tag{20}$$

In this case both domains give an insulator. But if the amplitude of the dilaton on the boundary is weak enough, then the potential can be approximated as linear, Φ will be close to 0 at all times, and so the conductivity will be unaffected. This transition is truly driven by disorder strength, much like the standard metal-insulator transition in condensed matter physics.

3. Outlook

We have studied electrical transport in strongly coupled holographic quantum field theories at zero charge density. In particular, we reduced the computation of σ^{ij} to solving linear differential equations on the black hole horizon. We found analytic bounds on the resulting conductivity matrix, and proposed a disorder-driven holographic metal-insulator transition.

There are recent models [17, 18, **?**] of (quasi-2d) strange metals where momentum is not conserved past microscopic time scales. We have explicitly constructed examples of perfect metals in the presence of strong disorder. Our results may therefore have important implications for the feasibility of more realistic models of strongly disordered strange metals. We encourage searching for non-holographic field theories where σ_{Q} is immune to disorder, and the extension of our holographic approach to charged black holes.

Acknowledgements

We especially thank Ed Witten for providing his code for solving elliptic partial differential equations. This research was funded through Nvidia.

Appendix A. Membrane Paradigm

In this appendix we derive (3) and (4). To compute the conductivity, we must solve a linear response problem for the gauge field, in the black hole background (1). As the (uncharged) matter and gravity sectors will only be sourced at second order in the gauge field, no matter or metric perturbations will be sourced. We need only solve the bulk Maxwell's equations:

$$\nabla_M \left(ZF^{MN}\right) = \frac{1}{\sqrt{-g}}\partial_M \left(Z\sqrt{-g}F^{MN}\right) = 0. \tag{21}$$

Without loss of generality, the asymptotics on the metric ansatz (1) are:

$$P(\mathbf{x}, r \to 0) = \frac{S(\mathbf{x})}{4\pi T r} + \cdots , \tag{22a}$$

$$Q(\mathbf{x}, r \to 0) = -4\pi T r S(\mathbf{x}) + \cdots, \tag{22b}$$
$$G_{ij}(\mathbf{x}, r \to 0) = \gamma_{ij}(\mathbf{x}) + \cdots, \tag{22c}$$
$$P(\mathbf{x}, r \to \infty) = \frac{1}{r^2} + \cdots, \tag{22d}$$
$$Q(\mathbf{x}, r \to \infty) = r^2 + \cdots \tag{22e}$$
$$G_{ij}(\mathbf{x}, r \to \infty) = \delta_{ij} r^2 + \cdots. \tag{22f}$$

Here T is the Hawking temperature of the black hole, and the temperature of the dual field theory. The boundary conditions we impose on A are analogous to [4]. We write

$$A = a(\mathbf{x}, r) - E_j t \mathrm{d}x^j, \tag{23}$$

and the boundary conditions on the form a are:

$$a_r(\mathbf{x}, r \to \infty) = a_t(\mathbf{x}, r \to \infty) = 0, \tag{24a}$$
$$a_i(\mathbf{x}, r \to \infty) = -E_i t, \tag{24b}$$
$$4\pi T r a_r(\mathbf{x}, r \to 0) = a_t(\mathbf{x}, r \to 0) = \alpha(\mathbf{x}), \tag{24c}$$
$$a_i(\mathbf{x}, r \to 0) = -\frac{E_i}{4\pi T} \log(4\pi T r) + \text{finite}. \tag{24d}$$

The function $\alpha(\mathbf{x})$ is undetermined, and so (24c) demonstrates the proper asymptotics. These boundary conditions serve to produce a constant electric field $\mathbf{E}$ in the boundary theory, and are in-falling at the horizon. The direct conductivity matrix is found by the standard holographic dictionary [19], by computing

$$\sigma^{ij} \mathbb{E}[\partial_t A^j(r \to 0)] = -\sigma^{ij} E_j = \frac{L^{d-2}}{e^2} \mathbb{E}[r^d \partial_r A^i(r \to 0)]. \tag{25}$$

We also assume that $Z(r = 0) = 1$. This is generically the case when Z takes the form (6) (or something analogous), and the dilaton is dual to a relevant operator in the boundary theory.

The holographic membrane paradigm states that σ^{ij} can be computed by finding a quantity that is independent of bulk radius r, which equals the conductivity as $r \to \infty$. One then evaluates this quantity at the horizon and is able to use the boundary conditions to uniquely fix σ^{ij} in terms of horizon data – for us, the metric at $r = 0$, and the dilaton coupling. It is easy to find such a quantity: plugging the ansatz (23) into (21) we obtain the equations (r and i components respectively):

$$\partial_i \left(Z\sqrt{-g} g^{rr} g^{ij} (\partial_r a_j - \partial_j a_r) \right) = 0, \tag{26a}$$
$$\partial_r \left(Z\sqrt{-g} g^{rr} g^{ij} (\partial_r a_j - \partial_j a_r) \right) + \partial_k \left(Z\sqrt{-g} g^{kl} g^{ij} (\partial_l a_j - \partial_j a_l) \right) = 0. \tag{26b}$$

Using the second of these equations we see that

$$J^i \equiv \frac{1}{e^2} \mathbb{E} \left[-Z\sqrt{-g} g^{rr} g^{ij} (\partial_r a_j - \partial_j a_r) \right]. \tag{27}$$

is independent of r. And as $r \to \infty$, J^i simplifies to

$$J^i = \frac{L^{d-2}}{e^2} \mathbb{E} \left[-r^d \partial_r a_j (r \to \infty) \right], \tag{28}$$

and so we recognize this as the expected value of the spatially averaged current operator in the field theory.

When evaluating J^i in the limit $r \to 0$, the only nonvanishing terms are

$$J^i = \frac{L^{d-2}}{e^2}\mathbb{E}\left[-ZS\sqrt{\gamma}\frac{4\pi Tr}{S}\gamma^{ij}\left(-\frac{E_j}{4\pi Tr} - \frac{\partial_j \alpha}{4\pi Tr}\right)\right] = \frac{L^{d-2}}{e^2}\mathbb{E}\left[Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right]. \tag{29}$$

Thus, we have recovered (3). (4) follows straightforwardly from (26a):

$$0 = \partial_i\left(ZS\sqrt{\gamma}\frac{4\pi Tr}{S}\gamma^{ij}\left(-\frac{E_j}{4\pi Tr} - \frac{\partial_j\alpha}{4\pi Tr}\right)\right) = -\frac{1}{\sqrt{\gamma}}\partial_i\left(Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right)\right). \tag{30}$$

Thus α obeys the Poisson equation with a peculiar source. If we compactify our spatial directions, then up to a constant shift, there is a unique solution to this equation – on the infinite plane, we expect there is only one solution (up to a constant) which is well-behaved and finite. Physically this constant is simply a gauge redundancy. It is also worth noting that

$$\mathcal{J}^i(\mathbf{x}) = \frac{L^{d-2}}{e^2}Z\sqrt{\gamma}\gamma^{ij}(E_j + \partial_j\alpha) \tag{31}$$

is also conserved: $\partial_i\mathcal{J}^i = 0$. Though $\mathcal{J}^i \neq J^i$, $\mathbb{E}[\mathcal{J}^i] = \mathbb{E}[J^i]$.

The above derivation is valid for any black hole, with any translational symmetry breaking, regardless of how strong. There is, however, one exception which we note: if the horizon is disconnected, then it may be impossible to perform our membrane paradigm inspired calculation without pushing J^i through an event horizon of a "floating black hole" – see Figure **??**. Scenarios where this may occur are described in [5, 6].

A.1. Proof of Symmetry of Conductivity Tensor

[NEED TO UPDATE FOR FINITE Z] An explicit proof of the symmetry of σ^{ij}, in any d, can be found by writing down the explicit solution for α in terms of the Green's functions $\mathcal{G}(\mathbf{x};\mathbf{y})$ on the torus:

$$\triangle_{\mathbf{x}}\mathcal{G}(\mathbf{x};\mathbf{y}) = \triangle_{\mathbf{y}}\mathcal{G}(\mathbf{x};\mathbf{y}) = -\frac{1}{\sqrt{\gamma}}\delta(\mathbf{x}-\mathbf{y}) + \frac{1}{\tilde{V}_d}, \tag{32}$$

where

$$\tilde{V}_d = \int\limits_{\mathrm{T}^d} \mathrm{d}^d\mathbf{x}\,\sqrt{\gamma}. \tag{33}$$

We require this additional constant factor so that a single-valued $\mathcal{G}$ exists. In any case, standard manipulations demonstrate

$$\sigma^{ij} = \frac{1}{V_d}\left[\int \mathrm{d}^d\mathbf{x}\,\sqrt{\gamma}\gamma^{ij} + \int \mathrm{d}^d\mathbf{x}\mathrm{d}^d\mathbf{y}\,\sqrt{\gamma(\mathbf{x})\gamma(\mathbf{y})}\gamma^{ik}(\mathbf{x})\gamma^{jl}(\mathbf{y})\frac{\partial^2\mathcal{G}}{\partial x^k\partial y^l}\right] \tag{34}$$

which shows that σ^{ij} is a symmetric matrix for any d.

Appendix B. Two Dimensions

In this appendix we analyze the consequences of (29) and (30). For simplicity we set $e = 1$, which removes some clutter. The key observation is as follows: there must exist a differential form "$\partial_j\mathit{\Omega}$" such that

$$Z\sqrt{\gamma}\gamma^{ij}\left(E_j + \partial_j\alpha\right) = \epsilon^{ij}\partial_j\mathit{\Omega} \tag{35}$$

where $\epsilon^{xy} = -\epsilon^{yx} = 1$ is the Levi-Civita "tensor" without any metric pre-factors.

$$\partial_i \varOmega = \psi_i + \partial_i \varPsi(\mathbf{x}), \tag{36}$$

with $\varPsi$ a single-valued function and ψ_i a set of two constants, the elements of the non-trivial cohomology group of the torus.

Some simple manipulations, along with the fact that

$$\sqrt{\gamma}\gamma^{ij} = -\epsilon^{ik}\frac{\gamma_{kl}}{\sqrt{\gamma}}\epsilon^{lj}, \tag{37}$$

lead to the following "dual" equation:

$$-\epsilon^{ij}(E_j + \partial_j \alpha) = \frac{1}{Z}\sqrt{\gamma}\gamma^{ij}\left(\psi_j + \partial_j \varPsi\right). \tag{38}$$

We immediately obtain

$$0 = \frac{1}{\sqrt{\gamma}}\partial_i\left(\frac{\sqrt{\gamma}}{Z}\gamma^{ij}\left(\psi_j + \partial_j \varPsi\right)\right). \tag{39}$$

This equation is identical to (30), up to the factor $Z(\mathbf{x}) \to 1/Z(\mathbf{x})$. Since these equations are all linear in E_j, we may also write

$$\alpha \equiv \alpha^j E_j, \tag{40a}$$
$$\psi_i \equiv \psi_i^j E_j, \tag{40b}$$
$$\varPsi \equiv \varPsi^i \psi_i^j E_j. \tag{40c}$$

and solve equations for α^j, $\varPsi^j$ and ψ_i^j. It follows from (29) that

$$\epsilon^{ik}\psi_k^j = \sigma^{ij}[Z, \gamma_{ij}]. \tag{41}$$

Plugging (40) into (35), we obtain

$$Z\sqrt{\gamma}\gamma^{ij}\left(\delta_j^k + \partial_j \alpha^k\right) = \epsilon^{ij}\left(\psi_j^k + \psi_l^k \partial_j \varPsi^l\right) = \epsilon^{ij}\left(\delta_j^l + \partial_j \varPsi^l\right)\psi_l^k \tag{42}$$

But we can also write

$$\frac{1}{Z}\sqrt{\gamma}\gamma^{ij}\left(\delta_j^k + \partial_j \varPsi^k\right) = -\epsilon^{ij}\left(\delta_j^m + \partial_j \alpha^m\right)\left(\psi^{-1}\right)_m^k = -\epsilon^{ij}\left(\delta_j^m + \partial_j \alpha^m\right)\left(\sigma^{-1}\right)^{mn}\epsilon^{nk}. \tag{43}$$

The constant term on the right hand side of (43) is the conductivity

$$\sigma^{ik}\left[\frac{1}{Z};\gamma\right] = -\epsilon^{ij}\left(\sigma[Z,\gamma]^{-1}\right)^{jn}\epsilon^{nk} = \frac{\sigma^{ik}[Z;\gamma]}{\det(\sigma[Z;\gamma])}, \tag{44}$$

from which (5) follows.

Appendix C. Random Resistor Lattices

Here we detail the analogy between our computation of σ and the resistance of a random resistor lattice. The argument below is not made rigorously, but we emphasize that the bounds on σ that we obtain, using intuition from resistor lattices, have been checked numerically and always hold in our simulations. In this appendix, we take $\gamma_{ij} = \delta_{ij}$, but that the dilaton coupling Z is arbitrary.

We assume that our black hole is disordered and that the thermodynamic limit has been attained, so that the conductivity matrix is isotropic. Let us suppose that Z is smooth on all length scales smaller than l. Then we fix a UV length scale $a \ll l$, and "discretize" the black hole horizon as a cubic lattice with spacing a between all points, which we henceforth refer to as vertices, labeled as u, v, etc. Now, we imagine placing a resistor in between all neighboring vertices, of resistance

$$R_{uv} = \frac{e^2}{L^{d-2}a^{d-2}}\frac{1}{Z}, \tag{45}$$

with Z evaluated "close to the resistor" (since Z is approximately a constant on the length scale a, we will not worry about making this more precise). The factors of a are necessary to convert between resistivity and resistance. We choose a so that our sample has a volume $(Na)^d$ with $N \gg 1$ – the number of vertices in the lattice is $(N+1)^d$.

We claim that the conductivity σ must be related to the effective resistance of this resistor lattice, if each resistor obeys Ohm's Law:

$$V_u - V_v = R_{uv} I_{uv}, \tag{46}$$

where V_u is the voltage at node u, and I_{uv} is the current flowing from u to v. This equation only need hold if u and v are connected (we denote as $u \sim v$).[1] The claim is slightly subtle for two reasons. Mathematically, we want to obtain a single-valued voltage function V on our lattice to employ theorems below. This is equivalent (in the continuum language) to asking that boundary coordinates have trivial topology, so that E_j is the gradient of a scalar function. We employed periodic boundary conditions in the rest of the paper. Nonetheless, because we are interested in large systems, we expect universal results to emerge independent of boundary conditions. Physically, there is no local expression on the black hole horizon for the local current density. Luckily, the "current" $\mathcal{J}^i(\mathbf{x})$ defined in (31) is expressible in terms of local horizon data, and, using current conservation for both $\langle J^i\rangle$ and $\mathcal{J}^i$, we derive

$$\int \mathrm{d}^{d-1}\mathbf{x}_\perp \; \mathcal{J}^i(\mathbf{x}) = I_0 = \int \mathrm{d}^{d-1}\mathbf{x}_\perp \; \left\langle J^i(\mathbf{x})\right\rangle. \tag{47}$$

with the integral running over all directions perpendicular to i, at any fixed value of x^i.

Let us define a continuum voltage function

$$V(\mathbf{x}) = -\mathbf{E}\cdot\mathbf{x} - \alpha, \tag{48}$$

so that

$$\mathcal{J}^i = -\frac{L^{d-2}}{e^2} Z \partial^i V. \tag{49}$$

If $\mathbf{x}_u$ is the location of the vertex u in the discretized graph, then we expect the resistance R_{uv} should be chosen so that (46) is obeyed, as are

$$V_u = V(\mathbf{x}_u), \tag{50a}$$

$$I_{uv} = a^{d-1} n^i_{uv} \mathcal{J}^i\left(\frac{\mathbf{x}_u + \mathbf{x}_v}{2}\right), \tag{50b}$$

where n^i_{uv} is a unit vector pointing from u to v. Using (49) we obtain (45). In summary, we are simply discretizing our continuum problem with the resistor lattice.

[1] Otherwise, we interpret $R_{uv} = \infty$ and $I_{uv} = 0$, and then $V_u - V_v$ can be arbitrary.

This will prove sufficient to compute σ, as we physically wish to add a net current of $I_0 = (Na)^{d-1}\mathbb{E}[J]$ to one surface of the cubic lattice, and collect that current at the opposite surface. We achieve this by adding two special vertices to our graph: a vertex i with an incoming current I_0 (from an external source), and a vertex f with outgoing current I_0: see Figure **??**b. At all other vertices, we must have current conservation, so ingoing and outgoing currents balance. We connect the vertex i to every vertex on one surface of the lattice, and f to every vertex on the opposite surface. Every vertex u connected to i (denoted with $u \sim i$) has $R_{ui} = 0$; similarly, if $f \sim u$ then $R_{uf} = 0$.

Suppose we set the voltage $V_i = V_0$, and $V_f = 0$, and that this induces a current

$$I_0 \equiv \frac{V_0}{R_{\text{eff}}}. \tag{51}$$

Then by Ohm's Law, $V_u = V_0$ if $u \sim i$, and $V_u = 0$ if $u \sim f$. I_0 is the total current flowing through the lattice, and thus R_{eff} is the effective resistance of our lattice along the appropriate axis. In the thermodynamic limit, this resistance must be related to the conductivity as

$$R_{\text{eff}} = \frac{1}{(Na)^{d-2}\sigma}. \tag{52}$$

Now that we have argued that the computation of the conductivity should be approximated by finding the effective resistance between the two special resistors above, we may exploit mathematical results to provide both upper and lower bounds on σ. For more details on this approach, see [10].

C.1. Thomson's Bound

Let us begin with the lower bound on the conductivity, which is a upper bound on the resistance. This is found using Thomson's principle, which mathematically is stated as follows. Define a flow function $\mathcal{I}_{uv} = -\mathcal{I}_{vu}$, where u and v are vertices of the lattice. $\mathcal{I}$ represents a proposed distribution of electrical currents through the lattice. We are interested in flows with a net current of I_0 flowing between the vertices i and f. Thus, we demand by current conservation that

$$\sum_{u\sim i} \mathcal{I}_{iu} = I_0, \tag{53a}$$

$$\sum_{u\sim f} \mathcal{I}_{fu} = -I_0, \tag{53b}$$

$$\sum_{u\sim v,\ v\neq i,f} \mathcal{I}_{uv} = 0, \tag{53c}$$

where $i \sim u$ implies that i and u are connected by a resistor. One can prove that [10]

$$\mathcal{R}[\mathcal{I}] = \frac{1}{I_0^2}\sum R_{uv}\mathcal{I}_{uv}^2 \geq R_{\text{eff}}, \tag{54}$$

with the bound saturated precisely on the physical flow $\mathcal{I}$, which has the property that one can define a singly-valued voltage function, with Ohm's Law obeyed for each resistor. The physical intuition behind this bound is that any other current flow has a loop of current, and one can show that such a loop dissipates unnecessary power. As $\mathcal{R}$ computes a power dissipated (normalized by the net current), $\mathcal{R}$ is thus larger than the true resistance, due to these spurious loops.

A simple projection for $\mathcal{I}_{uv}$ is:

$$\mathcal{I}_{uv} = \frac{I_0}{N^{d-1}} \times \begin{cases} 1 & u \sim v \text{ oriented parallel to } if \text{ axis, and } f \text{ closer to } v \\ -1 & u \sim v \text{ oriented parallel to } if \text{ axis, and } f \text{ closer to } u \\ 0 & \text{otherwise} \end{cases}. \tag{55}$$

Since $R_{ui} = R_{uf} = 0$ we do not care about $\mathcal{I}(u,i)$ or $\mathcal{I}(u,f)$. Thomson's bound gives

$$R_{\text{eff}} \le \mathcal{R}[\mathcal{I}] = \frac{e^2}{L^{d-2}a^{d-2}I_0^2} \sum_{\text{vertical}} \frac{1}{Z}\left(\frac{I_0}{N^{d-1}}\right)^2 \approx \frac{e^2}{L^{d-2}a^{d-2}} \frac{N^d}{N^{2(d-1)}} \mathbb{E}\left[\frac{1}{Z}\right]. \tag{56}$$

Using (52) we obtain our lower bound on the conductivity in (13) and (15).

This is not an optimal projection, because we have assumed that current flows over regions where $Z \to 0$. If at all possible, the currents will avoid these regions. If the resistors where $Z \to 0$ do not percolate across the lattice, then one can easily construct a modified flow $\mathcal{I}$ which avoids regions of low Z, as shown in Figure **??**. Precisely when it is impossible to find a single path from i to f that avoids regions where $Z \to 0$, corresponding to the onset of the percolation transition, must our lower bound scale as $\sigma \sim \min(Z)$.

C.2. Nash-Williams Bound

Now we obtain our upper bound on the conductivity. To do this, we use a Nash-Williams bound, which proceeds as follows. Define a cutset C of the edges in our lattice to be a group of edges such that every path from i to f must pass through at least one edge in C. Suppose that we have a disjoint set of cutsets, C_α. Then one can prove that [10]

$$R_{\text{eff}} \ge \sum_\alpha \left[\sum_{uv\in C_\alpha} \frac{1}{R_{uv}}\right]^{-1}. \tag{57}$$

Intuitively, we have found bunches of parallel resistors, and then placed each bunch in series, and assumed every other resistor in the lattice has zero resistance, which leads to a lower bound. Choosing our disjoint cutsets to be the N sets of $(N+1)^{d-1}$ resistors equidistant from both i and f, and oriented along the if axis, we obtain

$$R_{\text{eff}} \ge N\left[(N+1)^{d-1}\frac{L^{d-2}a^{d-2}}{e^2}\mathbb{E}[Z]\right]^{-1} \approx \left[\frac{(NLa)^{d-2}}{e^2}\mathbb{E}[Z]\right]^{-1}, \tag{58}$$

which leads to our lower bound on σ in (13) and (15).

Appendix D. Dilaton Domain Walls

In this appendix, we give arguments that there are choices of $V(\Phi)$ for which the domain walls persist all the way to zero temperature. We make the standard coordinate choice for AdS-Schwarzchild:

$$\mathrm{d}s^2 = \frac{L^2}{z^2}\left[\mathrm{d}z^2 - \mathrm{d}t^2 + \mathrm{d}\mathbf{x}^2\right]. \tag{59}$$

with

$$f(z) = 1 - \left(\frac{z}{z_0}\right)^{d+1}. \tag{60}$$

Let us suppose that we source the dilaton as

$$\Phi(z\to 0) = zA\sin(kx) + \cdots. \tag{61}$$

At $T = 0$, the geometry is AdS_4. We wish to argue that there is a solution to the dilaton equation of motion for which

$$\Phi(z\to\pm\infty) \approx \operatorname{sign}(\sin(kx))\Phi_0. \tag{62}$$

To begin with, let us suppose that $k \to 0$, so that A is the only dimensional parameter. Let us note that the scale at which nonlinear corrections to the equation of motion kick in is at $z = z_0 \sim 1/A$. And further, if $k \to 0$, then the thickness of any possible domain wall must be $\xi \sim 1/A$.

Now, any solution to the static equation of motion for Φ must be an extremum of the functional

$$\mathcal{E} = \int \mathrm{d}z\mathrm{d}^d\mathbf{x}\, \frac{1}{z^{d+2}} \left(\frac{z^2}{2} \left((\partial_z \Phi)^2 + (\partial_j \Phi)^2 \right) + V(\Phi) \right). \tag{63}$$

After the appropriate addition of counterterms, which are irrelevant for minimization purposes as we have fixed the boundary conditions on Φ, $\mathcal{E}$ is bounded from below. Let us now estimate, the value of $\mathcal{E}$ associated with domain walls. The contributions from the domain walls scale as, per surface area of the domain wall:

$$\int\limits_{z_0}^{\infty} \frac{\mathrm{d}z}{z^{d+2}} \xi \left(z^2 \left(\frac{\Phi_0}{\xi} \right)^2 \right) \sim \frac{\Phi_0^2}{\xi z_0^{d-1}} \sim \Phi_0^2 A^d.$$

The contributions from the domains are (per surface area)

$$\int\limits_{z_0}^{\infty} \frac{\mathrm{d}z}{z^{d+2}} \frac{1}{k} V(\Phi_0) \sim -|V_{\mathrm{min}}| \frac{A^{d+1}}{k}.$$

We need the modulus of this second term to be much larger than the modulus of the first term.

We also need to ensure that once $z \sim 1/k$, it does not become favorable for these domains to merge back together.

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