# Notes on strong disorder

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## I. CONSTRUCTION OF THE METRIC

### Solving for the vector field

The theory considered in [1] is

$$S_{bulk} = \frac{1}{8\pi\ell_p^2}\int d^4x\sqrt{-g}\left(R - \frac{3}{2}(\partial\chi)^2 + \frac{6}{\ell^2}\cosh\chi\right) - \frac{1}{8\pi}\int d^4x\sqrt{-g}\left(\frac{1}{6}v^3e^{3\chi}F_0^2 - \frac{1}{2}ve^{\chi}F_1^2\right), \tag{1}$$

which we can further truncate for our purposes to

$$S_{bulk} = \frac{1}{8\pi\ell_p^2}\int d^4x\sqrt{-g}\left(R + \frac{6}{\ell^2} - 2\pi\ell_p^2\sigma(z)F_{\mu\nu}F^{\mu\nu}\right). \tag{2}$$

In the presence of a point charge, the full action becomes

$$S = S_{bulk} + Q\int A_\mu dx^\mu. \tag{3}$$

Assume that the point charge is located at $(z, x, y) = (z_p, 0, 0)$, where $x$ and $y$ (or the polar $r$ and $\theta$) parametrise the flat boundary space. By assuming the background

$$ds^2 = -g_{tt}(z)dt^2 + g_{zz}(z)dz^2 + g_{xx}(z)\left(dx^2 + dy^2\right), \tag{4}$$

and the electrostatic field with $A_x = A_y = 0$, the Maxwell's equations give

$$\partial_z\left(\frac{\sigma g_{xx}}{\sqrt{g_{zz}g_{tt}}}\partial_z A_t\right) + \sigma\sqrt{\frac{g_{zz}}{g_{tt}}}\left(\partial_x^2 + \partial_y^2\right)A_t = Q\delta(z - z_p)\delta(x)\delta(y). \tag{5}$$

To find the solution for the vector field we follow [1] and use the WKB approximation

$$A_t(z, x, y) = \int \frac{d^2k}{(2\pi)^2}e^{-i\vec{k}\cdot\vec{x}}\exp\left\{\frac{1}{\lambda}\left(W_0 + \lambda W_1 + \cdots\right)\right\}, \tag{6}$$

$$\partial_z \to \lambda\partial_z, \tag{7}$$

with $\lambda \ll 1$. The solution is

$$A_t(z, x, y) = \zeta(z)^{-1/4}\int \frac{d^2k}{(2\pi)^2}e^{-i\vec{k}\cdot\vec{x}}\left(c_k e^{|\vec{k}|v(z)} + d_k e^{-|\vec{k}|v(z)}\right), \tag{8}$$

where

$$\zeta = \sigma^2\frac{g_{xx}}{g_{tt}}, \qquad \gamma = \frac{g_{zz}}{g_{xx}}, \qquad v = \int dz\sqrt{\gamma(z)}. \tag{9}$$

The momentum-dependent constants can be fixed by matching the near-boundary and near-horizon geometries, as was done in [1].

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**Solving for the metric backreaction**

The Einstein's equation derived from is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = \kappa_4^2\sigma\left(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \tag{10}$$

where $\Lambda = -3/\ell^2$ and the four-dimensional Newton's constant is $\kappa_4^2 = 4\pi\ell_p^2$. We will set $\ell = 1$.

In solving for $A_t$, the background was considered as fixed. Hence, we can state that $A_t$ is of order $\epsilon$, where $\epsilon \ll 1$. Now, to look for a backreaction of the metric to $A_t$, we assume that the metric can be written as

$$ds^2 = -\left(g_{tt} + \epsilon^2 h_{tt}\right)dt^2 + \left(g_{zz} + \epsilon^2 h_{zz}\right)dz^2 + 2\epsilon^2 h_{zr}dzdr + \left(g_{xx} + \epsilon^2 h_{rr}\right)dr^2 + r^2\left(g_{xx} + \epsilon^2 h_{\theta\theta}\right)d\theta^2. \tag{11}$$

The metric will be rotationally symmetric around the point charge's spatial origin at $r = 0$. As a further simplification, we introduce a WKB parameter for slowly varying spatial dependence of the metric, alongside slowly varying radial bulk dependence,

$$\partial_z \to \lambda\partial_z, \qquad \partial_r \to \mu\partial_r, \tag{12}$$

with $\lambda, \mu \ll 1$. Next, we expand the Einstein's equation to orders $\{\mathcal{O}(\epsilon^2), \mathcal{O}(\lambda^0), \mathcal{O}(\mu^2)\}$, $\{\mathcal{O}(\epsilon^2), \mathcal{O}(\lambda), \mathcal{O}(\mu)\}$ and $\{\mathcal{O}(\epsilon^2), \mathcal{O}(\lambda^2), \mathcal{O}(\mu^0)\}$, [i.e. this is the same as setting $\mu = \lambda$ and expanding to $\mathcal{O}(\lambda^2)$], giving us a very simple solution

$$h_{tt} = \frac{\kappa_4^2\sigma}{6g_{zz}g_{xx}}\left[g_{xx}\left(\partial_z A_t\right)^2 + g_{zz}\left(\partial_r A_t\right)^2\right], \tag{13}$$

$$h_{zz} = \frac{\kappa_4^2\sigma}{6g_{tt}g_{xx}}\left[g_{xx}\left(\partial_z A_t\right)^2 - g_{zz}\left(\partial_r A_t\right)^2\right], \tag{14}$$

$$h_{zr} = \frac{\kappa_4^2\sigma}{3g_{tt}}\left(\partial_z A_t\right)\left(\partial_r A_t\right), \tag{15}$$

$$h_{rr} = -\frac{\kappa_4^2\sigma}{6g_{tt}g_{zz}}\left[g_{xx}\left(\partial_z A_t\right)^2 - g_{zz}\left(\partial_r A_t\right)^2\right], \tag{16}$$

$$h_{\theta\theta} = -\frac{\kappa_4^2\sigma}{6g_{tt}g_{zz}}\left[g_{xx}\left(\partial_z A_t\right)^2 + g_{zz}\left(\partial_r A_t\right)^2\right], \tag{17}$$

where the four-dimensional Newton's constant naturally suppresses the $h_{\mu\nu}$ terms in comparison to $g_{\mu\nu}$.

What will be more useful for future is to solve for the metric in Cartesian form, which gives us

$$\begin{aligned} ds^2 = &-\left(g_{tt} + \epsilon^2 h_{tt}\right)dt^2 + \left(g_{zz} + \epsilon^2 h_{zz}\right)dz^2 + \left(g_{xx} + \epsilon^2 h_{xx}\right)dx^2 + \left(g_{xx} + \epsilon^2 h_{yy}\right)dy^2 \\ &+ 2\epsilon^2 h_{zx}dzdx + 2\epsilon^2 h_{zy}dzdy + 2\epsilon^2 h_{xy}dxdy, \end{aligned} \tag{18}$$

with

$$h_{tt} = \frac{\kappa_4^2\sigma}{6g_{xx}g_{zz}}\left[g_{zz}\left(\left(\partial_x A_t\right)^2 + \left(\partial_y A_t\right)^2\right) + g_{xx}\left(\partial_z A_t\right)^2\right], \tag{19}$$

$$h_{zz} = -\frac{\kappa_4^2\sigma}{6g_{tt}g_{xx}}\left[g_{zz}\left(\left(\partial_x A_t\right)^2 + \left(\partial_y A_t\right)^2\right) - g_{xx}\left(\partial_z A_t\right)^2\right], \tag{20}$$

$$h_{xx} = \frac{\kappa_4^2\sigma}{6g_{tt}g_{zz}}\left[g_{zz}\left(\left(\partial_x A_t\right)^2 - \left(\partial_y A_t\right)^2\right) - g_{xx}\left(\partial_z A_t\right)^2\right], \tag{21}$$

$$h_{yy} = \frac{\kappa_4^2\sigma}{6g_{tt}g_{zz}}\left[g_{zz}\left(\left(\partial_y A_t\right)^2 - \left(\partial_x A_t\right)^2\right) - g_{xx}\left(\partial_z A_t\right)^2\right], \tag{22}$$

$$h_{zx} = \frac{\kappa_4^2\sigma}{3g_{tt}}\partial_z A_t\partial_x A_t, \tag{23}$$

$$h_{zy} = \frac{\kappa_4^2\sigma}{3g_{tt}}\partial_z A_t\partial_y A_t, \tag{24}$$

$$h_{xy} = \frac{\kappa_4^2\sigma}{3g_{tt}}\partial_x A_t\partial_y A_t. \tag{25}$$

### A charge near AdS infinity

All bulk geometries of (present) interest are asymptotically AdS with $g_{tt} = g_{zz} = g_{xx} = 1/z^2$. For simplicity, we will consider $\sigma(z) = \sigma$. The relevant black hole solution in such as setup (with the scalar $\chi = 0$) is the AdS-RN$_4$ black hole. An electric charge placed far from the horizon, $z_p \ll z_h$ then induces an electric field with an approximate form of an electric charge in pure AdS,

$$A_t = q\left(\frac{1}{\sqrt{(z-z_p)^2+x^2+y^2}} - \frac{1}{\sqrt{(z+z_p)^2+x^2+y^2}}\right). \tag{26}$$

[Precise value of $q$ to be fixed later] Near the boundary, for $z \lesssim z_p \ll z_h$, the metric becomes

$$\begin{aligned} ds^2 = &-\left(\frac{f(z)}{z^2} + \left(x^2+y^2\right)h(z,x,y)\right)dt^2 + \left(\frac{1}{z^2 f(z)} - \frac{\left(x^2+y^2\right)h(z,x,y)}{f(z)^2}\right)dz^2 \\ &+\left(\frac{1}{z^2} + \frac{\left(x^2-y^2\right)h(z,x,y)}{f(z)}\right)dx^2 + \left(\frac{1}{z^2} + \frac{\left(y^2-x^2\right)h(z,x,y)}{f(z)}\right)dy^2 + 2\frac{2xyh(z,x,y)}{f(z)}dxdy, \end{aligned} \tag{27}$$

with

$$h(z,x,y) = \frac{1}{6}\kappa_4^2\sigma q^2 z^2\left[\frac{1}{\left((z-z_p)^2+x^2+y^2\right)^{3/2}} - \frac{1}{\left((z+z_p)^2+x^2+y^2\right)^{3/2}}\right]^2 \tag{28}$$

and the AdS-RN$_4$ emblackening factor

$$f(z) = 1 - M\left(\frac{z}{z_h}\right)^3 + Q_{BH}^2\left(\frac{z}{z_h}\right)^4. \tag{29}$$

Now, the point charge correction to the background can be re-summed into the emblackening factor, giving

$$\begin{aligned} ds^2 = &-\frac{\bar{f}(z,x,y)}{z^2}dt^2 + \frac{1}{z^2\bar{f}(z,x,y)}dz^2 + \left(\frac{1}{z^2} + \frac{\left(x^2-y^2\right)h(z,x,y)}{f(z)}\right)dx^2 \\ &+\left(\frac{1}{z^2} + \frac{\left(y^2-x^2\right)h(z,x,y)}{f(z)}\right)dy^2 + 2\frac{2xyh(z,x,y)}{f(z)}dxdy, \end{aligned} \tag{30}$$

with

$$\bar{f}(z,x,y) = 1 - M\left(\frac{z}{z_h}\right)^3 + Q_{BH}^2\left(\frac{z}{z_h}\right)^4 + \frac{\kappa_4^2\sigma q^2 z_h^4}{6}\left(x^2+y^2\right)\left(\frac{z}{z_h}\right)^4\left[\frac{1}{\left((z-z_p)^2+x^2+y^2\right)^{3/2}} - \frac{1}{\left((z+z_p)^2+x^2+y^2\right)^{3/2}}\right]^2. \tag{31}$$

### The full metric for a charge in a black brane geometry

The next step is to find the metric backreaction to the point charge in the full black brane geometry with the metric stated in Eq. (4). It was found in [1] that the vector field in such a geometry is

$$A_t^{(\pm)}(z,k) = \zeta(z)^{-1/4}\left(c_k^{(\pm)}e^{|\vec{k}|v(z)} + d_k^{(\pm)}e^{-|\vec{k}|v(z)}\right), \tag{32}$$

where $(+)$ stands for the quantities in the region of $0 \le z \le z_p$ and $(-)$ stands for the $z_p \le z \le z_h$ region, with

$$v^{(+)} = \int_0^z dz\sqrt{\gamma(z)}, \qquad v^{(-)} = \int_z^{z_h} dz\sqrt{\gamma(z)}. \tag{33}$$

Let us begin by evaluating the vector field in the (+) region, where

$$A_t^{(+)} = \frac{Q}{(2\pi)^2 \zeta_p^{1/4} \zeta^{1/4}(z)} \int_0^{2\pi} d\theta \int_0^{\infty} dk k e^{-ikr\cos\theta} \frac{\sinh\left(k v_h^{(+)} - k v_p^{(+)}\right) \sinh\left(k v^{(+)}(z)\right)}{k \sinh\left(k v_h^{(+)}\right)}. \tag{34}$$

Performing first the integral over $\theta$, we get the expression

$$A_t^{(+)} = \frac{Q}{(2\pi) \zeta_p^{1/4} \zeta^{1/4}(z)} \int_0^{\infty} dk k J_0(kr) \frac{\sinh\left(k v_h^{(+)} - k v_p^{(+)}\right) \sinh\left(k v^{(+)}(z)\right)}{k \sinh\left(k v_h^{(+)}\right)}, \tag{35}$$

which is in fact the Hankel transform of $\sinh\left(k v_h^{(+)} - k v_p^{(+)}\right) \sinh\left(k v^{(+)}(z)\right) / k \sinh\left(k v_h^{(+)}\right)$. In order to perform the integral transform, we assume that $e^{-k v_h} \ll 1$ and expand the integrand in this quantity, finding

$$A_t^{(+)} = \frac{Q}{2\pi \zeta_p^{1/4} \zeta^{1/4}(z)} \int_0^{\infty} dk \frac{k J_0(kr)}{2k} \left[ e^{-k\left(v_p^{(+)} - v^{(+)}(z)\right)} - e^{-k\left(v_p^{(+)} + v^{(+)}(z)\right)} - \sum_{m_1, m_2 = \pm 1} m_1 m_2 \sum_{n=1}^{\infty} e^{-k\left(2n v_h^{(+)} + m_1 v_p^{(+)} + m_2 v^{(+)}(z)\right)} \right], \tag{36}$$

which leads to,

$$\begin{aligned} A_t^{(+)} = \frac{Q}{4\pi \zeta_p^{1/4} \zeta^{1/4}(z)} \int_0^{\infty} dk \frac{k J_0(kr)}{k} \sum_{n=0}^{\infty} \Big[ & e^{-k\left(v_p^{(+)} - v^{(+)}(z) + 2n v_h^{(+)}\right)} - e^{-k\left(v_p^{(+)} + v^{(+)}(z) + 2n v_h^{(+)}\right)} \\ & - e^{+k\left(v_p^{(+)} + v^{(+)}(z) - 2(n+1) v_h^{(+)}\right)} + e^{+k\left(v_p^{(+)} - v^{(+)}(z) - 2(n+1) v_h^{(+)}\right)} \Big]. \end{aligned} \tag{37}$$

By noting that all exponential functions have the form $e^{-ka}$ with $a \geq 0$, we can find the full expression for the vector field in the region of $0 \leq z \leq z_p$,

$$\begin{aligned} A_t^{(+)} = \frac{Q}{4\pi \zeta_p^{1/4} \zeta^{1/4}(z)} \sum_{n=0}^{\infty} \Bigg[ & \frac{1}{\sqrt{\left(v_p^{(+)} - v^{(+)}(z) + 2n v_h^{(+)}\right)^2 + x^2 + y^2}} - \frac{1}{\sqrt{\left(v_p^{(+)} + v^{(+)}(z) + 2n v_h^{(+)}\right)^2 + x^2 + y^2}} \\ & - \frac{1}{\sqrt{\left(v_p^{(+)} + v^{(+)}(z) - 2(n+1) v_h^{(+)}\right)^2 + x^2 + y^2}} + \frac{1}{\sqrt{\left(v_p^{(+)} - v^{(+)}(z) - 2(n+1) v_h^{(+)}\right)^2 + x^2 + y^2}} \Bigg]. \end{aligned} \tag{38}$$

Next we solve for the vector field in the $z_p \leq z \leq z_h$ region where

$$\begin{aligned} A_t^{(-)} &= \frac{Q}{(2\pi)^2 \zeta_p^{1/4} \zeta^{1/4}(z)} \int_0^{2\pi} d\theta \int_0^{\infty} dk k e^{-ikr\cos\theta} \frac{e^{-k v_h^{(+)}}}{k} \left[ e^{k v_p^{(+)}} - \frac{\sinh\left(k v_h^{(+)} - k v_p^{(+)}\right)}{\sinh\left(k v_h^{(+)}\right)} \right] \sinh\left(k v^{(-)}(z)\right) && (39) \\ &= \frac{Q}{2\pi \zeta_p^{1/4} \zeta^{1/4}(z)} \sum_{n=0}^{\infty} \sum_{m_1, m_2 = \pm 1} m_1 m_2 \int_0^{\infty} dk \frac{k J_0(kr)}{2k} e^{-k\left((2n+1) v_h^{(+)} + m_1 v_p^{(+)} + m_2 v^{(-)}(z)\right)} && (40) \\ &= \frac{Q}{4\pi \zeta_p^{1/4} \zeta^{1/4}(z)} \sum_{n=0}^{\infty} \sum_{m_1, m_2 = \pm 1} m_1 m_2 \frac{1}{\sqrt{\left((2n+1) v_h^{(+)} + m_1 v_p^{(+)} + m_2 v^{(-)}(z)\right)^2 + x^2 + y^2}} && (41) \\ &= \frac{Q}{4\pi \zeta_p^{1/4} \zeta^{1/4}(z)} \sum_{n=0}^{\infty} \sum_{m_{p,h} = \pm 1} \frac{m_p}{\sqrt{\left(v^{(-)}(z) + m_p v_p^{(+)} + (2n+1) m_h v_h^{(+)}\right)^2 + x^2 + y^2}} && (42) \\ &= \frac{Q}{4\pi \zeta_p^{1/4} \zeta^{1/4}(z)} \sum_{n=0}^{\infty} \sum_{m_{p,h} = \pm 1} \frac{m_p}{\sqrt{\left(v^{(+)}(z) - m_p v_p^{(+)} - 2n m_h v_h^{(+)} - (m_h + 1) v_h^{(+)}\right)^2 + x^2 + y^2}}, && (43) \end{aligned}$$

where we have used the identity $v^{(+)}(z)+v^{(-)}(z)=v_h^{(+)}$ in going to the last line. Hence, we find

$$A_t^{(-)}=\frac{Q}{4\pi\zeta_p^{1/4}\zeta^{1/4}(z)}\sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{\left(v^{(+)}(z)-v_p^{(+)}+2nv_h^{(+)}\right)^2+x^2+y^2}}-\frac{1}{\sqrt{\left(v_p^{(+)}+v^{(+)}(z)+2nv_h^{(+)}\right)^2+x^2+y^2}}\right.$$
$$\left.-\frac{1}{\sqrt{\left(v_p^{(+)}+v^{(+)}(z)-2(n+1)v_h^{(+)}\right)^2+x^2+y^2}}+\frac{1}{\sqrt{\left(v^{(+)}(z)-v_p^{(+)}-2(n+1)v_h^{(+)}\right)^2+x^2+y^2}}\right]. \tag{44}$$

By comparing Eqs. (37) and (44) we see that the expression differ only in the signs in front of the difference $v^{(+)}(z)-v_p^{(+)}$. This comes from the fact that $v^{(+)}\le v_p^{(+)}$ in the $(+)$ region and $v^{(+)}\ge v_p^{(+)}$ in the $(-)$ region. Clearly both results match at $z=z_p$. By defining all quantities with the index $(+)$ as ones without any $(\pm)$ index, i.e.

$$v(z)\equiv v^{(+)}(z)=\int_0^z dz'\sqrt{\frac{g_{zz}(z')}{g_{xx}(z')}},\qquad v_p\equiv v(z_p),\qquad v_h\equiv v(z_h), \tag{45}$$

we can write the final expression for the vector field valid throughout the entire bulk geometry, $0\le z\le z_h$ as

$$A_t(z,x,y)=\frac{Q}{4\pi\zeta_p^{1/4}\zeta^{1/4}(z)}\sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{(|v_p-v(z)|+2nv_h)^2+x^2+y^2}}-\frac{1}{\sqrt{(v_p+v(z)+2nv_h)^2+x^2+y^2}}\right.$$
$$\left.-\frac{1}{\sqrt{(v_p+v(z)-2(n+1)v_h)^2+x^2+y^2}}+\frac{1}{\sqrt{(|v_p-v(z)|-2(n+1)v_h)^2+x^2+y^2}}\right]. \tag{46}$$

As a further check on this result, we see that by taking the limit of $v_h\to\infty$, only the first two terms at $n=0$ can contribute. Hence, we recover the pure AdS result.

## Multiple charges

We are interested in introducing multiple charges into the black brane geometry. Since Maxwell's equations are linear, we can simply add the various vector fields corresponding to a set of $N$ point charges of charge $Q_i$ placed at $(z_1,x_1,y_1)$, … $(z_N,x_N,y_N)$. Similarly, we define $v_{p,i}=v(z_i)$. The total vector field is then

$$A_t(z,x,y)=\frac{1}{4\pi\zeta^{1/4}(z)}\sum_{i=1}^{N}\frac{Q_i}{\zeta_{p,i}^{1/4}}\left[\sum_{n=0}^{\infty}\left[\frac{1}{\sqrt{(|v_{p,i}-v(z)|+2nv_h)^2+(x_i-x)^2+(y_i-y)^2}}\right.\right.$$
$$-\frac{1}{\sqrt{(v_{p,i}+v(z)+2nv_h)^2+(x_i-x)^2+(y_i-y)^2}}-\frac{1}{\sqrt{(v_{p,i}+v(z)-2(n+1)v_h)^2+(x_i-x)^2+(y_i-y)^2}}$$
$$\left.\left.+\frac{1}{\sqrt{(|v_{p,i}-v(z)|-2(n+1)v_h)^2+(x_i-x)^2+(y_i-y)^2}}\right]\right]. \tag{47}$$

The final result for the metric, including the leading-order WKB backreaction, is then, as above,

$$ds^2=-(g_{tt}+h_{tt})\,dt^2+(g_{zz}+h_{zz})\,dz^2+(g_{xx}+h_{xx})\,dx^2+(g_{xx}+h_{yy})\,dy^2+2h_{xy}dxdy, \tag{48}$$

with

$$h_{tt} = \frac{\kappa_4^2\sigma}{6g_{xx}}\left[(\partial_x A_t)^2 + (\partial_y A_t)^2\right], \tag{49}$$

$$h_{zz} = -\frac{\kappa_4^2\sigma g_{zz}}{6g_{tt}g_{xx}}\left[(\partial_x A_t)^2 + (\partial_y A_t)^2\right], \tag{50}$$

$$h_{xx} = \frac{\kappa_4^2\sigma}{6g_{tt}}\left[(\partial_x A_t)^2 - (\partial_y A_t)^2\right], \tag{51}$$

$$h_{yy} = \frac{\kappa_4^2\sigma}{6g_{tt}}\left[(\partial_y A_t)^2 - (\partial_x A_t)^2\right], \tag{52}$$

$$h_{xy} = \frac{\kappa_4^2\sigma}{3g_{tt}}\partial_x A_t \partial_y A_t. \tag{53}$$

## II. CHARGE DIFFUSION IN THE PRESENCE OF A SINGLE CHARGE

In this section we compute the charge diffusion in the background geometry with a single charge. The vector field, as a result of a single charge, is

$$\begin{aligned} A_t(z,r) = \frac{Q}{4\pi\zeta_p^{1/4}\zeta^{1/4}(z)}\sum_{n=0}^{\infty}\Bigg[&\frac{1}{\sqrt{(|v_p - v(z)| + 2nv_h)^2 + r^2}} - \frac{1}{\sqrt{(v_p + v(z) + 2nv_h)^2 + r^2}} \\ &- \frac{1}{\sqrt{(v_p + v(z) - 2(n+1)v_h)^2 + r^2}} + \frac{1}{\sqrt{(|v_p - v(z)| - 2(n+1)v_h)^2 + r^2}}\Bigg], \end{aligned} \tag{54}$$

or in terms of the Fourier space momentum,

$$\begin{aligned} A_t = \frac{Q}{4\pi\zeta_p^{1/4}\zeta^{1/4}(z)}\int_0^{\infty} dk J_0(kr)\sum_{n=0}^{\infty}\Big[&e^{-k(|v_p - v(z)| + 2nv_h)} - e^{-k(v_p + v(z) + 2nv_h)} \\ &- e^{+k(v_p + v(z) - 2(n+1)v_h)} + e^{+k(|v_p - v(z)| - 2(n+1)v_h)}\Big]. \end{aligned} \tag{55}$$

The metric has the form

$$ds^2 = -(g_{tt} + h_{tt})\,dt^2 + (g_{zz} + h_{zz})\,dz^2 + 2h_{zr}dzdr + (g_{xx} + h_{rr})\,dr^2 + r^2(g_{xx} + h_{\theta\theta})\,d\theta^2, \tag{56}$$

where $h_{\mu\nu}$ are all proportional to $\mathcal{O}(A_0^2)$, hence $h_{\mu\nu} \propto \epsilon^2$. We will use

$$g_{tt} = \frac{f(z)}{z^2}, \qquad g_{zz} = \frac{1}{z^2 f(z)}, \qquad g_{xx} = \frac{1}{z^2}. \tag{57}$$

To find non-trivial effects of $h_{\mu\nu}$ on charge diffusion, we need to use the following scaling of the perturbations of the vector field,

$$\epsilon A_\mu \to \epsilon A_\mu + \epsilon^3 a_\mu, \tag{58}$$

where we will use the radial gauge, $a_z = 0$. Note also that $A_i = 0$ and only $A_t \neq 0$. We can then use the Maxwell's equation

$$\nabla_\mu F^{\mu\nu} = 0, \tag{59}$$

expanded to $\mathcal{O}(\lambda^3)$ in the WKB expansion, with $\mu = \lambda$ to find the fluctuation equations for $a_t$ and $a_r$,

$$rf^2\partial_z^2 a_t + rf\partial_r^2 a_t + f\partial_r a_t - rf\partial_t\partial_r a_r - f\partial_t a_r = \frac{1}{3}\kappa_4^2\sigma z^4(\partial_r A_t)\left[(\partial_r A_t)^2 + f(\partial_z A_t)^2\right], \tag{60}$$

$$r\partial_t\partial_z a_t - f\partial_r(r\partial_z a_r) = 0, \tag{61}$$

$$\partial_t\partial_r a_t - \partial_t^2 a_r + f^2\partial_z^2 a_r + f\partial_z f\partial_z a_r = 0. \tag{62}$$

Now, this is a coupled set of non-homogeneous differential equations. To find the solution, let us write

$$a_t = e^{-i\omega t} J_0(kr)\mathfrak{a}_t(z) + \alpha_t(t,z,r), \tag{63}$$

$$a_r = e^{-i\omega t} J_1(kr)\mathfrak{a}_r(z) + \alpha_r(t,z,r), \tag{64}$$

where $\mathfrak{a}_t$ and $\mathfrak{a}_r$ will solve the homogeneous part of the linear differential equations. Furthermore, we can define a gauge-invariant variable

$$Z = k\mathfrak{a}_t - i\omega\mathfrak{a}_r, \tag{65}$$

which enables us to rewrite the homogeneous part of the system in terms of a single differential equation

$$\partial_z^2 Z + \frac{\omega^2 \partial_z f}{f\left(\omega^2 - k^2 f\right)}\partial_z Z + \frac{\omega^2 - k^2 f}{f^2} Z. \tag{66}$$

The solution has a standard hydrodynamic expansion in $\omega$ and $k^2$ and a diffusive quasi-normal mode in the longitudinal direction outwards from the point charge at $r = 0$.

Let us now move on to finding the solutions for the non-homogeneous $\alpha_t$ and $\alpha_r$ functions. By looking at equation (60), we can first analyse the time dependence of $\alpha_t$ and $\alpha_r$. Since the right-hand-side has no time dependence, it is clear that $\alpha_t$ cannot be time-dependent unless it exactly satisfies $rf^2\partial_z^2\alpha_t + rf\partial_r^2\alpha_t + f\partial_r\alpha_t = 0$. Let us assume that $\alpha_t$ does indeed satisfy this equation. We then have to solve

$$-rf\partial_t\partial_r\alpha_r - f\partial_t\alpha_r = \frac{1}{3}\kappa_4^2\sigma z^4\left(\partial_r A_t\right)\left[\left(\partial_r A_t\right)^2 + f\left(\partial_z A_t\right)^2\right], \tag{67}$$

which could be done with $\alpha_r$ linear in $t$. Eq. (61) would the imply that

$$\alpha_r = \frac{t\beta(z)}{r}, \tag{68}$$

which is inconsistent with Eq. (60) because this solution makes the left-hand-side vanish identically. This contradiction leads us to conclude that $\alpha_t$ has no time dependence.

Now, let us assume that $\alpha_r$ has non-trivial time dependence. Then, according to Eq. (60), it must be true that

$$\partial_t\partial_r\left(r\alpha_r\right) = 0, \tag{69}$$

which can be solved by

$$\alpha_r = \frac{\beta(t,z)}{r}. \tag{70}$$

This expression also automatically satisfies Eq. (61). The remaining equation, Eq. (62) becomes

$$f^2\partial_z^2\beta + f\partial_z f\partial_z\beta - \partial_t^2\beta = 0, \tag{71}$$

which implies that $\beta(t,z) = e^{-i\omega t}\gamma(z)$, i.e.

$$\partial_z^2\gamma + \frac{\partial_z f}{f}\partial_z\gamma + \frac{\omega^2}{f^2}\gamma = 0. \tag{72}$$

The function $\gamma$ has a hydrodynamical expansion in $\omega$ and we can impose in-falling boundary conditions on its near-horizon behaviour. Hence,

$$\alpha_r(t,z,r) = C_r\frac{e^{-i\omega t}}{r}\left(1 - \frac{z}{z_h}\right)^{-i\mathfrak{w}/2}\left[1 + \mathfrak{w}\gamma_1(z) + \mathcal{O}\left(\omega^2\right)\right], \tag{73}$$

where $\mathfrak{w} = \omega/2\pi T$. Of course, Eq. (72) is exactly the same as Eq. (66) with $k = 0$. The main difference is the $1/r$ dependence of $\alpha_r$. We also note that $C_r$ is an arbitrary constant, so we could just set $C_r = 0$.

What remains to be solved is a very complicated differential equation

$$rf^2\partial_z^2\alpha_t + rf\partial_r^2\alpha_t + f\partial_r\alpha_t = \frac{1}{3}z^4\left(\partial_r A_t\right)\left[\left(\partial_r A_t\right)^2 + f\left(\partial_z A_t\right)^2\right]. \tag{74}$$

By examining the right-hand-side, we see that it vanishes at $z = 0$ but it has a sharp peak at $r = 0$ and $z = z_p$, i.e. at the point charge. If we keep $C_r \neq 0$ then $a_r$ diverges at $r = 0$.

## III. SOME IDENTITIES OF POTENTIAL USE

We can expand the "bracketed" part of the integrand in a Taylor series in small $k$, finding

$$A_t = \frac{Q}{4\pi\zeta_p^{1/4}\zeta^{1/4}(z)} \int_0^\infty dk k^2 J_0(kr) \left[ \Big( |v_p - v(z)| - v_p - v(z) \Big) \Big( |v_p - v(z)| + v_p + v(z) - 2v_h \Big) + \mathcal{O}(k) \right], \tag{75}$$

and finally

$$A_t = \begin{cases} \frac{Q}{\pi\zeta_p^{1/4}\zeta^{1/4}(z)} v(z)\,(v_h - v_p) \int_0^\infty dk \left(k^2 + \ldots\right) J_0(kr), & 0 \le v(z) \le v_p, \\ \frac{Q}{\pi\zeta_p^{1/4}\zeta^{1/4}(z)} v_p\,(v_h - v(z)) \int_0^\infty dk \left(k^2 + \ldots\right) J_0(kr), & v_p \le v(z) \le v_h. \end{cases} \tag{76}$$

For convenience, we define

$$A_t \equiv \mathcal{A}_t(z) \int_0^\infty dk \left(k^2 + \ldots\right) J_0(kr), \tag{77}$$

which gives

$$\partial_r A_t = -\mathcal{A}_t(z) \int_0^\infty dk \left(k^3 + \ldots\right) J_1(kr). \tag{78}$$

Now, an inspection of plots (and eventually a more rigorous argument) shows that the dominant part of the $A_t$ integral comes from the small $k$ region. Similarly, this is the case for $\partial_r A_t$ unless $v(z)$ is close to $v_p$. Let as assume that $v(z)$ is not close to $v_p$. It is then reasonable to introduce a small dimensionless cut-off $kr \le \Lambda \ll 1$, so that

$$A_t \approx \frac{\mathcal{A}_t(z)\Lambda^3}{3\,r^3}, \qquad \partial_r A_t \approx -\frac{\mathcal{A}_t(z)\Lambda^5}{10\,r^4}. \tag{79}$$

A useful combination is then

$$\left(\partial_r A_t\right)^2 + f(z)\left(\partial_z A_t\right)^2 \approx \frac{f(z)\left(\partial_z \mathcal{A}_t\right)^2 \Lambda^6}{9\,r^6} + \frac{\mathcal{A}_t^2 \Lambda^{10}}{100\,r^8} \tag{80}$$

## IV. NOTES

The spectrum has to be discrete in order for the system to localise.

[1] D. Anninos, T. Anous, F. Denef, and L. Peeters, (2013), arXiv:1309.0146 [hep-th].