Electron star ingredients

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I. SOLUTIONS OF THE SHEAR SECTOR MODES IN THE IR LIFSHITZ LIMIT

A. IR geometry and $k = 0$ solutions

The geometry in the IR ($r \to \infty$) approaches that of a pure Lifshitz geometry. In this limit we have

$$\begin{aligned} f(r) &\to 1/r^{2z} \\ g(r) &\to g_\infty/r^2 \\ h(r) &\to h_\infty/r^z. \end{aligned} \tag{1}$$

The two differential equation in the shear sector are

$$\begin{aligned} 0 = Z_1'' &+ 2kr^2h'Z_2' + \left(\frac{rg\sigma\mu}{2} + \frac{\omega^2 f' + 2k^2 r f^2}{f\left(\omega^2 - k^2r^2 f\right)}\right) Z_1' + \frac{g}{f}\left(\omega^2 - k^2r^2f\right) Z_1 \\ &+ 2kr^2\sqrt{f}\mu\left(\frac{2\omega^2 h'^2}{f\left(\omega^2 - k^2r^2f\right)} + \frac{g\sigma}{\mu}\right) Z_2, \end{aligned} \tag{2}$$

$$\begin{aligned} 0 = Z_2'' &+ \frac{1}{2}\left(\frac{f'}{f} - \frac{g'}{g}\right) Z_2' - \frac{kh'}{\omega^2 - k^2r^2f} Z_1' + \frac{g}{f}\left(\omega^2 - k^2r^2f\right) Z_2 \\ &- \left(\frac{2\omega^2 h'^2}{f\left(\omega^2 - k^2r^2f\right)} + \frac{g\sigma}{\mu}\right) Z_2 \end{aligned} \tag{3}$$

In the limit of $k \to 0$, the two equations decouple

$$0 = Z_1'' + \left(\frac{rg\sigma\mu}{2} + \frac{f'}{f}\right) Z_1' + \frac{\omega^2 g}{f} Z_1, \tag{4}$$

$$0 = Z_2'' + \frac{1}{2}\left(\frac{f'}{f} - \frac{g'}{g}\right) Z_2' + \left(\frac{\omega^2 g}{f} - \frac{2h'^2}{f} - \frac{g\sigma}{\mu}\right) Z_2. \tag{5}$$

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and we can find the asymptotic IR Lifshitz behaviour of Z_1 and Z_2 to be

$$Z_1 = \left(1 + \frac{i(z+1)}{2z\sqrt{g_\infty}}\frac{1}{\omega r^z}\right) r e^{i\sqrt{g_\infty}\omega r^z/z} \tag{6}$$

$$Z_2 = \left(1 + \frac{iz}{\sqrt{g_\infty}}\frac{1}{\omega r^z}\right) e^{i\sqrt{g_\infty}\omega r^z/z}, \tag{7}$$

which are series expansions of the full (irrelevant) solutions

$$Z_1 = r^{1+z/2} H^{(1)}_{\frac{z+2}{2z}}\left(g_\infty^{1/2}\frac{\omega r^z}{z}\right) \tag{8}$$

$$Z_2 = r^{z/2} H^{(1)}_{3/2}\left(g_\infty^{1/2}\frac{\omega r^z}{z}\right). \tag{9}$$

We would now like to find the k-dependent corrections to the above solutions to analytically extract the hydrodynamical quasi-normal mode (QNM).

B. Cases with integer values of the exponent z

1. *Cases with exponents $z \geq 3$*

We would like to find analytic k-dependent corrections to Z_1 and Z_2 in the Lifshitz IR region. The corrections should be such that the limit of $k \to 0$ smoothly reproduces the above $k = 0$ results. On the other hand, the limit of $\omega \to 0$ is not analytic and our solutions will represent an asymptotic series in ω controlled by powers of r. Away from the in-falling boundary conditions at the horizon ($r \to \infty$), however, we expect that the limit of $\omega \to 0$ is defined as well. We anticipate the form

$$Z_1 = e^{i\sqrt{g_\infty}\omega r^z/z} r P_1(r, \omega, k) \tag{10}$$

$$Z_2 = e^{i\sqrt{g_\infty}\omega r^z/z} P_2(r, \omega, k), \tag{11}$$

where P_1 and P_2 are polynomials in ascending powers of $1/r$.

We can first expand equations (2) and (3) in $k^2 r^2 f \ll \omega^2$. Since we only work with $z > 1$, the expansion parameter tends to $k^2 r^2 f = k^2 r^{2(1-z)} \to 0$ in the IR. The expansion therefore makes sense for all non-vanishing values of ω and finite values of k. The limits of our approximation are

$$k^2 \ll \omega^2 r^{2(z-1)} \qquad \text{and} \qquad r \to \infty. \tag{12}$$

On top of that, we are interested in the hydrodynamical QNMs, hence we may think of both ω and k as small. We find, up to $\mathcal{O}(k^4)$,

$$
\begin{aligned}
0 = Z_1'' - \left(\frac{z+1}{r} + \frac{2(z-1)k^2}{\omega^2 r^{2z-1}} + \frac{2(z-1)k^4}{\omega^4 r^{4z-3}}\right) Z_1' + g_\infty \left(\omega^2 r^{2(z-1)} - k^2\right) Z_1 \\
- \frac{2\sqrt{z(z-1)}k}{r^{z-1}} Z_2' + 4\sqrt{z(z-1)} \left(\frac{zk}{r^z} + \frac{(z-1)k^3}{\omega^2 r^{3z-2}}\right) Z_2
\end{aligned}
\tag{13}
$$

and

$$
\begin{aligned}
0 = Z_2'' - \frac{z-1}{r} Z_2' + \left[g_\infty \left(\omega^2 r^{2(z-1)} - k^2\right) - \frac{2z^2}{r^2} - \frac{2z(z-1)k^2\left(\omega^2 r^{2(z-1)} + k^2\right)}{\omega^4 r^{4z-2}}\right] Z_2 \\
+ \frac{\sqrt{z(z-1)}k\left(\omega^2 r^{2(z-1)} + k^2\right)}{\omega^4 r^{3z-1}} Z_1'
\end{aligned}
\tag{14}
$$

Using a power series expansion in $1/r$ for Z_1 and Z_2 shows that we can recursively solve differential equations (13) and (14), order-by-order in r, with two series of form

$$
\begin{aligned}
P_1 = 1 + \sum_{i=z-2}^{\infty} \frac{a_i(\omega, k)}{r^i} \\
P_2 = 1 + \sum_{i=z-2}^{\infty} \frac{b_i(\omega, k)}{r^i}
\end{aligned}
\tag{15}
$$

The non-zero terms in both series begin at order $1/r^{z-2}$. In the limit of $k \to 0$ we find that $a_{z-2} = a_{z-1} = b_{z-2} = b_{z-1} = 0$, $a_z = \frac{i(z+1)}{2z\sqrt{g_\infty}\omega}$ and $b_z = \frac{iz}{\sqrt{g_\infty}\omega}$, as required.
If we only seek the leading ω and k behaviour it suffices to consider the series with three terms between $i = z-2$ and $i = z$. In that case the equation (13) will be solved up to order $\mathcal{O}\left(\frac{1}{r}\right)$, leaving terms of order $\mathcal{O}\left(\frac{1}{r^2}\right)$ and higher unsolved. Equation (14) will be solved up to order $\mathcal{O}\left(\frac{1}{r^2}\right)$, leaving terms of order $\mathcal{O}\left(\frac{1}{r^3}\right)$ and higher unsolved. Further extending polynomials $P_{1,2}$ by n terms is then able to solve the two differential equations by further n orders.

2. *Special case with $z = 2$*

A special case, which cannot be solved by the above ansatz is the case when $z - 2 = 0$, i.e. $z = 2$. To solve the system we can use the following modified ansatz:

$$
Z_1 = e^{\frac{i\sqrt{g_\infty}\omega r^z}{z} + f(r)} r P_1(r, \omega, k) \tag{16}
$$

$$
Z_2 = e^{\frac{i\sqrt{g_\infty}\omega r^z}{z} + f(r)} P_2(r, \omega, k). \tag{17}
$$

It is clear that since equations (13) and (14) have no constant terms, the functions in the exponents must equal, so it is sufficient to find a single $f(r)$ for both Z_1 and Z_2. To only find $f(r)$, it is sufficient to simply set $Z_1 = 0$ and use equation (14) to leading order in k. We are left with

$$0 = Z_2'' - \frac{1}{r} Z_2' + \left[g_\infty r^2 \left(\omega^2 - \frac{k^2}{r^2} \right) - \frac{8}{r^2} \right] Z_2 \tag{18}$$

to which the full solution is [completely irrelevant, but it's fun to play with special functions :-)]

$$Z_2 = r^4 e^{\frac{1}{2} i\omega\sqrt{g_\infty} r^2} \left[C_1 U \left(2 + \frac{i\sqrt{g_\infty} k^2}{4\omega}, 4, -i\sqrt{g_\infty}\omega r^2 \right) + C_2 L^3_{-2-\frac{i\sqrt{g_\infty}k^2}{4\omega}} \left(-i\sqrt{g_\infty}\omega r^2 \right) \right], \tag{19}$$

where U is the confluent hypergeometric function and $L_n^\lambda(z)$ the Laguerre polynomial.
Analysing its asymptotics near $r \to \infty$, we find that $C_2 = 0$ in order to only keep $e^{+\frac{1}{2} i\omega\sqrt{g_\infty} r^2}$ terms (the in-falling b.c.). To match this solution onto the $k = 0$ solution we must set $C_1 = -g_\infty \omega^2$. There is of course still the freedom of multiplying the entire solution by a constant. Expanding in $1/r$ we find

$$\begin{aligned} Z_2 &= -g_\infty \omega^2 r^4 e^{\frac{1}{2} i\sqrt{g_\infty}\omega r^2} U \left[2 + \frac{i\sqrt{g_\infty} k^2}{4\omega}, 4, -i\sqrt{g_\infty}\omega r^2 \right] \\ &= e^{\frac{1}{2} i\sqrt{g_\infty}\omega r^2} \left(-i\sqrt{g_\infty}\omega r^2 \right)^{-\frac{i\sqrt{g_\infty}k^2}{4\omega}} [1 + \ldots] \\ &= \exp \left\{ \frac{i\sqrt{g_\infty}\omega}{2} \left(r^2 - \frac{k^2}{2\omega^2} \log \left(-i\sqrt{g_\infty}\omega r^2 \right) \right) \right\} [1 + \ldots] . \end{aligned} \tag{20}$$

Therefore

$$e^{f(r)} = \left(-i\sqrt{g_\infty}\omega r^2 \right)^{-\frac{i\sqrt{g_\infty}k^2}{4\omega}} = e^{-\frac{i\sqrt{g_\infty}k^2}{4\omega} \log\left(-i\sqrt{g_\infty}\omega r^2 \right)}. \tag{21}$$

Note that this structure is similar to the more usual AdS cases at finite temperature...
We can now use, as before, polynomials $P_{1,2}$ to find

$$\begin{aligned} Z_1 &= e^{\frac{1}{2} i\sqrt{g_\infty}\omega r^2 - \frac{i\sqrt{g_\infty}k^2}{4\omega} \log\left(-i\sqrt{g_\infty}\omega r^2 \right)} r \left(1 - \frac{\sqrt{2}k}{r} + \frac{12 i\omega^2 - 12\sqrt{g_\infty}\omega k^2 + i g_\infty k^4}{16\sqrt{g_\infty}\omega^3 r^2} \right. \\ &\quad \left. - \frac{32 i\omega^2 k - 4\sqrt{g_\infty}\omega k^3 + i g_\infty k^5}{8\sqrt{2 g_\infty}\omega^3 r^3} + \ldots \right) \\ Z_2 &= e^{\frac{1}{2} i\sqrt{g_\infty}\omega r^2 - \frac{i\sqrt{g_\infty}k^2}{4\omega} \log\left(-i\sqrt{g_\infty}\omega r^2 \right)} \left(1 + \frac{k}{\sqrt{2}\omega^2 r} + \frac{32 i\omega^2 - 4\sqrt{g_\infty}\omega k^2 + i g_\infty k^4}{16\sqrt{g_\infty}\omega^3 r^2} + \ldots \right) \end{aligned} \tag{22}$$

so that both (13) and (14) are satisfied to $\mathcal{O}(1/r^2)$.

II. QUASI-NORMAL MODES

We would like to find the hydrodynamical QNM in the shear sector of the electron star background at $T = 0$.

A. Flux with real ω^2

To find the conserved flux in this system, consider the off-shell Lagrangian

$$\mathcal{L}_{off-shell} = \frac{L^2}{\kappa^2}\left(Z_i'^* A_{ij} Z_j' + Z_i^* B_{ij} Z_j' + \text{non-derivative terms}\right) \tag{23}$$

where

$$A_{11} = \frac{\sqrt{f}}{4r^2\sqrt{g}\left(\omega^2 - k^2 r^2 f\right)}, \qquad A_{22} = -\frac{\sqrt{f}}{2\sqrt{g}}, \qquad A_{12} = A_{21} = 0, \tag{24}$$

$$B_{11} = \frac{(rf' - 2f)}{2\omega^2 r^3 \sqrt{fg}}, \qquad B_{21} = -\frac{k\left(rf' + 2f\right)}{2r\mu\sqrt{g}\left(\omega^2 - k^2 r^2 f\right)}, \qquad B_{12} = B_{22} = 0. \tag{25}$$

$$\tag{26}$$

This Lagrangian is invariant under simultaneous global $U(1)$ transformations of both Z_1 and Z_2. The reason for this is the cross-term $Z_2^* B_{21} Z_1'$. Assuming that $(r, \omega^2, k) \in \mathbb{R}$, the flux can then be found to be

$$\mathcal{F} = 2i\left[-Z_1^* A_{11} Z_1' + Z_1 A_{11} Z_1'^* + Z_2^* A_{22} Z_2' - Z_2 A_{22} Z_2'^* + \frac{1}{2} B_{21}\left(Z_1^* Z_2 - Z_2^* Z_1\right)\right]. \tag{27}$$

$\mathcal{F}$ is conserved along the radial direction, i.e. $\partial_r \mathcal{F} = 0$.

Now, in the UV part of the geometry the fields can be expanded as

$$\begin{aligned} Z_1 &= Z_1^{(0)} + r^2 Z_1^{(2)} + r^3 Z_1^{(3)} + \dots \\ Z_2 &= Z_2^{(0)} + r Z_2^{(1)} + \dots, \end{aligned} \tag{28}$$

where $Z_2^{(1)}$ is related to the vev of the QFT current J_μ, while $Z_1^{(2)}$ is completely determined by the sources of the $T_{\mu\nu}$ components of $Z_1^{(0)}$. The vev of $T_{\mu\nu}$ comes in at the order of r^3. The value of the flux at the AdS boundary is

$$\begin{aligned} \lim_{r\to 0} \mathcal{F}(r) = 2i \lim_{r\to 0}\left(Z_1 A_{11} Z_1'^* - Z_1^* A_{11} Z_1'\right) + \\ + 2i A_{22}(0)\left(Z_2^{(0)*} Z_2^{(1)} - Z_2^{(0)} Z_2^{(1)*}\right) + i B_{21}(0)\left(Z_1^{(0)*} Z_2^{(0)} - Z_1^{(0)} Z_2^{(0)*}\right) \end{aligned} \tag{29}$$

which along with the limiting values

$$\begin{aligned} \lim_{r\to 0} A_{11} &= -\lim_{r\to 0} \frac{\sqrt{f}}{4r^2\sqrt{g}\left(\omega^2 - k^2 r^2 f\right)} = \lim_{r\to 0} \frac{c}{4\left(\omega^2 - c^2 k^2\right) r^2} \\ \lim_{r\to 0} A_{22} &= -\lim_{r\to 0} \frac{\sqrt{f}}{2\sqrt{g}} = -\frac{c}{2} \\ \lim_{r\to 0} B_{21} &= -\lim_{r\to 0} \frac{k\left(rf' + 2f\right)}{2r\mu\sqrt{g}\left(\omega^2 - k^2 r^2 f\right)} = \frac{3c\hat{M}}{2\hat{\mu}} \frac{k}{\omega^2 - c^2 k^2} \end{aligned} \tag{30}$$

gives the conserved flux

$$\mathcal{F}=ic\left[\frac{1}{\omega^2-c^2k^2}\left(\lim_{r\to 0}\frac{1}{r}\left(Z_1^{(0)}Z_1^{(2)*}-Z_1^{(2)}Z_1^{(0)*}\right)+\frac{3}{2}\left(Z_1^{(0)}Z_1^{(3)*}-Z_1^{(3)}Z_1^{(0)*}\right)\right)\right.$$
$$\left.+Z_2^{(0)}Z_2^{(1)*}-Z_2^{(0)*}Z_2^{(1)}+\frac{3Mk}{2\hat{\mu}\left(\omega^2-c^2k^2\right)}\left(Z_1^{(0)*}Z_2^{(0)}-Z_1^{(0)}Z_2^{(0)*}\right)\right]. \tag{31}$$

To impose the Dirichlet boundary conditions at the boundary we need to fix $Z_1^{(0)}$ and $Z_2^{(0)}$ to some constants. However, to find only the QNMs, without the full Green's functions, it is particularly useful to set $Z_1^{(0)}=Z_2^{(0)}=0$. Generally, the values of $Z_1^{(0)}$ and $Z_2^{(0)}$ can be thought of as functions of ω and k at some fixed physical parameters $\hat{M}$, $\hat{Q}$, $\hat{\mu}$, etc. describing the star geometry. Given some propagating modes that satisfy $Z_1^{(0)}=Z_2^{(0)}=0$, we can see that the flux vanishes away from the light-cone ($\omega^2=c^2k^2$) for such $\omega(k)$. Therefore

$$\text{For a quasinormal mode } \tilde{\omega}(k) \qquad \Longrightarrow \qquad \mathcal{F}\left(\tilde{\omega}(k)\right)=0 \tag{32}$$

It is interesting to note that the flux actually diverges unless we set $Z_1^{(0)}=0$ or alternatively if $Z_1^{(0)}Z_1^{(2)*}-Z_1^{(2)}Z_1^{(0)*}$ vanishes.

We would like to use this fact to find QNMs from the IR part of the geometry. The question we need to answer is therefore in what other cases can $\mathcal{F}=0$? We can always set $Z_1^{(0)}$ and $Z_2^{(0)}$ to be real. Then the flux vanishes if $Z_1^{(2)}$, $Z_1^{(3)}$ and $Z_2^{(1)}$ are real as well. This is something we would, however, not generically expect to be true.

B. Flux with complex frequency

We should look for the flux of $\omega\in\mathbb{C}$ fluctuations to find the value of $\mathcal{F}$ on the QNMs. The off-shell action is

$$S^{(2)}=\frac{L^2}{\kappa^2}\int d^4k dr\left\{Z_i'(-k)A_{ij}(k)Z_j'(k)+Z_i(-k)B_{ij}(k)Z_j'(k)+\cdots\right\} \tag{33}$$

Because only A_{11}, A_{22}, B_{11} and B_{21} are non-zero the symmetry of this action is

$$Z_i(k)\to e^{i\alpha}Z_i(k)$$
$$Z_i(-k)\to e^{-i\alpha}Z_i(-k) \tag{34}$$

We are using $-k$ for $(-\omega, -k)$. The Nöther current (flux) is then

$$\begin{aligned}\mathcal{F} = i \Big\{ & \left[Z_1'(-k)Z_1(k) - Z_1(-k)Z_1'(k)\right]\left[A_{11}(k) + A_{11}(-k)\right] + \\ & + \left[Z_2'(-k)Z_2(k) - Z_2(-k)Z_2'(k)\right]\left[A_{22}(k) + A_{22}(-k)\right] \\ & + Z_1(-k)Z_1(k)\left[B_{11}(k) - B_{11}(-k)\right] + \\ & + Z_1(k)Z_2(-k)B_{21}(k) - Z_1(-k)Z_2(k)B_{21}(-k)\Big\}\end{aligned} \tag{35}$$

Now A_{11}, A_{22} and B_{11} are invariant under $k \to -k$, whereas $B_{21}(-k) = -B_{21}(k)$.

$$\begin{aligned}\mathcal{F} = i \Big\{ & 2A_{11}(k)\left[Z_1'(-k)Z_1(k) - Z_1(-k)Z_1'(k)\right] + 2A_{22}(k)\left[Z_2'(-k)Z_2(k) - Z_2(-k)Z_2'(k)\right] + \\ & + B_{21}(k)\left[Z_1(-k)Z_2(k) + Z_1(k)Z_2(-k)\right]\Big\}\end{aligned} \tag{36}$$

Imagine that $\mathcal{F}(\omega, k)$ is a polynomial defined over the complex plane of which zeroes we denote by $\tilde{\omega}_i(k)$. From our construction above I claim that these are the QNMs of the electron star system. Hence

$$\mathcal{F}(\omega, k) = \prod_{i=1}^{\infty}\left(\omega - \tilde{\omega}_i(k)\right) \tag{37}$$

III. EXTERIOR OF THE STAR

Outside the star the geometry is that of the Reissner-Nordström-AdS. We have $\hat{\sigma} = \hat{\rho} = \hat{p} = 0$ and

$$f = \frac{c^2}{r^2} - \hat{M}r + \frac{r^2\hat{Q}^2}{2}, \qquad g = \frac{c^2}{r^4 f}, \qquad h = \hat{\mu} - r\hat{Q}. \tag{38}$$

Also, as everywhere along the geometry,

$$\mu(r) = \frac{h(r)}{\sqrt{f(r)}}. \tag{39}$$

Equations (2) and (3) become

$$\begin{aligned}0 = {} & Z_1'' + 2kr^2h'Z_2' + \frac{\omega^2 f' + 2k^2rf^2}{f\left(\omega^2 - k^2r^2f\right)}Z_1' + \frac{g}{f}\left(\omega^2 - k^2r^2f\right)Z_1 \\ & + 2kr^2\sqrt{f}\mu\left(\frac{2\omega^2h'^2}{f\left(\omega^2 - k^2r^2f\right)}\right)Z_2,\end{aligned} \tag{40}$$

$$\begin{aligned}0 = {} & Z_2'' + \frac{1}{2}\left(\frac{f'}{f} - \frac{g'}{g}\right)Z_2' - \frac{kh'}{\omega^2 - k^2r^2f}Z_1' + \frac{g}{f}\left(\omega^2 - k^2r^2f\right)Z_2 \\ & - \frac{2\omega^2h'^2}{f\left(\omega^2 - k^2r^2f\right)}Z_2\end{aligned} \tag{41}$$

IV. SMALL STAR LIMIT

The easiest case to tract analytically is the limit when the star becomes small. Fermionic excitations in this scenario were analysed in [1].

The profile of the star is characterised by three functions $\hat{\sigma}$, $\hat{\rho}$ and $\hat{p}$. They all reach their maximum value in the IR at $r \to \infty$ limit, where the geometry is pure Lifshitz. They monotonically decrease with decreasing r and reach $\hat{\sigma} = \hat{\rho} = \hat{p} = 0$ at the boundary of the star ($r = r_s$). The small star limit is characterised by

$$\lambda^2 \equiv h_\infty^2 - \hat{m}^2 \ll 1 \tag{42}$$

where $\lambda^2 = \frac{6^{4/3}\hat{m}^{2/3}(1-\hat{m}^2)^{2/3}}{(2\hat{m}^4-7\hat{m}^2+6)^{2/3}}\frac{1}{\hat{\beta}^{2/3}}$. Therefore at an arbitrary $\hat{m}$, the small star limit is achieved by taking large $\hat{\beta}$. The exponent z becomes

$$z = \frac{1}{1-\hat{m}^2} + \frac{\lambda^2}{\left(1-\hat{m}^2\right)^2} + \ldots \tag{43}$$

The correction to the Lifshitz geometry inside the star is

$$\begin{aligned} f &= \frac{1}{r^{2z}}\left(1 + f_1\frac{1}{r^{|\alpha|}} + \ldots\right) \\ g &= \frac{g_\infty}{r^2}\left(1 + g_1\frac{1}{r^{|\alpha|}} + \ldots\right) \\ h &= \frac{h_\infty}{r^z}\left(1 + h_1\frac{1}{r^{|\alpha|}} + \ldots\right) \end{aligned} \tag{44}$$

where

$$|\alpha| = \frac{\hat{m}\sqrt{3\left(2-\hat{m}^2\right)}}{\sqrt{1-\hat{m}^2}}\frac{1}{\lambda} - 1 - \frac{1}{2\left(1-\hat{m}^2\right)} + \ldots \tag{45}$$

and

$$g_\infty = \frac{6-7\hat{m}^2+2\hat{m}^4}{6\left(1-\hat{m}^2\right)^2} + \frac{\left(6-7\hat{m}^2+2\hat{m}^4\right)\left(1+4\hat{m}^2\right)}{12\hat{m}^2\left(1-\hat{m}^2\right)^3}\lambda^2 + \ldots \tag{46}$$

Corrections to the pure Lifshitz geometry inside the star therefore become exponentially suppressed for $r > 1$ when $\lambda \ll 1$. It is shown in [1] that f_1, g_1 and h_1 can be normalised in such a way that to leading order in λ the boundary of the star is at $r_s = 1$, while the correction to the pure Lifshitz geometry remains exponentially suppressed.

[1] S. A. Hartnoll, D. M. Hofman and D. Vegh, JHEP **1108** (2011) 096 [arXiv:1105.3197 [hep-th]].