Shear excitations of the electron star - diffusion out of thermal equilibrium

Matthew Stephenson[a]

[a] *Stanford University*

E-mail: matthewjstephenson@icloud.com

Abstract:

The electron star is a holographic bulk setup which consists of a non-extremal AdS-Reissner-Nordström black brane and an ideal gas of electrons. The gravitational system is dual to a field theory with interacting fractionalised and mesonic degrees of freedom, and is thermodynamically favoured over a pure black brane scenario. The electron gas in this Einstein-Maxwell-fluid theory is treated as being at zero temperature. The system is thus gravitationally stable but is *not* in thermodynamic equilibrium. After analysing thermodynamic properties of the background, we compute the quasi-normal mode spectrum and correlation functions of gauge-invariant quantities on the boundary to study momentum and charge transport in the shear sector. We perform a detailed analysis of the hydrodynamic diffusion mode dispersion relation and compare our numerical results with thermodynamic predictions. We show that they only agree at very low temperature and near transition to a purely black brane background. We thus conclude that in accordance with expectations, hydrodynamics and thermodynamics cannot successfully describe a system out of thermal equilibrium. This provides further evidence for the importance of holographic studies of thermalisation, hydrolysation and out-of-equilibrium phenomena.

Contents

1 Introduction

We begin our work by presenting the electron star background, which was introduced in [1–3] as a system that allows for an investigation of interactions between bulk charges and a charged black hole. This provides a holographic dual to a theory with interacting fractionalised charges, dual to the charge stored behind the horizon, and gauge-invariant mesonic degrees of freedom dual to bulk charges. Furthermore, it provides a solution to a well-known problem of the holographic dual to a Reissner-Nordström (RN) black brane, which has finite zero-temperature entropy due to the non-zero size of the extremal RN horizon. In fact, the entropy of the electron star vanishes at zero temperature.

The electron star system consists of gravity, a Maxwell vector field and free charged bulk fermions treated in the ideal fluid approximation. This means that the electron cloud stabilises solely through Pauli exclusion principle, in the absence of any interactions. The corresponding problem of constructing an uncharged star was addressed in the context of general relativity by Tolman, Oppenheimer and Volkov [4, 5]. To find the electron star background in a space-time with a negative cosmological constant, we need to look for a solution of the Einstein's equation

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \frac{3}{L^2} g_{\mu\nu} = \kappa^2 \left[\frac{1}{e^2} \left(F_{\mu\sigma} F_\nu{}^\sigma - \frac{1}{4} g_{\mu\nu} F_{\rho\tau} F^{\rho\tau} \right) + T_{\mu\nu} \right], \tag{1.1}$$

and the Maxwell's equation

$$\nabla_\nu F^{\mu\nu} = e^2 J^\mu. \tag{1.2}$$

The stress-energy tensor and current of an ideal fluid can be written as

$$T_{\mu\nu} = (p+\rho) u_\mu u_\nu + p g_{\mu\nu}, \qquad J_\mu = \sigma u_\mu. \tag{1.3}$$

Here, p, ρ, σ are the pressure, energy and charge density. Furthermore, u_μ is the velocity field normalised to $u^2 = -1$, L is the characteristic length scale of the AdS space, e is the electromagnetic coupling and κ Newton's constant in four space-time dimensions. To prepare for numerical calculations, we also introduce the 'hatted' dimensionless quantities

$$\hat{p} = L^2 \kappa^2 p, \qquad \hat{\rho} = L^2 \kappa^2 \rho, \qquad \hat{\sigma} = e L^2 \kappa \sigma. \tag{1.4}$$

In order to solve the Einstein-Maxwell system for the electron star, we adopt the following metric ansatz,

$$ds^2 = L^2 \left[-f(r) dt^2 + \frac{1}{r^2}(dx^2 + dy^2) + g(r) dr^2 \right], \tag{1.5}$$

with the one-form gauge field written as

$$A = \frac{eL}{\kappa} h(r) dt. \tag{1.6}$$

In these coordinates the geometry has an asymptotical AdS infinity at $r = 0$. A static planar black brane that can form in this theory is precisely the AdS-Reissner-Nordström black brane. The ideal fluid of fermions, which may also be present in the bulk, should then be in thermal equilibrium with the non-extremal RN horizon. The *local* temperature felt by the fluid at each radial slice along the bulk is given by the expression

$$T_{loc} = \frac{T}{\sqrt{g_{tt}}} = \frac{T}{L\sqrt{f}}, \tag{1.7}$$

where T is the Hawking temperature of the non-extremal RN horizon at $r = r_+$,

$$T = \frac{1}{4\pi c}\left|\frac{df}{dr}\right|_{r=r_+}. \tag{1.8}$$

Charge density σ must then obey the Fermi-Dirac distribution controlled by the local temperature and chemical potential,

$$\sigma = 2\pi^2\beta\sum_{\pm}\int\frac{d^3p}{(2\pi)^3}\frac{\pm 1}{1+e^{(E\mp\mu_{loc})/T_{loc}}}, \tag{1.9}$$

where β is a free parameter that counts the microscopic degrees of freedom of the fermions and the local chemical potential is given by the scaled temporal component of the gauge field

$$\mu_{loc} = \frac{A_t}{L\sqrt{f}} = \frac{e}{\kappa}\frac{h}{\sqrt{f}}. \tag{1.10}$$

Since our fluid is made of only electrons (and no positrons), only the $\sum_+$ term will be included. Using (1.7), (1.10) and introducing dimensionless quantities $\hat{E} = \frac{\kappa}{e}E$, $\hat{\beta} = \frac{e^4L^2}{\kappa^2}\beta$ and the electron star mass $\hat{m}^2 = \frac{\kappa^2}{e^2}m^2$, we find

$$\hat{\sigma} = \hat{\beta}\int d\hat{E}\frac{\hat{E}\sqrt{\hat{E}^2-\hat{m}^2}}{1+\exp\left\{\frac{Le}{\kappa T}\left(\sqrt{f}\hat{E}-h\right)\right\}}. \tag{1.11}$$

The regime of validity of our theory [2] in the Thomas-Fermi approximation requires the magnitudes of parameters κ, L and e to scale as

$$e^2 \sim \frac{\kappa}{L} \ll 1. \tag{1.12}$$

By construction, the geometry should be dual to a charged thermal theory at the boundary. The IR of the bulk is therefore required to contain a non-extremal RN horizon at some r_+. Furthermore, the thermal electron cloud in the bulk can stretch all the way to the horizon and exist between some r_s and r_0, such that $r_s < r_0 = r_+$. Let us first assume that the solution remains approximately AdS-RN$_4$ near horizon even in the presence of the electron star. The background geometry is then

$$f = \frac{1}{r^2} - \left(\frac{1}{r_+^2}+\frac{\hat{\mu}^2}{2}\right)\frac{r}{r_+} + \frac{\hat{\mu}^2}{2}\frac{r^2}{r_+^2}, \tag{1.13}$$

$$g = \frac{1}{r^4 f}, \tag{1.14}$$

$$h = \hat{\mu}\left(1-\frac{r}{r_+}\right), \tag{1.15}$$

where r_+ is the location of the event horizon and $\hat{\mu}$ the chemical potential. To check the validity of our AdS-RN$_4$ assumption, we can study the thermal charge density (1.11) near horizon in terms of a new coordinate ξ, defined as

$$r = r_+ - \frac{\kappa^2 \xi^2}{4\pi e^2 L^2}. \tag{1.16}$$

Since $\frac{\kappa}{eL} \ll 1$ by (1.12), the definition (1.16) permits that ξ be an $\mathcal{O}(1)$ variable. We find

$$\hat{\sigma} = \hat{\beta} \int_{\hat{m}}^{\infty} d\hat{E} \frac{\hat{E}\sqrt{\hat{E}^2 - \hat{m}^2}}{1 + \exp\left\{\frac{\xi}{\sqrt{T}}\hat{E} + \mathcal{O}\left(\frac{\kappa}{eL}\right)\right\}}, \tag{1.17}$$

where the $\mathcal{O}\left(\frac{\kappa}{eL}\right)$ term is $\hat{E}$-independent and small, so we neglect it. We find that near horizon, when ξ is small, $\hat{\sigma}$ becomes large and diverges at $\xi = 0$. Note that the charge density can be decreased at a fixed non-zero ξ by lowering T. As we will later discover, the RN horizon can only exists in the electron star system at strictly $T \neq 0$. We can therefore conclude that $\hat{\sigma}$ always diverges at the horizon. Because $\hat{\sigma}$ is also large near the horizon, our treating of the background as approximately RN is invalid since the metric would receive large corrections near r_+ from its coupling to the fluid.

Finding the full thermal numerical background is hard because the integrands $\hat{\sigma}$, $\hat{\rho}$ and $\hat{p}$ depend on functions f and h. This would requires us to solve a set of coupled integro-differential equations for the Einstein-Maxwell system. However, the system can be greatly simplified if we assume, as in [2], that the electron star solution is approximated by a cloud at zero temperature. The electron star now exists between r_s and r_0 away from the horizon, so that $0 < r_s < r_0 < r_+$. In the parameter regime (1.12), we find that $T_{loc}/\mu_{loc} \ll 1$. Charge density (1.11) then reduces to its zero-temperature form, along with the energy density and the equation of state

$$\hat{\sigma} = \hat{\beta} \int_{\hat{m}}^{\frac{h}{\sqrt{f}}} d\hat{E}\hat{E}\sqrt{\hat{E}^2 - \hat{m}^2}, \quad \hat{\rho} = \hat{\beta} \int_{\hat{m}}^{\frac{h}{\sqrt{f}}} d\hat{E}\hat{E}^2\sqrt{\hat{E}^2 - \hat{m}^2}, \quad \hat{p} = \frac{h}{\sqrt{f}}\hat{\sigma} - \hat{\rho}. \tag{1.18}$$

The undetermined field variables f and h now only appear in the limits of simpler definite integrals, which makes the calculation more tractable. Our assumption of treating the electron gas as being at zero temperature is well motivated from the perspective that it ensures gravitational equilibrium. The solution is, however, *not* in thermal equilibrium and, as we found above, thermal corrections would induce large corrections to geometry in the deep IR near-horizon region. From the point of view of the boundary theory, we expect that the hydrodynamic behaviour of the IR excitations of the model will not agree with thermodynamic predictions.[1]

Let us nevertheless proceed by treating the electron cloud as if at zero temperature. An important motivation for considering such a system is to shed light on the applicability of hydrodynamic derivative expansion. Since our system would eventually equilibrate,

[1] This claim is supported by vast literature on holographic (Wilsonian) renormalisation group, showing that the near-horizon properties of the bulk are reflected in the IR excitations of the boundary field theory. For references see [6–10] and the references therein.

and certainly obey hydrodynamics at the end, it would be interesting to study how the IR excitations asymptote to those predicted by phenomenological hydrodynamics, which always require thermodynamic input in order for the equations to close. This could help us understand the difference between hydrodynamic and thermodynamic scales.

Using the ansätze in equations (1.3, 1.5, 1.6) for the Einstein-Maxwell system, we obtain the following system of coupled differential equations for fields f, g, h, $\hat{p}$, $\hat{\rho}$ and $\hat{\sigma}$,

$$\hat{p}' + (\hat{p}+\hat{\rho})\frac{f'}{2f} - \frac{h'\hat{\sigma}}{\sqrt{f}} = 0, \tag{1.19}$$

$$\frac{1}{r}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{4}{r}\right) + (\hat{p}+\hat{\rho})g = 0, \tag{1.20}$$

$$\frac{f'}{rf} - \frac{h'^2}{2f} + g(3+\hat{p}) - \frac{1}{r^2} = 0, \tag{1.21}$$

$$h'' + \frac{rh'}{2}g(\hat{p}+\hat{\rho}) - g\sqrt{f}\hat{\sigma} = 0. \tag{1.22}$$

Using equations (1.18), we can eliminate three of the unknown fields from the equations of motion and reduce them down to three equation, involving three fields f, g, and h,

$$\frac{1}{r}\left(\frac{f'}{f} + \frac{g'}{g} + \frac{4}{r}\right) + \frac{gh\hat{\sigma}}{\sqrt{f}} = 0, \tag{1.23}$$

$$\frac{f'}{rf} - \frac{h'^2}{2f} + g(3+\hat{p}) - \frac{1}{r^2} = 0, \tag{1.24}$$

$$h'' + \frac{g\hat{\sigma}}{\sqrt{f}}\left(\frac{rhh'}{2} - f\right) = 0. \tag{1.25}$$

As discussed above, the deep IR of the bulk between r_0 and r_+ is described by the AdS-RN$_4$ geometry (1.15). By looking at the integration limits of expressions in (1.18), it is clear that the star cannot be supported for any temperature of the black hole. For a given mass $\hat{m}$, the star forms at a radius r_0 where the AdS-RN parameters f, g, h are

$$\hat{m} = \frac{h}{\sqrt{f}}, \qquad\qquad \frac{d}{dr}\frac{h}{\sqrt{f}} = 0. \tag{1.26}$$

Following the profile of the star along the radial coordinate towards AdS infinity at $r = 0$, we find that the star ends at its surface r_s, which is also a function of the temperature. The surface is determined to be at a radial coordinate r_s where all p, ρ and σ vanish. Furthermore, the exterior of the star between $r = 0$ and r_s must also take the form of an AdS-RN solution, but with modified parameters for the speed of light, mass and charge. The parameters of the solution can be determined by matching the functions f, g, h at r_s. More explicitly, we write down generic RN-AdS functions

$$f_{RN-AdS} = c^2\left(\frac{1}{r^2} - \hat{M}r + \frac{r^2}{2}\hat{Q}^2\right), \quad g_{RN-AdS} = \frac{c^2}{r^4 f}, \quad h_{RN-AdS} = c\left(\hat{\mu} - r\hat{Q}\right) \tag{1.27}$$

where c is the exterior (boundary) speed of light which is not necessarily equal to one. Then by writing

$$f_{RN-AdS}|_{r=r_s} = f_{ES}|_{r=r_s}\,, \tag{1.28}$$

$$g_{RN-AdS}|_{r=r_s} = g_{ES}|_{r=r_s}\,, \tag{1.29}$$

$$g_{RN-AdS}|_{r=r_s} = g_{ES}|_{r=r_s}\,, \tag{1.30}$$

we obtain a system of equations that can be solved to compute $\hat{Q}, \hat{M}$ and c.

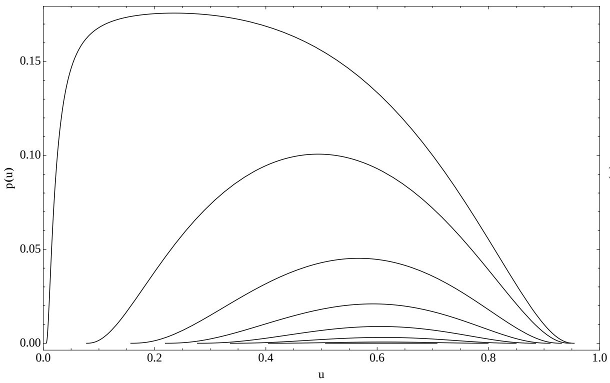
We can now numerically find the background solutions for $T > 0$. To find the region of parameter space when the star is able to form, we take the interior near-horizon AdS-RN solution at progressively lower temperatures and solve the background equations (1.23, 1.24, 1.25) numerically. We use a new coordinate $u = r/r_+$ with the horizon at $u = 1$ and the AdS boundary at $u = 0$. Furthermore, the controlling parameter we use is the dimensionless T/μ. The results for $T/\mu = \{0.00003,\ 0.007,\ 0.027,\ 0.05,\ 0.07,\ 0.09,\ 0.12,\ 0.13\}$ are presented in fig. 1. What becomes immediately obvious is that there is a critical temperature T_c, determined by eq. (1.26), above which the electron star background has no solution and a star cannot be sustained. Once T_c is crossed so that $T < T_c$, the star emerges, grows and eventually dominates the entire space-time. In the limit of zero temperature, when all the bulk charge is contained in the fluid alone, the extremal uncharged black hole shrinks to zero.

We conclude this section by returning to the question of hydrodynamical excitations of the boundary theory. We argued that we do *not* expect the IR excitation to match predictions of phenomenological hydrodynamics and thermodynamics. The only exceptions should be the regions of very low T and temperatures near T_c. The former conclusion follows from the fact that as $T \to 0$, $\hat{\sigma}$ in (1.17) is exponentially suppressed even near horizon. The later follows from our observation that near T_c the star become very small and eventually disappears. The system becomes dominated by the RN black brane and therefore approaches thermal equilibrium when $T \approx T_c$.

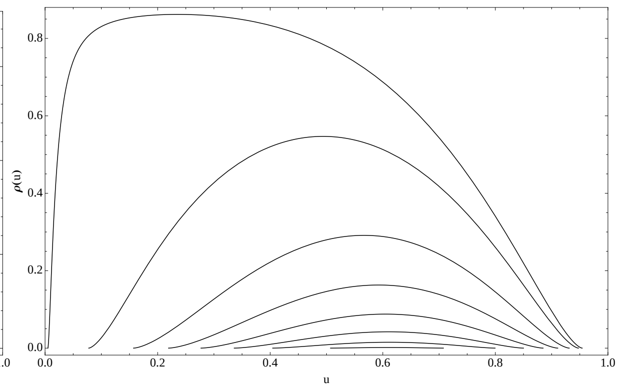
Finally, we can introduce an additional parameter that describes the electrons star background. The parameter is the scaling exponent z, which is fixed by the values of m and β. Its meaning can be seen by taking the limit of $T \to 0$ so that the RN black brane disappears and the electron star dominates the deep IR, making the space asymptotically Lifshitz [1],

$$f \to \frac{1}{r^2 z}, \qquad g \to \frac{g_\infty}{r^2}, \qquad h \to \frac{h_\infty}{r^z}. \tag{1.31}$$

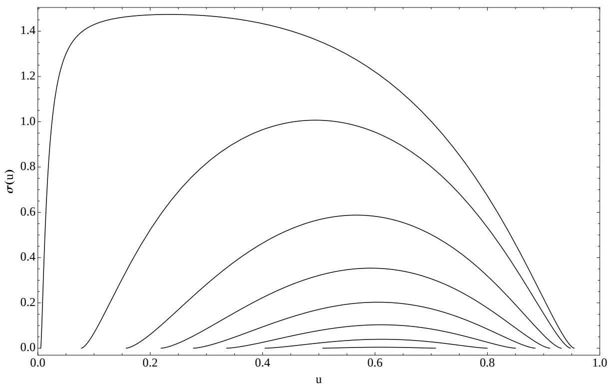
The parameter z is then precisely the non-relativistic Lifshitz scaling exponent $t \to \lambda^z t$, $\boldsymbol{x} \to \lambda \boldsymbol{x}$. In the limit of $z \to \infty$ the Lifshitz background becomes $AdS_2 \times \mathbb{R}_2$, which is also the local horizon geometry of the AdS-RN$_4$ black brane. Given that we argued that we expect the near-horizon bulk excitations to control the IR hydrodynamic regime, this implies that we expect to recover good agreement between our results and those predicted by thermodynamics at *large* z.



(a) Pressure



(b) Energy density



(c) Charge density

Figure 1: The electron star's development as a function of T/μ. The top curves correspond to $T/\mu = 0.00003$ and the bottom ones to $T/\mu = 0.13$. Here $\hat{m} = 0.36$, $\hat{\beta} = 19.951$.

2 Hydrodynamics

Equipped with a numerical solution for the electron star background, we proceed our analysis by computing various thermodynamical and hydrodynamical properties of the dual theory. The first step is to the find the free energy $\hat{\Omega}$ which equals the on-shell Euclidean bulk action. From a purely thermodynamical point of view, the free energy is

$$\hat{\Omega} = \hat{M} - \hat{\mu}\hat{Q} - \hat{s}T, \tag{2.1}$$

where $\hat{s}$ is the Bekenstein-Hawking entropy, $\hat{M}$ the mass parameter of the AdS-RN black brane, $\hat{\mu}$ the field theory chemical potential and $\hat{Q}$ the charge parameter of the black brane. From standard thermodynamics it is also known that

$$\hat{\Omega} = -\hat{p}\hat{V}, \tag{2.2}$$

where $\hat{p}$ is the pressure and $\hat{V}$ the volume of the system. The Euclidean bulk action reads

$$S_E = \int d^4x\sqrt{g}\left[\frac{1}{2\kappa^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4e^2}F^2 + \hat{p}\right] + S_{GH} + S_{c.t.}, \tag{2.3}$$

where S_{GH} is the Gibbons-Hawking term, necessary to make the variational problem well-defined and $S_{c.t.}$ is the counter-term prescribed by the 'minimal-subtraction' holographic renormalisation scheme, necessary to cancel the divergences at AdS-inifinity, i.e. the ∂AdS at $r = 0$ [11, 12]. In particular,

$$S_{GH} = -\frac{1}{2\kappa^2}\int_{\partial AdS} d^3x\sqrt{\gamma}\, 2K, \tag{2.4}$$

and

$$S_{c.t.} = -\frac{1}{2\kappa^2}\int_{\partial AdS} d^3x\frac{4}{L}\sqrt{\gamma} + L\sqrt{\gamma}^{(3)}R, \tag{2.5}$$

where γ is the determinant of the induced boundary metric, K the extrinsic curvature and ${}^{(3)}R$ the three-dimensional Ricci scalar of the induced boundary metric, which in our case vanishes. Written out explicitly, the Lagrangian is

$$\begin{aligned}\mathcal{L}_E = L^2\sqrt{f(r)g(r)}\,&\left(r^2g(r)f'(r)^2 + rf(r)\left(rf'(r)g'(r) + 2g(r)\left(-rf''(r) + 2f'(r)\right.\right.\right.\\ &\left.\left.\left.+rh'(r)^2\right)\right) + 4f(r)^2\left(g(r)\left(r^2g(r)\left(\hat{p}(r)+3\right)-5\right) - rg'(r)\right)\right)/\left(4\kappa^2r^4f(r)^2g(r)^2\right).\end{aligned} \tag{2.6}$$

After applying the background equations of motion, we find

$$\mathcal{L}_E = \frac{L^2}{\kappa^2}\frac{d}{dr}\frac{f'(r) - 2h(r)h'(r)}{2r^2\sqrt{f(r)g(r)}}. \tag{2.7}$$

The Gibbons-Hawking term and the counter-term Lagrangian further reduce to

$$\mathcal{L}_{GH} + \mathcal{L}_{c.t.} = \frac{L^2\left(\varepsilon f'(\varepsilon) + 4f(\varepsilon)\left(\varepsilon\sqrt{g(\varepsilon)} - 1\right)\right)}{2\varepsilon^3\kappa^2\sqrt{f(\varepsilon)}\sqrt{g(\varepsilon)}}, \tag{2.8}$$

where ε is an infinitesimal positive number, i.e. $\varepsilon \to 0$.

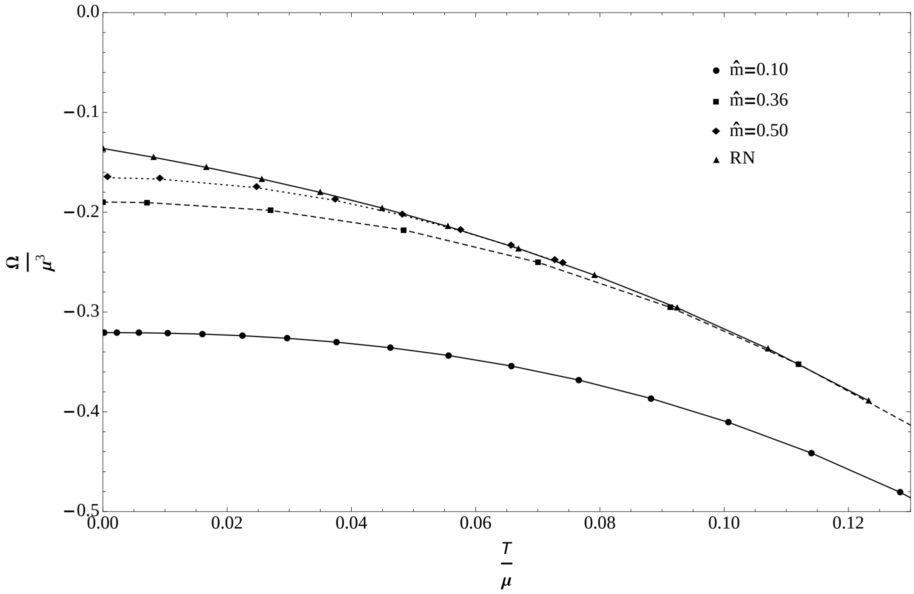


Figure 2: Free energy of the Electron Star system for three electron masses $\hat{m}$. The RN result is overlaid for comparison.

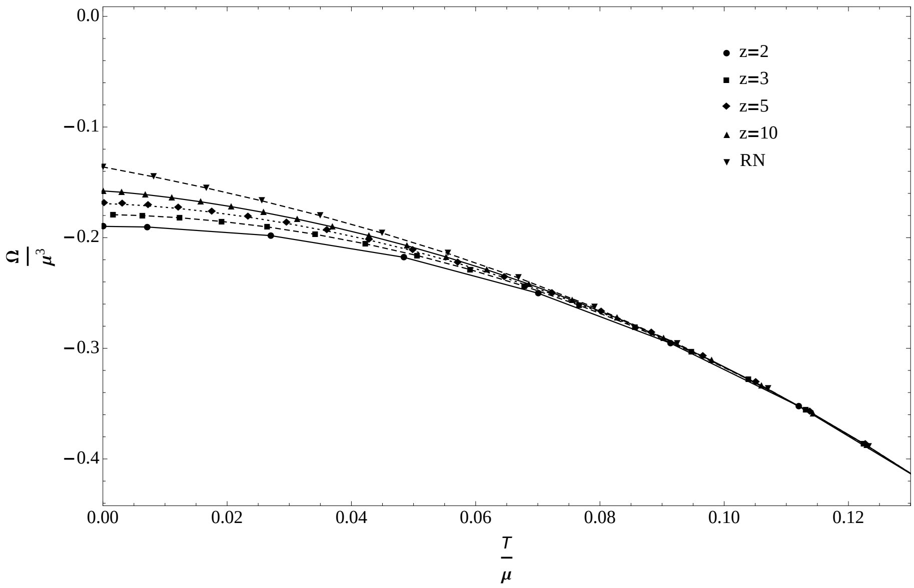


Figure 3: Free energy of the Electron Star system for four critical exponents z. The RN result is overlaid for comparison.

Given that the background can be numerically computed, so can the action and thus the free energy. The resulting plots of dimensionless $\hat{\Omega}/\hat{\mu}^2$ are presented in fig. 2 where we have varied the electron mass $\hat{m}$, and in fig. 3 where we have varied the critical exponent z. The chemical potential is extracted from the outer RN part of the bulk between $0 \leq r \leq r_s$. What we find is that as the electron mass $\hat{m}$ decreases at low temperatures, so does the free energy.[2] As the temperature is increased, a transition to the pure AdS-RN appears. For each $\hat{m}$ there is an associated critical transition temperature T_c. Near that temperature, the free energies of electron star with different physical parameters converge to the AdS-RN free energy. Said differently, for every electron star mass, there is a temperature above which the RN black brane becomes sufficiently large that the electron cloud collapses and the black brane dominates the space-time. Furthermore, by varying the critical exponent z we see that the higher z is, the closer to the RN results one gets. This is consistent with our discussion at the end of section 1, which argued that our results should converge to those of pure AdS-RN$_4$ in the limit of $z \to \infty$. It should also be noted that the AdS-RN free energy is always larger then that of the mixed AdS-RN-electron star system. The configurations with the electron star present are therefore thermodynamically preferred to the pure AdS-RN state.

We can also numerically compute the dimensionless entropy density,[3] which is given by

$$\frac{\hat{s}}{\hat{\mu}^2} = \frac{2\pi}{r_+^2 \hat{\mu}^2} = \frac{2\pi}{\hat{\mu}^2}, \tag{2.9}$$

in the coordinate system where the horizon is fixed at $r_+ = 1$. This can be done either by using (2.1) or the Bekenstein-Hawking entropy formula in terms of the area of the event horizon. The results are plotted in fig. 4, where the electron mass is varied and fig. 5, where the critical exponent dependence is examined. The most important feature of the electron star's $\hat{s}/\hat{\mu}^3$ compared to the pure RN case, is that the dimensionless entropy density vanishes as temperature goes to zero, thinking of the chemical potential as fixed. This circumvents a major issue with the thermodynamics of a theory dual to a pure RN-AdS system. In that case, as seen by the RN curve in fig. 4, the entropy density remains finite even at zero temperature, which is in contradiction with our expectations of physical theories should behave. Regarding the electron star parameters, we again find that higher electron mass and higher critical exponent lead to faster convergence to the RN result.

With the knowledge of the free energy and entropy density, we can proceed to compute any thermodynamical quantity of interest. The hydrodynamical quantity of main interest for this work is the diffusion constant $\mathcal{D}$, for viscous fluids. It is known from phenomenological hydrodynamics (e.g. [13]) that the diffusion constant is related to the shear viscosity, the energy density and pressure, through

$$\mathcal{D} = \frac{c^2 \eta}{\epsilon + p/c^2}, \tag{2.10}$$

where we had to restore the dependence of $\mathcal{D}$ on the speed of light, since the boundary theory does not have it equal to one. The standard holographic practice is to use the KSS[4] relation [14]

$$\frac{\eta}{\hat{s}} = \frac{\hbar}{4\pi k_B} = \frac{1}{4\pi}, \tag{2.11}$$

[2]Here temperature is measured with respect to the chemical potential through the dimensionless parameter T/μ

[3]As in the case of temperature and free energy, the dimensionless entropy density is also measured in units of chemical potential, which is the relevant scale.

[4]Kovtun-Son-Starinets

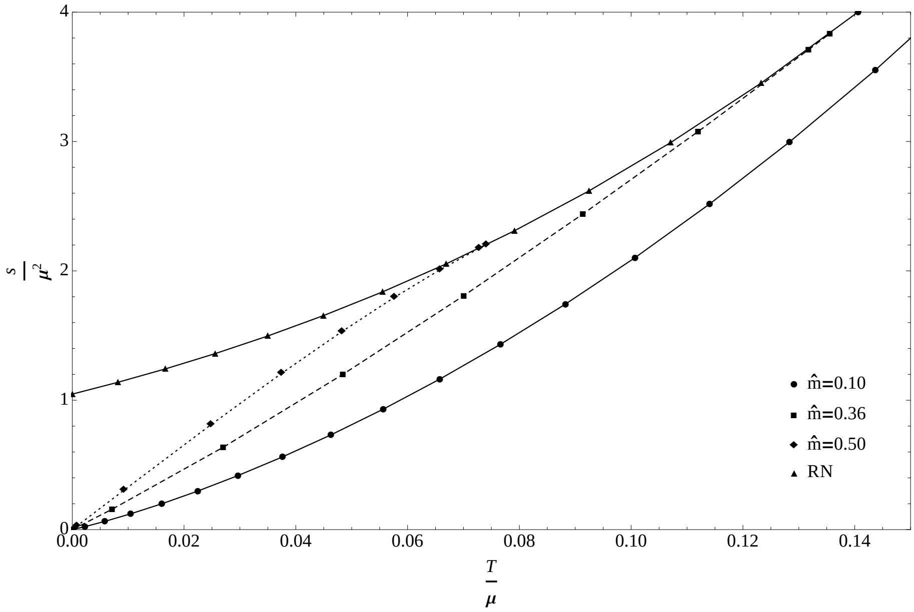


Figure 4: The entropy density of the electron star for three electron masses $\hat{m}$. The RN result is overlaid for comparison.

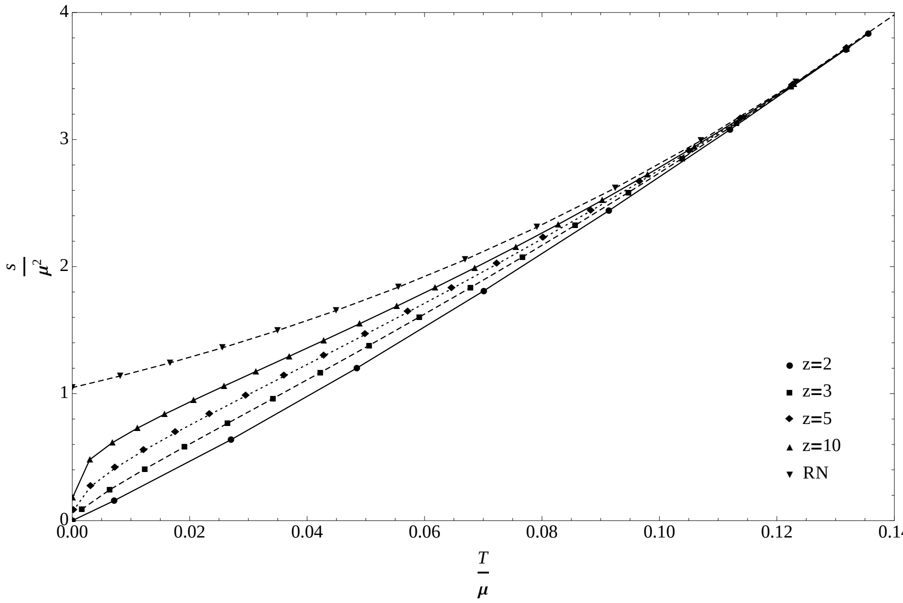


Figure 5: The entropy density for the electron star for four critical exponents z. The RN result is overlaid for comparison.

where in the last step we have used the natural units $k_B = \hbar = 1$. Moreover, the asymptotic AdS symmetries indicate that the boundary theory should be conformal, which imposes the relation

$$\epsilon - 2p = 0, \tag{2.12}$$

on energy density and pressure. Since the electron star system had not been examined in this way before, we will now verify both these statements. To examined the conformality condition, the pressure can be extracted from the on-shell action through eq. (2.2). Energy corresponds to the mass parameter of the outer RN part of the space-time. From the numerical solution one indeed sees in fig. 6 that the condition holds.

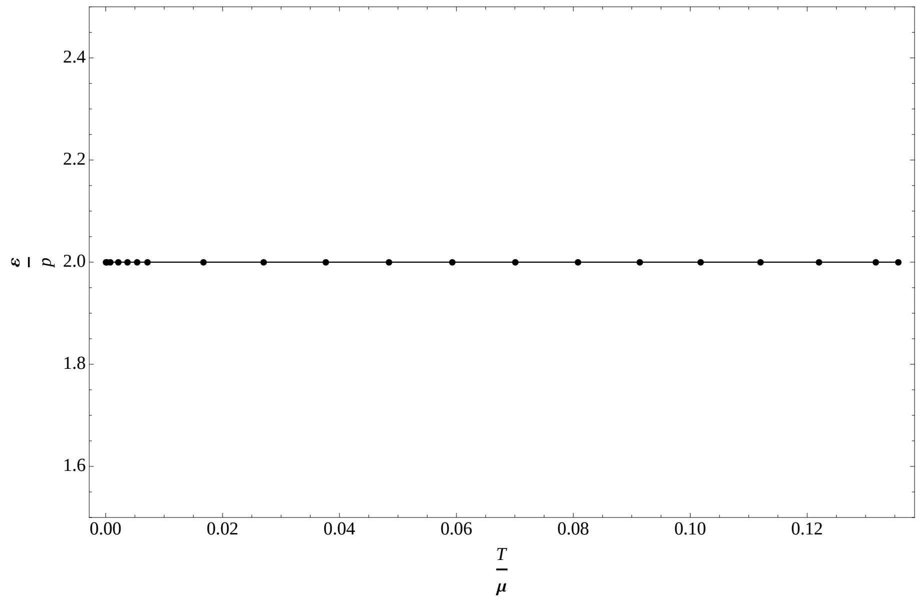


Figure 6: Verification of the conformality condition (2.12).

The verification of the KSS relation is somewhat more involved. Since the entropy (2.9) for this system is already known, we need to calculate the shear viscosity in an independent way. The easiest way is to make use of the Kubo formula that in four bulk dimensions relates the shear viscosity with the retarded Green's function of a shear component of the stress-energy tensor. The two-point function is found by solving the Einstein's equation for the graviton mode h_{xy} perturbation of the metric tensor and using the standard holographic techniques [15, 16]. In particular,

$$\eta = \lim_{\omega\to 0} \frac{1}{2\omega} \int dt d\boldsymbol{x} e^{i\omega t} \langle [T_{xy}(x), T_{xy}(0)] \rangle_R = -\lim_{\omega\to 0} \frac{1}{\omega} G_R(\omega, \mathbf{0}), \tag{2.13}$$

where $\mathbf{0}$ stands for zero spatial two-momentum. Fortunately, at zero momentum, which is relevant for this calculation, the Einstein's equation (1.1) for the h_{xy} mode decouples from the other fluctuations. We find that

$$h_x{}^{y\prime\prime}(r) + \frac{f'_{out}(r)}{f_{out}(r)} h_x{}^{y\prime}(r) + \omega \frac{g_{out}(r)}{f_{out}(r)} h_x{}^y(r) = 0, \tag{2.14}$$

where f_{out}, g_{out}, h_{out} are the RN parameters for the outer RN part of the space-time determined by the matching criteria eq. (1.30). By expanding this equation near the boundary we find that its solution admits a series expansion of the form

$$h_x{}^y = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + \dots. \tag{2.15}$$

The Frobenius method indices of the equation are determined by setting ${h_x}^y = r^\nu f(r)$ and solving the resulting equation so that f is regular at the regular singular point $r = 0$. We find that $\nu = \{0, 3\}$. Therefore with the exception of the two undetermined coefficients in front of the leading and sub-leading terms, $a_0 r^0$ and $a_3 r^3$, the remaining coefficients are completely determined in terms of a_0 and a_3. In our case,

$$a_1 = 0, \qquad a_2 = a_0 \frac{\omega^2}{2c^2}, \qquad a_4 = \frac{9 a_3 c^2 \hat{M} - a_0 \hat{Q}^2 \omega^2}{\omega^2}, \tag{2.16}$$

where c, $\hat{M}$, and $\hat{Q}$ are the outer RN parameters from eq. (1.27). These relations will be used as checks of our numerical solution. In order to proceed, we solve eq. (2.14) numerically with the appropriate in-falling boundary conditions at the horizon. This ensures that the resulting Green's function is retarded [15, 16]. Once the solution is found, the checks mentioned above are performed to verify the numerical stability of the solution. The solution is fitted to the series expression, in order to determine the free parameters a_0 and a_3. Having determined these then one can immediately compute the retarded Green's function through the standard holographic procedure (see e.g. [17])

$$G_R(\omega, \boldsymbol{k}) = (2\Delta - d) \frac{a_3(\omega, \boldsymbol{k})}{a_0(\omega, \boldsymbol{k})}, \tag{2.17}$$

where d is the dimensionality of the boundary (in this case $d = 3$) and Δ the scaling dimension of the operator appearing in the two-point function. Stress-energy tensor has dimension $\Delta = d = 3$, which is consistent with form of the Frobenius expansion found above. Combining eq. (2.13) and eq. (2.17), the results that verify the KSS relation are plotted in fig. 7.

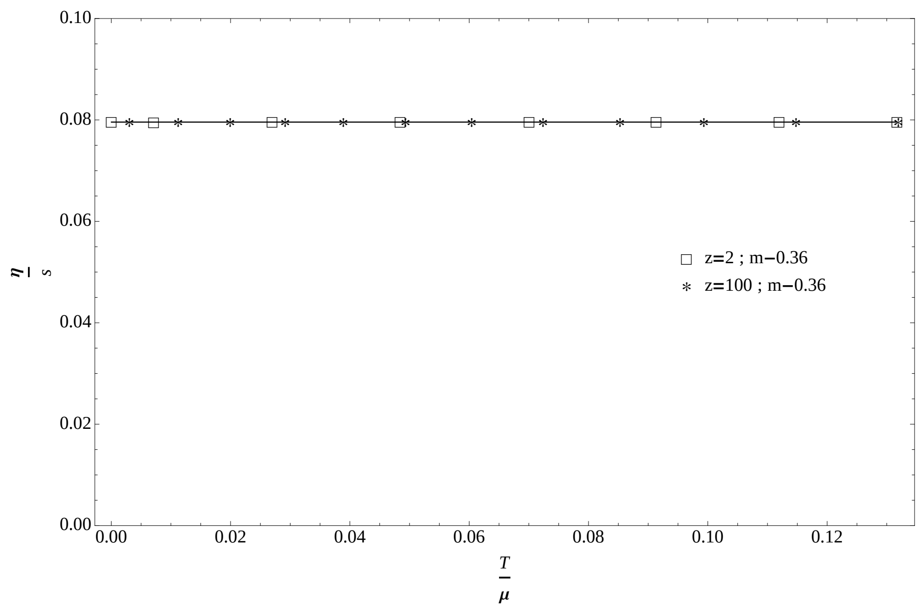


Figure 7: $\frac{\eta}{s}$ for two critical exponents. The solid line corresponds to the value $\frac{1}{4\pi}$.

We can now compute the diffusion coefficient $\mathcal{D}$ through eq. (2.10). The results are presented in fig. 8, where the electron mass $\hat{m}$ is varied and in fig. 9, where the critical exponent z is varied.

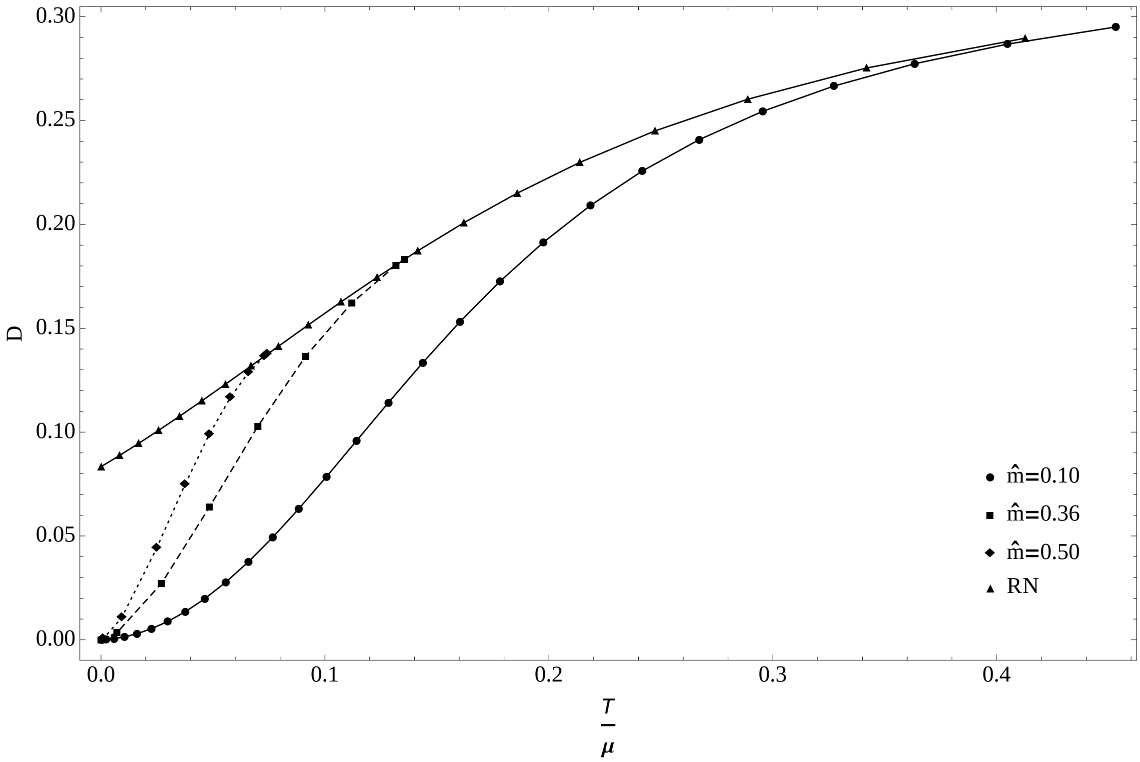


Figure 8: Diffusion coefficient for three electron masses and the pure RN case.

We see that $\mathcal{D}$ converges to the RN values above a certain temperature which is distinct for each electron mass. This observation is consistent with the already established fact that the electron clouds disappear at high enough temperatures. What can also be determined from these results is that for stars with more massive constituent electrons, $\mathcal{D}$ converges to RN faster, i.e. at lower T/μ. Similarly, the geometries with higher critical exponents z converge to RN quicker. This is again consistent with the fact that as $z \to \infty$, the horizon geometry becomes that of the RN black brane.

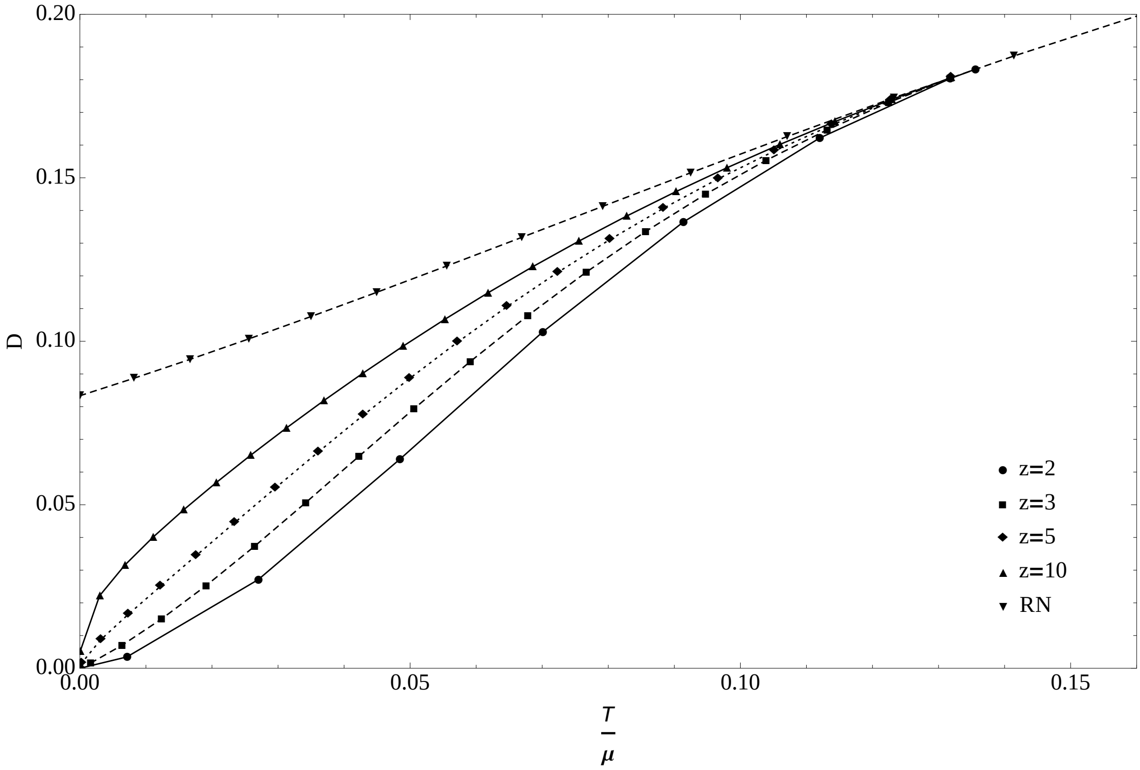


Figure 9: Diffusion coefficient for four critical exponents and the pure RN case.

3 Conclusion

In this section we analyse the full linear perturbation in the shear sector of the electron star system, focusing on the low-energy excitations of the dual boundary theory. The holographic method which we will employ to extract them numerically will be the calculation of quasi-normal modes [18]. We begin by writing the action, which reproduces the Einstein-Maxwell system given in eqs. (1.1) and (1.2) with the appropriate ideal fluid stress-energy tensor eq. (1.3),

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}\left(R + \frac{6}{L^2}\right) - \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} - \rho(\sigma) + \sigma u^\mu(\partial_\mu\phi + A_\mu)\right.$$
$$\left. + \lambda(u^\mu u_\mu + 1)\right] + S_{GH} + S_{c.t.}. \tag{3.1}$$

The part of action that describe an ideal fluid at zero temperature is known as the Schutz action [19]. The full action should be varied with respect to the metric $g_{\mu\nu}$, the $U(1)$ gauge field A_μ, the four-velocity of the fluid u_μ, the charge density of the fluid σ, the Clebsch potential ϕ as well as the Lagrange multiplier λ, which is introduced into the action to ensure requirement for a relativistic fluid that $u^2 = -1$. For convenience, we rescale the fields, along with their fluctuations, so that the action is proportional to L^2/κ^2,

$$A_\mu \to \frac{eL}{\kappa}A_\mu, \qquad \phi \to \frac{eL}{\kappa}\phi, \qquad u_\mu \to Lu_\mu, \qquad \sigma \to \frac{1}{eL^2\kappa}\sigma, \qquad \lambda \to \frac{1}{L^2\kappa^2}\lambda. \tag{3.2}$$

If we now use the physically motivated definitions of the chemical potential μ, charge density σ and pressure p,

$$\mu(\sigma) \equiv \rho'(\sigma) = u^\mu(\partial_\mu\phi + A_\mu), \qquad\qquad p(\sigma) \equiv -\rho(\sigma) + \sigma\mu(\sigma) \tag{3.3}$$

then the stress-energy tensor from the Schutz action takes the form of a perfect fluid

$$T^{\mu\nu} = (p + \rho)u^\mu u^\nu + pg^{\mu\nu}, \tag{3.4}$$

where p is pressure and ρ the energy density of the fluid.

In order to find the equations of motion for the perturbations, we have to excite all the fields as

$$g_{\mu\nu}(r) \to g_{\mu\nu}(r) + h_{\mu\nu}(r,t,\boldsymbol{x}), \qquad\qquad A_\mu(r) \to A_\mu(r) + a_\mu(r,t,\boldsymbol{x}), \tag{3.5}$$
$$u_\mu(r) \to u_\mu(r) + \delta u_\mu(r,t,\boldsymbol{x}), \qquad\qquad \phi(r) \to \phi(r) + \delta\phi(r,t,\boldsymbol{x}), \tag{3.6}$$
$$\sigma(r) \to \sigma(r) + \delta\sigma(r,t,\boldsymbol{x}), \qquad\qquad \lambda(r) \to \lambda(r) + \delta\lambda(r,t,\boldsymbol{x}), \tag{3.7}$$

and vary the equations of motion to first order. We use the Fourier decomposition of the space-time directions transverse to the radial bulk direction,

$$\phi(r,t,\boldsymbol{x}) = \int \frac{d\omega d^2k}{(2\pi)^3}e^{-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}}\,\phi(r,\omega,\boldsymbol{k}), \tag{3.8}$$

where ϕ stands for each field in the problem. We will choose the momentum to be in the y direction. This procedure results in twenty one equations. They naturally split into two decoupled sets of equations - the shear equations of motion, involving fields that are odd under $y \to -y$, and the longitudinal equations of motion, involving fields that are even under $y \to -y$. This decoupling is guaranteed at linear order in the fluctuations as the theory is invariant under $y \to -y$.

We will now focus only on the shear degrees of freedom, which include h_{xy}, h_{ty}, h_{ry}, a_y, δu_y. By varying the action with respect to these fields we find the equations

$$
\begin{aligned}
&{h_y}^{x\prime\prime} - ikr^2 g {h_y}^{r\prime} + \left(\frac{f'}{2f} - \frac{g'}{2g} - \frac{2}{r}\right){h_y}^{x\prime} - 2ikr^2 g\left(\frac{f'}{4f} + \frac{g'}{4g}\right){h_y}^{r} \\
&\qquad + \omega^2\frac{g}{f}{h_y}^{x} - \omega k r^2 g {h_t}^{t} = 0, \qquad (3.9)\\
&{h_y}^{t\prime\prime} - i\omega\frac{g}{f}{h_y}^{r\prime} + \left(\frac{3f'}{2f} - \frac{g'}{2g}\right){h_y}^{t\prime} - \frac{2h'}{f}a_y' - 2i\omega\frac{g}{f}\left(\frac{g'}{4g} - \frac{f'}{4f} - \frac{1}{r}\right){h_y}^{r} \\
&\qquad + \omega k\frac{g}{f}{h_y}^{x} - k^2r^2 g {h_y}^{t} + 2\frac{g}{\sqrt{f}}(p+\rho)\delta u_y + 2\left(\frac{h'^2}{f} + g(p+\rho)\right){h_y}^{t} = 0, \qquad (3.10)\\
&\left(\omega^2 - k^2r^2 f\right){h_y}^{r} + i\omega\frac{f}{g}{h_y}^{t\prime} - ik\frac{f}{g}{h_y}^{x\prime} + \frac{i\omega}{g}\left(\frac{2f}{r} + f'\right){h_y}^{t} - 2i\omega\frac{h'}{g}a_y = 0, \qquad (3.11)\\
&a_y'' + \frac{1}{2}\left(\frac{f'}{f} - \frac{g'}{g}\right)a_y' + \frac{g}{f}(\omega^2 - k^2r^2 f)a_y - h'{h_y}^{t\prime} + i\omega\frac{g}{f}h'{h_y}^{r} + g\sigma\delta u_y \\
&\qquad + \frac{h'}{2}\left(rg(p+\rho) - \frac{f'}{f} + \frac{g'}{g}\right){h_y}^{t} = 0, \qquad (3.12)\\
&\delta a_y + \mu\delta u_y = 0, \qquad (3.13)
\end{aligned}
$$

where the indices of the metric fluctuations $h_{\mu\nu}$ are raised and lowered using the background metric $g_{\mu\nu}$, a prime denotes a derivative with respect to r, and the dependence of the field fluctuations upon r, ω and k, and of the background fields upon r, have been suppressed for conciseness. The equations can be considerably simplified if we first solve a subset of them algebraically. We solve eqs. (3.11) and (3.13) for ${h_y}^r$ and δu_y and substitute the solutions into the remaining equations. What we find is that there are only two linearly-independent equations of motion in the shear sector, which can be naturally written in terms of the following gauge-invariant combinations of the fluctuations

$$Z_1(r,\omega,k) = \omega {h_y}^{x} - kr^2 f(r){h_y}^{t}(r,\omega,k), \qquad (3.14)$$
$$Z_2(r,\omega,k) = a_y(r,\omega,k). \qquad (3.15)$$

The variables Z_1 and Z_2 are invariant under both the bulk $U(1)$ gauge symmetry, which acts on the gauge field perturbation as

$$a_\mu(r,\omega,k) \to a_\mu(r,\omega,k) - \partial_\mu\Lambda(r,\omega,k), \qquad (3.16)$$

and the bulk diffeomorphism symmetry, which acts on the fields as

$$
\begin{aligned}
h_{\mu\nu}(r,x,t) &\to h_{\mu\nu}(r,x,t) - \nabla_\mu\xi_\nu(r,x,t) - \nabla_\nu\xi_\mu(r,x,t),\\
a_\mu(r,x,t) &\to a_\mu(r,x,t) - \xi^\alpha(r,x,t)\nabla_\alpha A_\mu(r) - A_\alpha\nabla_\mu\xi^\alpha(r,x,t),\\
\delta\sigma(r,x,t) &\to \delta\sigma(r,x,t) - \xi^\alpha(r,x,t)\nabla_\alpha\sigma(r). \qquad (3.17)
\end{aligned}
$$

to linear order in the fluctuations. We use the notation ∇_μ for the covariant derivative with respect to the background metric. These particular combinations are natural in the language of the dual field theory operators as they guarantee that the relevant field theory Ward identities are satisfied.

In terms of the gauge-invariant fluctuations the two linearly-independent equations of motion are

$$\begin{aligned} &Z_1'' + 2kr^2h'Z_2' + \left(\frac{rg\sigma\mu}{2} + \frac{\omega^2 f' + 2k^2rf^2}{f(\omega^2 - k^2r^2f)}\right)Z_1' + \frac{g}{f}(\omega^2 - k^2r^2f)Z_1 \\ &\quad + 2kr^2\sqrt{f}\mu\left(\frac{2\omega^2h'^2}{f(\omega^2 - k^2r^2f)} + \frac{g\sigma}{\mu}\right)Z_2 = 0, \end{aligned} \tag{3.18}$$

$$\begin{aligned} &Z_2'' + \frac{1}{2}\left(\frac{f'}{f} - \frac{g'}{g}\right)Z_2' - \frac{kh'}{\omega^2 - k^2r^2f}Z_1' + \frac{g}{f}(\omega^2 - k^2r^2f)Z_2 \\ &\quad - \left(\frac{2\omega^2h'^2}{f(\omega^2 - k^2r^2f)} + \frac{g\sigma}{\mu}\right)Z_2 = 0. \end{aligned} \tag{3.19}$$

These are the equations that need to be solved numerically in order for us to extract the quasi-normal modes. On top of that, we need the on-shell action, which is required for the calculation of the full Green's functions and spectral function. The on-shell action corresponds to the generating functional of the boundary theory [20, 21]. The action for the original and the gauge-invariant fields is presented in Appendix A.

It is further important to determine the off-shell action to quadratic order in the fluctuations in the gauge-invariant variables. The reason is that the off-shell enables us to find a Noether current, which is conserved along the radial direction of the bulk. It thus serves as a useful tool for an additional check on the stability of our numerics. This is further discussed in Appendix B.

In order to proceed with the numerics, we need to determine the boundary conditions for Z_1 and Z_2. At the horizon we first analytically impose the in-falling boundary conditions which are required to compute the retarded Green's functions,

$$Z = e^{\cdots}\mathcal{Z} \tag{3.20}$$

$$\tag{3.21}$$

Fixing the remaining initial conditions at the horizon required for numerical integration is somewhat more subtle because our system mixes the two gauge-invariant operators. The fact that we are solving a system of coupled differential equations means that it is not well-defined to talk about the 'one-to-one' operator-source $\int \mathcal{O}Z$ correspondence anywhere, but strictly on the boundary. In other words, it is only on the boundary that the solution of each equation of motion provides information about a single, specific field theory operator. Numerically, however, we always have to work on a cut-off surface close to the boundary where we must take into account the operator mixing effects. The recipe of for dealing with such operator mixings was given in [22] and consists of numerically finding a set of linearly independent solutions to the equations of motion. Out of them, a solution matrix is formed that reduces to a diagonal matrix at the boundary. This matrix enables us to extract all the relevant information about the boundary system.

The prescription is implemented in the following way. After we impose the in-falling boundary conditions (3.21), we expand the equations of motion close to the horizon and demand that the regular part of the solution is expressed as a series. We can write

$$\mathcal{Z}_1(u) = a_0 + a_1(1-u) + \dots, \tag{3.22}$$

$$\mathcal{Z}_2(u) = b_0 + b_1(1-u) + \dots. \tag{3.23}$$

Here, only the first two coefficients will be presented for conciseness. However, in the actual calculation the series was continued up to sixth order, which provides us with enhanced numerical accuracy. Plugging the ansatz into the equations of motion and iteratively solving, we find that every coefficient can be computed in terms of a_0 and b_0. Those two parameters will provide the two linearly independent solutions by setting $(a_0, b_0) = (1, 1)$ for one of them and $(a_0, b_0) = (1, -1)$ for the other. The first-order coefficients for each field are presented here. They are

$$\begin{aligned} a_1 = a_0 &\left[\varepsilon - \frac{i(\varepsilon-1)k^2}{\omega} - \frac{(\varepsilon-1)\left(2k^2+\hat{Q}^2-6\right)}{\hat{Q}^2+4i\omega-6} - \frac{8i(\varepsilon-1)\hat{Q}^2\omega}{\left(\hat{Q}^2-6\right)^2}\right] \\ &+ \frac{2b_0(\varepsilon-1)k\hat{Q}\left(\hat{Q}^2+2i\omega-6\right)}{\hat{Q}^2+4i\omega-6}, \end{aligned} \tag{3.24}$$

and

$$b_1 = -\frac{2ia_0(\varepsilon-1)k\hat{Q}}{\omega\left(\hat{Q}^2+4i\omega-6\right)} + \frac{2b_0(\varepsilon-1)\left(k^2+2\hat{Q}^2\right)}{\hat{Q}^2+4i\omega-6} - \frac{8ib_0(\varepsilon-1)\hat{Q}^2\omega}{\left(\hat{Q}^2-6\right)^2} + b_0. \tag{3.25}$$

The parameter ε is taken to be a very small, i.e. $\varepsilon \ll 1$, number determining how close to the horizon the integration starts. Its appearance in these expressions highlights the point that one cannot just set the initial conditions for $\mathcal{Z}_{1,2}$ to 1 or -1 because numerically we need to specify them away from the horizon. The knowledge of the relevant coefficients a_i and b_i allows us to set the initial conditions, i.e. the values and derivatives of $\mathcal{Z}_{1,2}$, which are required to numerically integrate two second-order differential equations.

The free parameters of the system that we work with are the dimensionless temperature T/μ, the dimensionless scaled momentum k/μ, the critical exponent z and the electron mass $\hat{m}$. We are now ready to find the quasi-normal mode corresponding to diffusion and to extract the diffusive properties of the dual system. Diffusion is described by the lowest lying pole on the imaginary axis in the complex frequency plane. The interest will be focused on extracting the diffusion coefficient of the dual theory and studying its dependence on temperature, momentum through the dispersion relation and then z and $\hat{m}$.

Putting all the ingredients together the complex frequency plane looks like fig. 10. In this plot the first three (in the sense that they are the lowest ones, or said differently they

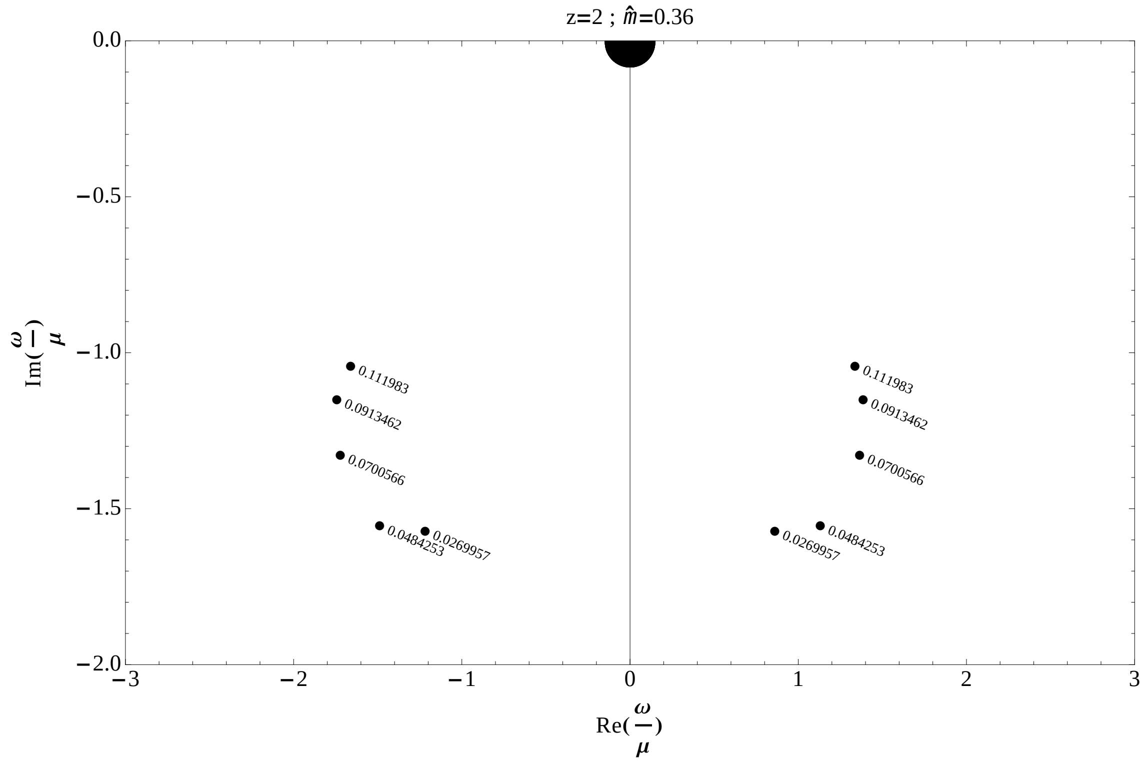


Figure 10: QNMs in the complex frequency (ω) plane for $\frac{k}{\mu} = 0.1$ and $\frac{T}{\mu} \simeq 0.11, 0.09, 0.07, 0.05, 0.03$.

have the smallest (absolute) imaginary part) poles are presented, for a series of temperatures. The blob near the origin represents the lowest lying poles, which because they have the smallest imaginary part they are the longest-living ones (i.e. they attenuate with the slowest rate). The scaling imposed by the simultaneous plotting of the poles with non-vanishing real part, obscures these poles, which are the ones corresponding to diffusion and are the most interesting. Zooming into the origin, the diffusion mode, for various temperatures, looks like fig. 11. Since the purely imaginary pole is of maximum interest, from now on plots will be presented in one-dimensional form, only depicting the imaginary axis.

The first numerical study will attempt to extract the diffusion coefficient. It is expected that the diffusion mode behaves like

$$\omega = -i\mathcal{D}(T)k^2 + \dots \tag{3.26}$$

or in the proper dimensionless form

$$\frac{\omega}{c\mu} = -i\bar{\mathcal{D}}(T)\left(\frac{k}{\mu}\right)^2 + \dots \tag{3.27}$$

In order to extract $\mathcal{D}$ therefore, one can track the diffusion pole for a very small $\frac{k}{\mu}$, against temperature and then $\mathcal{D}(T) = -\Im\frac{\omega/\mu c}{(k/\mu)^2}$. The momentum value chosen is $k/\mu = 0.001$. The remaining free parameters are then z and $\hat{m}$, that is each pair defines a $\mathcal{D}(T)$ curve. Firstly

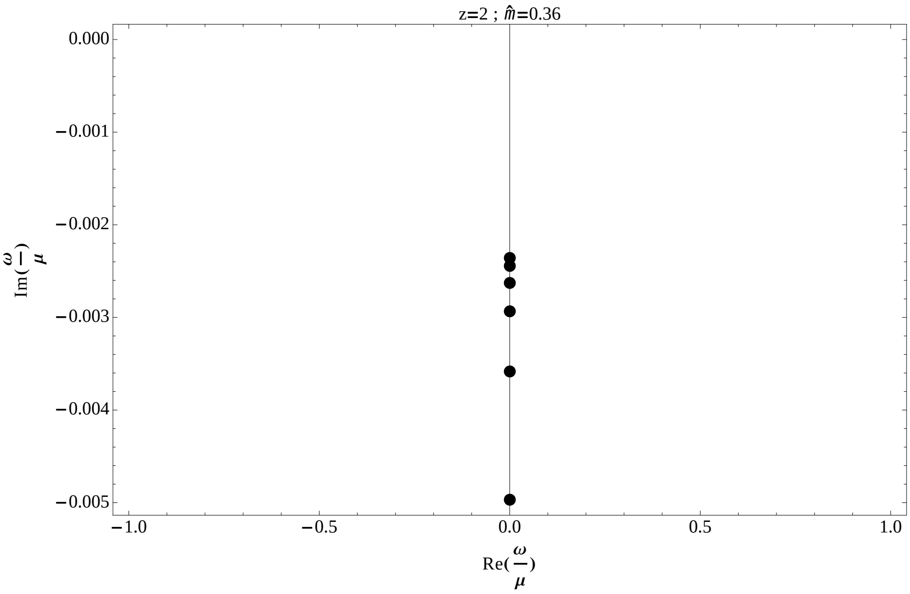
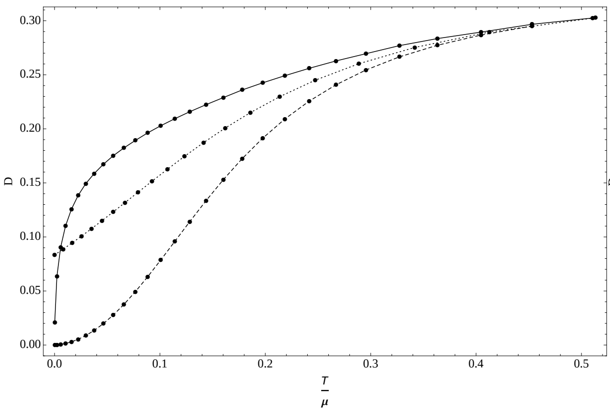
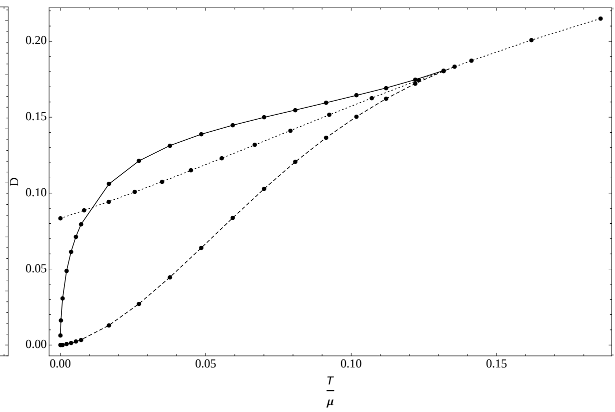


Figure 11: QNMs on the imaginary axis for $\frac{k}{\mu} = 0.1$ and $\frac{T}{\mu} \simeq 0.11, 0.09, 0.07, 0.05, 0.03$.

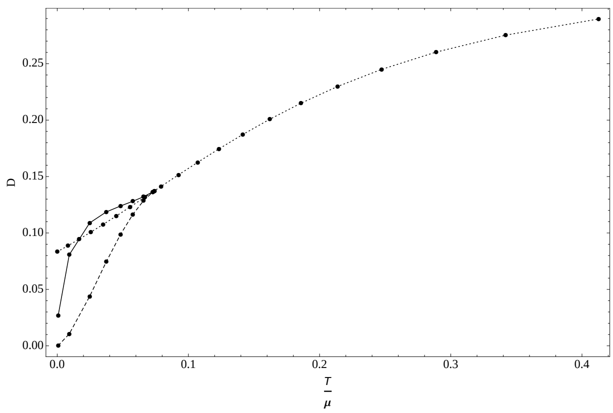
the critical exponent z will be kept constant at $z = 2$ and variations of the electron mass $\hat{m}$ will be presented. These results are presented in fig. 12, where the solid lines correspond to the actual results, derived from the diffusion pole, while the RN result (dotted line) along with hydrodynamics expectation (dashed line) have been overlaid for comparison. Similarly in fig. 13 the diffusion constant (again from the pole, from hydrodynamics and for AdS-RN) is presented for four different critical exponents ($z \in \{3, 5, 10, 100\}$), keeping the electron mass constant at $\hat{m} = 0.36$.



(a) $z = 2$; $\hat{m} = 0.10$

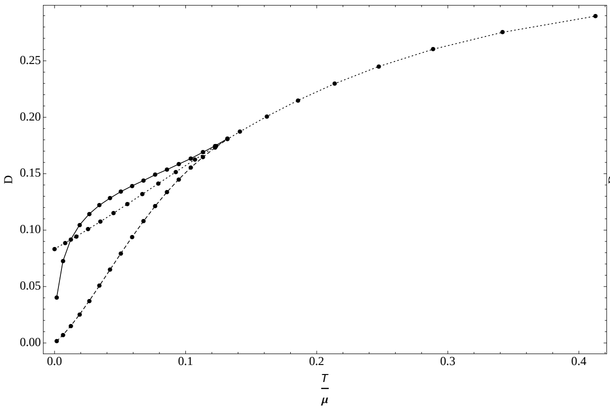


(b) $z = 2$; $\hat{m} = 0.36$

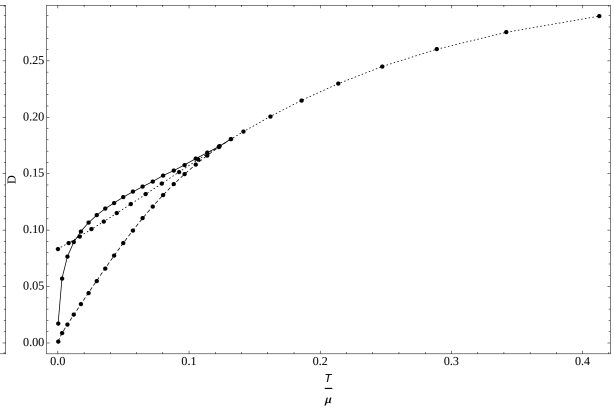


(c) $z = 2$; $\hat{m} = 0.50$

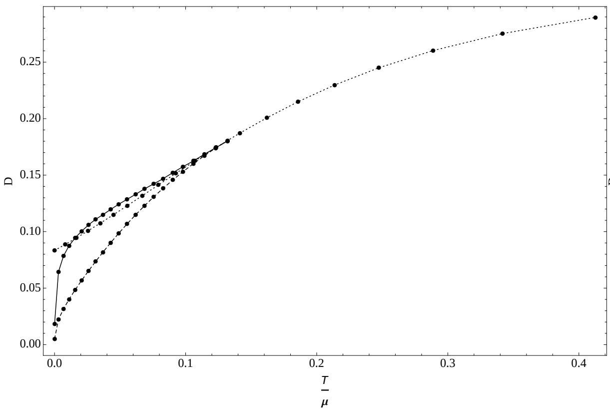
Figure 12: $\mathcal{D}(T)$ for $z = 2$ and $\hat{m} \in \{0.1, 0.36, 0.5\}$.



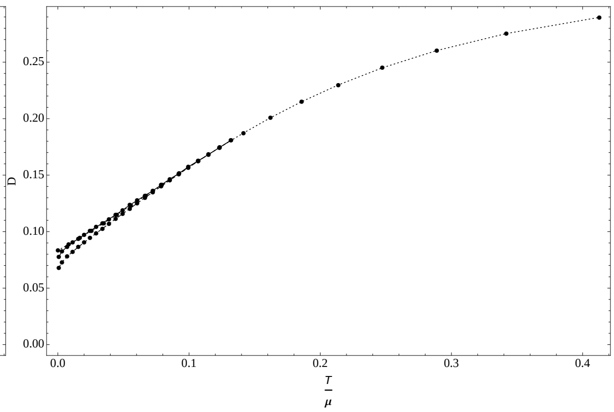
(a) $z = 3$; $\hat{m} = 0.36$



(b) $z = 5$; $\hat{m} = 0.36$



(c) $z = 10$; $\hat{m} = 0.36$



(d) $z = 100$; $\hat{m} = 0.36$

Figure 13: $D(T)$ for $\hat{m} = 0.36$ and $z \in \{3, 5, 10, 100\}$.

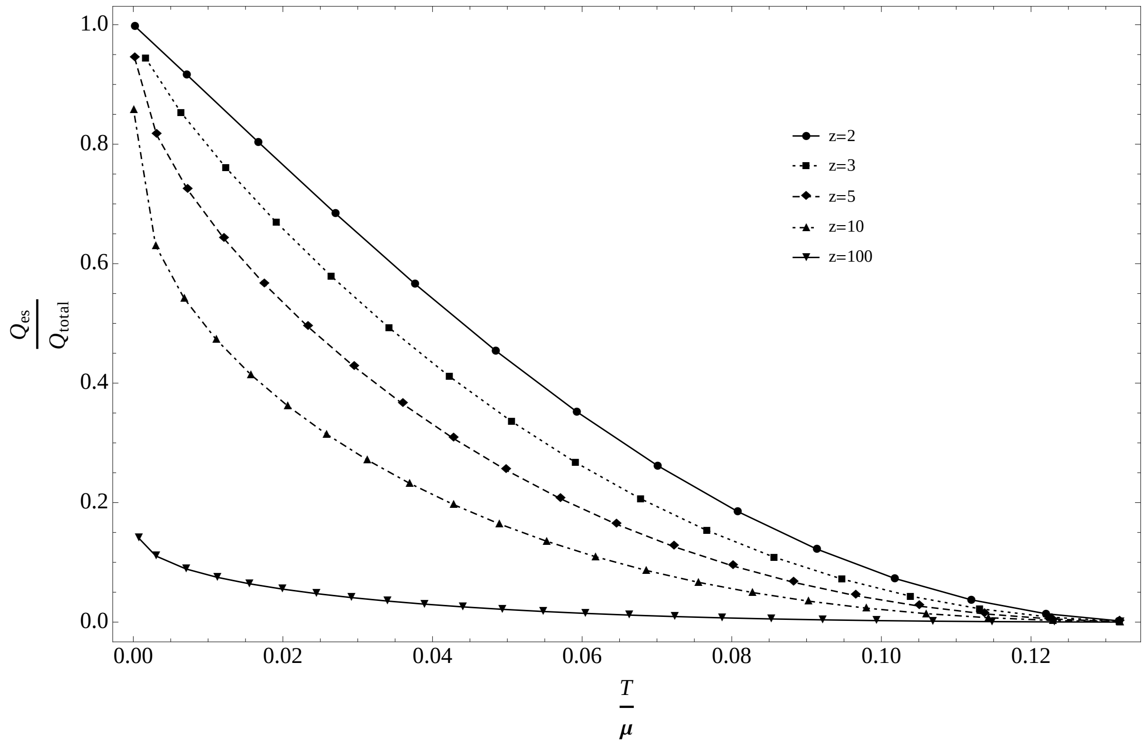


Figure 14: Fraction of Electron Star charge vs. z and $\frac{T}{\mu}$.

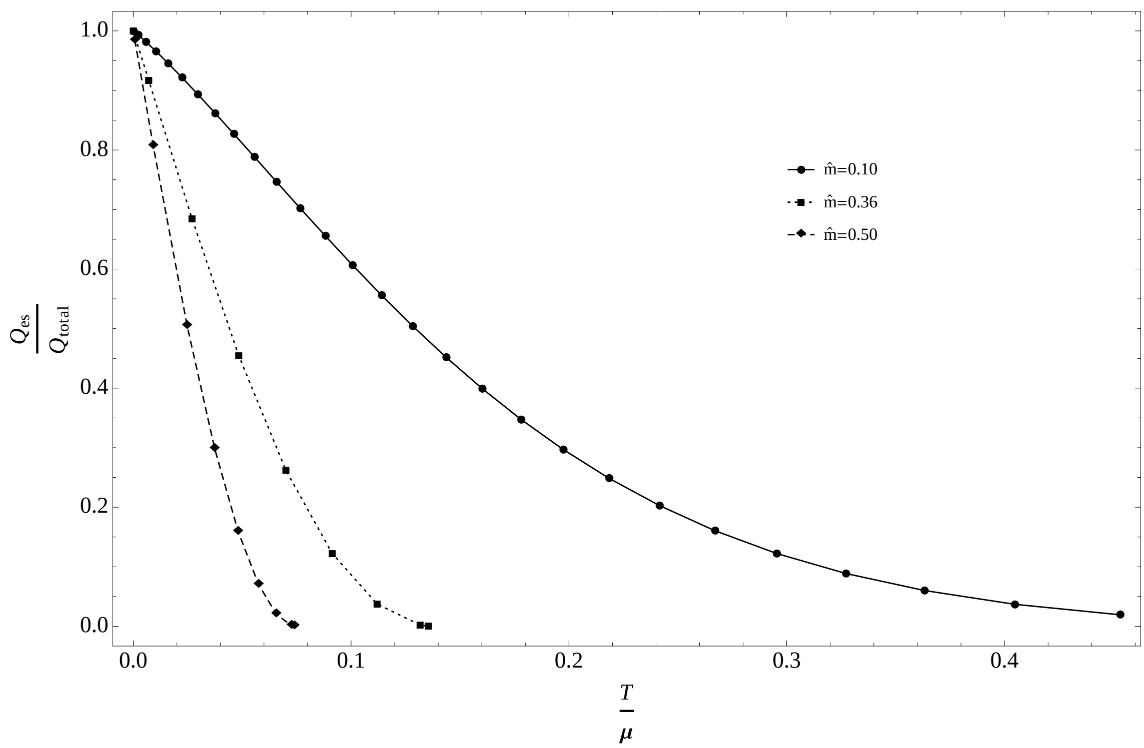


Figure 15: Fraction of Electron Star charge vs. $\hat{m}$ and $\frac{T}{\mu}$.

A The on-shell action for the shear modes

The on-shell action is found by perturbing the bulk action eq. (3.1) to second order in the fluctuations of the fields and imposing the equations of motion. We find

$$
\begin{aligned}
S^{(2)}_{on-shell} = \int_{r\to 0} \frac{d\omega dk}{(2\pi)^2}\frac{L^2}{\kappa^2} \Big[& \frac{1}{4r^4\sqrt{fg}}{h_t}^x{h_t}^{x\prime} + \frac{i\omega\sqrt{g}}{4r^2\sqrt{f}}({h_t}^x{h_x}^r - {h_x}^x{h_t}^r - {h_y}^y{h_t}^r) \\
& + \frac{\sqrt{f}}{8r^2\sqrt{g}}({h_t}^t{h_x}^{x\prime} + {h_x}^x{h_t}^{t\prime} + {h_y}^y{h_t}^{t\prime} + {h_t}^t{h_y}^{y\prime} + {h_x}^x{h_y}^{y\prime} + {h_y}^y{h_x}^{x\prime}) \\
& - \frac{ik\sqrt{fg}}{4}({h_t}^t{h_x}^r + {h_y}^y{h_x}^r) - \frac{ik\sqrt{g}}{4r^2\sqrt{f}}{h_t}^x{h_t}^r - \frac{1}{r^5\sqrt{fg}}{h_t}^{x2} \\
& + \frac{(rf'-2f)}{16r^3\sqrt{fg}}({h_t}^t{h_x}^x + {h_t}^t{h_y}^y + {h_x}^x{h_y}^y + {h_y}^y{h_x}^x - {h_r}^r{h_x}^x - {h_r}^r{h_y}^y - {h_x}^{22} - {h_y}^{y2}) \\
& \frac{\sqrt{f}}{4r^3\sqrt{g}}({h_t}^{t2} + {h_t}^t{h_r}^r - {h_t}^t{h_x}^x - {h_t}^t{h_y}^y) + \frac{1}{2r^2\sqrt{fg}}a_t(a_t' + i\omega a_r) \\
& - \frac{\sqrt{f}}{2\sqrt{g}}a_x(a_x' - ika_r) - \frac{h'}{4r^2\sqrt{fg}}a_t({h_r}^r + {h_t}^t - {h_x}^x - {h_y}^y) - \frac{h'}{2r^2\sqrt{fg}}a_x{h_T}^x \\
& - \frac{\sqrt{g}\sigma}{2r^2}{h_t}^r\delta\phi + \frac{\sqrt{f}\sigma}{2r^2\sqrt{g}}\delta u_r\phi \\
& - \frac{\sqrt{f}}{4r^2\sqrt{g}}{h_y}^x{h_y}^{x\prime} + \frac{f^{3/2}}{4\sqrt{g}}{h_y}^t{h_y}^{t\prime} - \frac{i\omega\sqrt{fg}}{4}{h_y}^t{h_y}^r + \frac{ik\sqrt{fg}}{4}{h_y}^x{h_y}^r - \frac{\sqrt{f}}{2\sqrt{g}}a_y a_y' \\
& + \frac{\sqrt{f}(rf'-2f)}{4r\sqrt{g}}{h_y}^{t2} - \frac{(rf'-2f)}{4r^3\sqrt{fg}}{h_x}^{x2} + \frac{\sqrt{f}h'}{2\sqrt{g}}{h_y}^t a_y \,\Big] \\
& + counterterms,
\end{aligned}
\tag{A.1}
$$

where in the quadratic products of fluctuation one has argument $(r, -\omega, -k)$ and the second has (r, ω, k). A prime denotes a derivative with respect to r.

We now wish to write the action in terms of the gauge-invariant variables (3.15). It is not possible to write the full on-shell action in terms of these variables. This does not mean that any of the bulk gauge symmetries are broken but simply reflects the fact that the variables (3.15) are valid to linear order in fluctuations, whereas the action is quadratic in fluctuations. However, the derivative terms in the on-shell action *can* be written purely in terms of the gauge-invariant variables. This ensures that the relevant Ward identities of the field theory are satisfied. The on-shell action then takes the form

$$
S^{(2)}_{on-shell} = \int_{r\to 0} \frac{d\omega dk}{(2\pi)^2}\frac{L^2}{\kappa^2}\left[Z_i(r,-\omega,-k)\mathcal{A}_{ij}Z_j'(r,\omega,k)\right] + \text{non-derivative terms}. \tag{A.2}
$$

The coefficients are now much simpler. They equal

$$
\mathcal{A}_{11} = \frac{\sqrt{f}}{4r^2\sqrt{g}(\omega^2 - k^2r^2f)}, \tag{A.3}
$$

$$
\mathcal{A}_{22} = -\frac{\sqrt{f}}{2\sqrt{g}}, \tag{A.4}
$$

$$
\mathcal{A}_{12} = \mathcal{A}_{21} = 0. \tag{A.5}
$$

B The off-shell action for the shear modes

In this appendix we present the off-shell action computed from the original action in eq. (3.1). As discussed in appendix A, the quadratic action cannot be written fully in terms of variables (3.15). This problem can be circumvented by adding purely real boundary counter-terms, different from the holographic renormalisation counter-terms, such that all the single-derivative terms may be written in terms of Z_1 and Z_2. These do not affect the equations of motion of the theory and hence do not change the poles of the Green's functions. Because the extra terms are real, they also leave the spectral function invariant. They only change the contact terms in the real part of the Green's functions. The off-shell action then takes the form

$$\begin{aligned} S^{(2)}_{off-shel} = \int dr \frac{d\omega dl}{(2\pi)^2}\frac{L^2}{\kappa^2}\,\big[Z_i'(r,-\omega,-k)A_{ij}Z_j'(r,\omega,k) + Z_i(r,-\omega,-k)B_{ij}Z_j'(r,\omega,k) \\ +\text{non-derivative terms}\big]\,. \end{aligned} \tag{B.1}$$

The coefficients are

$$A_{11} = \frac{\sqrt{f}}{4r^2\sqrt{g}(\omega^2 - k^2r^2f)}, \qquad A_{12} = A_{21} = 0, \qquad A_{22} = -\frac{\sqrt{f}}{2\sqrt{g}}, \tag{B.2}$$

and

$$B_{11} = \frac{rf' - 2f}{2\omega^2 r^3\sqrt{fg}}, \qquad B_{12} = 0, \qquad B_{21} = -\frac{k(rf' + 2f)}{2r\mu\sqrt{g}(\omega^2 - k^2r^2d)}, \qquad B_{22} = 0. \tag{B.3}$$

The action eq. (B.1) has an associated Noether current which corresponds to a global symmetry of the form

$$Z(r,\omega,k) \to e^{i\alpha}Z(r,\omega,k), \qquad Z(r,-\omega,-k) \to e^{-i\alpha}Z(r,-\omega,-k). \tag{B.4}$$

This results in the existence of a quantity which is invariant under translations in the radial direction and this invariance can be used as a check on the numerical results obtained.

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