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Proving the Goldbach Conjecture: Algebraic Proofs and Predicting Prime Numbers

By Oussama Basta

***Abstract*—The Goldbach's Conjecture is an astonishing proposition that stands as one of the most renowned and enduring unsolved problems in number theory and mathematics. This research aims to provide a proof for this remarkable conjecture. The approach to be followed for the proof is yielded by using a predefined system of equations, and with a relatively simple analysis. The proof is quite simple compared to the size of the problem.**

In the second part of this study, we leverage the same system of equations to develop a general mathematical framework for predicting prime numbers within the known sequence, laying down a general mathematical framework that is computationally concise and can just achieve the objective. With proper selection of the coefficients of the equations in the algorithm, it's guaranteed that prime numbers are among the outputs. The algorithm consists of basic arithmetic operations which is by itself ground breaking. The proof of the algorithm is also astoundingly straightforward and compact.

***Index Terms*— Goldbach's Conjecture, Number theory, Prime numbers, Prediction**

I. THE PROOF OF QUATERNARY, TERNARY, AND BINARY GOLDBACH'S CONJECTURES

Definitions; Assuming ζ is an even number greater than 4 and P_c is a prime number greater than 2, we define our system of equations as follows:

$$F_1 = \zeta - P_c \tag{1}$$
$$F_2 = \zeta + P_c \tag{2}$$

A. Proving The Quaternary Goldbach's Conjecture (QGC)

While as stated in *Theorem 1* from [1]:*"Every even integer is a sum of at most 6 primes"* In this section, we will prove a variant of *Theorem 1* referred to as the Quaternary Goldbach's Conjecture (QGC). The QGC in here is defined as follows: *"Every even integer greater than 6 can be expressed as the sum of 4 primes."*

From to [2] it is stated that: "*An arbitrary natural number, with the exception of 1, can, for sufficiently large k, be represented as the sum of at most k primes*." The Schnirelmann constant, denoted as k, is the resultant of this statement. The smallest confirmed value of k is known as the Schnirelmann constant and in [3] it is k≤6. Additionally, we have, that every odd number greater than 1 is the sum of at most five primes [4].

With all that said, and by writing ζ, F_1 and F_2 under the light of these definitions: we have from equations (1) and (2) that:

$$\zeta = P_{\zeta 1} + P_{\zeta 2} + P_{\zeta 3} + P_{\zeta 4} + P_{\zeta 5} + P_{\zeta 6} \tag{3}$$

According to k≤6 Because it's even, then:

$$F_1 = P_{\zeta 1} + P_{\zeta 2} + P_{\zeta 3} + P_{\zeta 4} + P_{\zeta 5} + P_{\zeta 6} - P_c \tag{4}$$

$$F_2 = P_{\zeta 1} + P_{\zeta 2} + P_{\zeta 3} + P_{\zeta 4} + P_{\zeta 5} + P_{\zeta 6} + P_c \tag{5}$$

LHS of equation (5) can be redefined as the sum of five prime numbers since F_2 is odd. In this case, we have:

$$P_{F2,1}+P_{F2,2}+P_{F2,3}+P_{F2,4}+P_{F2,5}=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4}+P_{\zeta 5}+P_{\zeta 6}+P_c \qquad (6)$$

And P_c can always be chosen, so it can be that $P_c=P_{F2,1}$ or $P_{F2,2}$ or $P_{F2,3}$ or $P_{F2,4}$ or $P_{F2,5}$ in any expansion of 5 prime numbers of F_2. As defined P_c is a prime number and F_2 is any odd number ≥11. So, if for example $P_c=P_{F2,5}$ *is* chosen and taking it to the LHS, then:

$$P_{F2,1}+P_{F2,2}+P_{F2,3}+P_{F2,4}=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4}+P_{\zeta 5}+P_{\zeta 6} \qquad (7)$$

From the LHS of (7) we can see that the QGC is valid for even numbers greater than 10, as the smallest prime number is two and plugging that we get the RHS equals 12. The remaining cases for (7) are $LHS=8$ and $LHS=10$ so we prove them by plugging prime numbers. So, 8 =2+2+2+2 and 10= 2+2+3+3. Then we have it that QGC holds and k is updated to k≤5.

B. Proving the Ternary Goldbach's Conjecture (TGC)

The famous TGC that is being proven here is defined as:
"Every odd integer greater than 5 is the sum of three primes."
And according to QGC proven earlier,

$$\zeta=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4} \qquad (8)$$

Because it's even, if one of $P_{\zeta 1}$, $P_{\zeta 2}$, $P_{\zeta 3}$ or $P_{\zeta 4}$ is chosen to be equal to P_c, in which P_c is defined as a prime number and F_1 in this case is any odd number ≥11. In this case and following the same logic of proving QGC:

$$F_1=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4}-P_c \qquad (9)$$
$$F_2=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4}+P_c \qquad (10)$$

Let's establish that $P_c=P_{\zeta 1}$ *or* $P_{\zeta 2}$ *or* $P_{\zeta 3}$ *or* $P_{\zeta 4}$ in any expansion of 4 prime numbers of ζ so the work is done on the RHS this time, where:

$$P_{F1,1}+P_{F1,2}+P_{F1,3}+P_{F1,4}+P_{F1,5}=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4}-P_c \qquad (11)$$

For k≤5 since F_1 is odd, so if $P_c=P_{\zeta 4}$ is chosen then,

$$P_{F1,1}+P_{F1,2}+P_{F1,3}+P_{F1,4}+P_{F1,5}=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3} \qquad (12)$$

And since $F_1\geq 11$ the remaining cases of $F_2=7$=2+2+3 and $F_2=9=3+3+3$. Then TGC holds for this case. And Schnirelman's constant is indeed k≤4.

C. Proving the Binary Goldbach's Conjecture (BGC)

Let,

$$\zeta=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4} \qquad (13)$$

Because It's now established that Schnirelman's constant is k≤4, in this case, we have:

$$F_1=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4}-P_c \qquad (14)$$
$$F_2=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4}+P_c \qquad (15)$$

Again since $P_c=$ is a prime number and F_2 is any odd integer >5. P_c can be always chosen, so, $P_c=P_{F2,1}$ or $P_{F2,2}$ or $P_{F2,3}$ in any expansion of 3 prime numbers of F_2, which are the TGC primes of F_2; This time we will use the LHS of equation (15):

$$F_2=\quad P_{F2,1}+P_{F2,2}+P_{F2,3} \qquad (16)$$

So,

$$P_{F2,1}+P_{F2,2}+P_{F2,3}=P_{\zeta 1}+P_{\zeta 2}+P_{\zeta 3}+P_{\zeta 4}+Pc \qquad (17)$$

$P_{F2,1}+ P_{F2,2}+ P_{F2,3}- Pc= P_{\zeta 1}+ P_{\zeta 2}+ P_{\zeta 3}+ P_{\zeta 4}$ (18)

So if we choose $P_c = P_{F2,3}$,

$P_{F2,1}+ P_{F2,2}= P_{\zeta 1}+ P_{\zeta 2}+ P_{\zeta 3}+ P_{\zeta 4}$ (19)

And since ζ is any even number sufficiently large>6, for the case of F_2 =6, 6=3+3 and for F_2 =4, 4=2+2. Then BGC holds for this case and k≤3 now. And Finally The Binary Goldbach's Conjecture is proven.

II. THE ALGORITHM

Define the following three equations:

$$P_c = |\alpha_1 P_1 - \beta_1 P_{i+1} - \gamma_1| \quad (20)$$
$$F_1 = |\alpha_2 P_i + \beta_2 P_{i+1} + \gamma_2 - P_c| = \zeta - P_c \quad (21)$$
$$F_2 = \alpha_2 P_1 + \beta_2 P_{i+1} + \gamma_2 + P_c = \zeta + P_c \quad (22)$$

Where P_i is a prime number with index i from the known sequence of prime numbers and the coefficients $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2$, and γ_2 are natural numbers.

After defining those equations, the following algorithm shall be used:

1. Get two consecutive prime numbers.
2. Sieve through the coefficients.
3. If equation (20) outputs a prime number and $\zeta \gg P_c$, then one of the equations (21) and (22) must yield a prime number.

III. PROOF OF THE ALGORITHM

Define:

$$F_1 = \zeta + P_c \quad (23)$$
$$F_2 = \zeta - P_c \quad (24)$$

Where,

$$P_c = \alpha_1 P_i + \beta_1 P_{i+1} + \gamma_1 \quad (25)$$
$$\zeta = \alpha_2 P_i + \beta_2 P_{i+1} + \gamma_2 \quad (26)$$

Where $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2$, and γ_2 are natural numbers.
With simple adding, subtracting and multiplying F_1 and F_2 from equations (23) and (24) we have these three equations,

$$F_2 - F_1 = 2P_c \quad (27)$$
$$F_2 + F_1 = 2\zeta \quad (28)$$
$$\frac{\zeta^2 - F_1 F_2}{{P_c}^2} = 1 \quad (29)$$

First of all equation (29) shows that $\zeta \gg P_c$.Now, there are three cases, (A) both F_1 and F_2 are not prime numbers, (B) either one of them is a prime number, or (C) both are prime numbers.
For the first case (A) where both are not prime
By TGC

$$F_1 F_2 = (P_1 + P_2 + P_3)(P_4 + P_5 + P_6) = P_1 P_4 + P_1 P_5 + P_1 P_6 + P_2 P_4 + P_2 P_5 + P_2 P_6 + P_3 P_4 + P_3 P_5 + P_3 P_6 \quad (30)$$

$$F_1 F_2 = (\zeta + Pc)(\zeta - Pc) = \zeta^2 - {P_c}^2 \quad (31)$$
$$= \zeta^2 - (\alpha_1 P_i + \beta_1 P_{i+1} + \gamma_1)^2 \ (substituting\ Pc\ from\ equation\ (20))$$
$$= (\alpha_2 P_i + \beta_2 P_{i+1} + \gamma_2)^2 - (\alpha_1 P_i + \beta_1 P_{i+1} + \gamma_1)^2 (substituting\ \zeta\ from\ equation\ (21)\ or\ (22))$$
$$= \alpha_2^2 P_i^2 + \beta_2^2 P_{i+1}^2 + \gamma_2^2 + 2\alpha_2\beta_2 P_i P_{i+1} + 2\alpha_2 \gamma_2 P_i + 2\beta_2\gamma_2 P_{i+1}$$
$$-(\alpha_1^2 P_i^2 - \beta_1^2 P_{i+1}^2 - \gamma_1^2 - 2\alpha_1 \beta_1 P_i P_{i+1} - 2\alpha_1\gamma_1 P_i - 2\beta_1\gamma_1 P_{i+1})$$

$$F_1 F_2 = (\alpha_2^2 - \alpha_1^2)P_i^2 + (\beta_2^2 - \beta_1^2)P_{i+1}^2 + 2(\alpha_2\beta_2 - \alpha_1\beta_1)P_i P_{i+1}$$
$$+2(\alpha_2\gamma_2 - \alpha_1\gamma_1)P_i + 2(\beta_2\gamma_2 - \beta_1\gamma_1)P_{i+1} + (\gamma_2^2 - \gamma_1^2) \quad (32)$$

It can be seen that by comparing equations (30) and (32) that there is no integral solution to the coefficients which we already defined as integral. To show that, we equate equations 25 and 27 and express them in matrix form as AX=B,

$$\begin{bmatrix} (\alpha_2^2-\alpha_1^2) & 2(\alpha_2\beta_2-\alpha_1\beta_1) & 2(\alpha_2\gamma_2-\alpha_1\gamma_1) \\ 2(\alpha_2\beta_2-\alpha_1\beta_1) & (\beta_2^2-\beta_1^2) & 2(\beta_2\gamma_2-\beta_1\gamma_1) \\ 2(\alpha_2\gamma_2-\alpha_1\gamma_1) & 2(\beta_2\gamma_2-\beta_1\gamma_1) & (\gamma_2^2-\gamma_1^2) \end{bmatrix} \begin{bmatrix} P_i \\ P_{i+1} \\ 1 \end{bmatrix} = \begin{bmatrix} P_1P_4 & P_1P_5 & P_1P_6 \\ P_2P_4 & P_2P_5 & P_2P_6 \\ P_3P_4 & P_3P_5 & P_3P_6 \end{bmatrix} \quad (33)$$

It can be noticed that the Matrix A is symmetrical, hence, the rows of the matrix A are linearly dependent, and the determinant of A is zero the there are no integral coefficients, one of F_1 and F_2 or both are prime numbers.

IV. A PRIMALITY TEST BASED ON THE ALGORITHM

The steps for creating a primality test algorithm using a predetermined matrix of coefficients; which are adjusted to identify whether either F_1, F_2, or both are prime numbers. Can be summarized as follows, based on the provided previously provided proof:

1. The first step is to create the matrix A. This matrix contains the coefficients found by iteration for the two equations (20), (21) and (22).
2. The second step is to create the vector B. This vector contains the values of N, N + 1 or any small even number, and 1.
3. The third step is to calculate the dot product of A and B. The dot product is a common operation in linear algebra, and it can be calculated in any programming language.
4. The fourth step is to check if either element of C is equal to 1. If either element of C is equal to 1, then N is prime. Otherwise, N is not prime.

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