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Solution to Infinity Problem Based on Scattering Matrix Using Time-Evolution Operators Without Needing Renormalization

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Abstract The purpose of our work is to achieve a new formulation which always ensures the convergence of the scattering matrix in such a way of preventing overlapping divergences of the scattering matrix in principle. We present a new nonperturbative representation of the scattering matrix in terms of so-called global time-evolution operator that is based on the improved Heisenberg picture. Our study demonstrates that there does not exist the infinity problem within the framework of our formulation that employs the global time-evolution operator and importantly the formulated theory satisfies all requirements of scattering. This interesting result is obtained successfully at the level of both quantum mechanics and quantum field theory. Ultimately, we draw the successful conclusion that it is possible to formulate a new scattering theory irrelevant to the infinity problem.

1 Introduction

The core of quantum scattering theory is the Dyson series and the Feynman diagram method. Dyson's formula for the scattering matrix (S-matrix) and Feynman's diagram rule had promoted quantum scattering theory to an elaborated theory [1,2,3]. However, we usually encounter formidable divergence problems when calculating the scattering matrix based on the Feynman diagram. For this reason, our main concern is to construct a new formulation which is consistent and perfect.

Renormalization acknowledged as an astounding mathematical trick enables one to overcome some overlapping divergences due to the Feynman diagrams. This sophisticated formulation provides an approach to calibrating fundamental physical quantities so that computational results of the scattering matrix could coincide with experiments. John Ward's approach, the Yang-Mills method and Salam's studies on the problem of overlapping divergences contributed to the early development of renormalization theory [4,5,6]. Subsequently, Stueckelberg, Green, Bogoliubov and Parasiuk's contribution developed renormalization theory into a systematized theory with more solid foundation [7,8]. On the other hand, Wolfhart Zimmermann, Bogoliubov and others' iterative method was finalized as Bogoliubov, Parasiuk, Hepp and Zimmermann's (BPHZ) method [9,10].

The recent researches show the ramifications of renormalization technique that covers a wide range of the studies: renormalization group flow [11,12,13], renormalization group function and equation [14,15,16,17], renormalized perturbation theory [18], renormalization theory on the perturbative Feynman graph expansion [19], and on the other hand, connections between these different techniques of renormalization [20]. The coverage of renormalization continues to extend inasmuch as it should satisfy the Lorentz and gauge symmetry and the requirement for cosmological spacetime as well [21,22,23,24].

Renormalization theory of dominant status has enjoyed so much successes in dealing with overlapping divergences of the scattering matrix. Nevertheless, the heart of renormalization theory still has not been completed and the center of research continues to shift [25,26]. The facts, in a sense, indicates that it is necessary to explore for a new sphere of scattering theory without infinity in parallel with the development of renormalization [27,29,28]. The best way is to find out a general method without the infinity problem available to all cases of calculations of the scattering matrix. In this regard, it is remarkable that there are attempts to con-

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struct new formulations of scattering theory without infinity [27,29].

In our view, one of the key questions of quantum field theory is whether renormalization theory is able to reach the ultimate goal to resolve the divergence problem of scattering matrix in a general way within the present theoretical framework. Such an opinion seems paradoxical and challenging but the present situation of research showing Odyssei in renormalization naturally causes it to burgeon. With this understanding, we present the theory on the scattering matrix based on the consistent time-evolution operator called the global time-evolution operator which, in essence, builds the improved Heisenberg picture. This operator is immune to the problem of overlapping divergence, thus not needing renormalization.

2 Consistent time-evolution operator and convergence of scattering matrix

2.1 Consistent time-evolution operator

The Heisenberg picture is an important mathematical formulation for investigating the time evolution of quantum states together with the Schrödinger picture. In particular, the Heisenberg picture plays a key role in the case of investigating the scattering problem. This is because the scattering operator essentially should be the time-evolution operator in the Heisenberg picture if the Heisenberg picture possesses generality.

Let us consider why and how to improve the Heisenberg picture. The state function Φ_S in the Schrödinger picture is determined by the Schrödinger equation:

$$\mathrm{i}\hbar\frac{\partial \Phi_S}{\partial t} = \hat{H}\Phi_S. \tag{1}$$

The formal solution to this equation is considered to be

$$\Phi_S = e^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}\Phi_H, \tag{2}$$

where Φ_H as a time-independent function is defined as the wave function in the Heisenberg picture. Here, subscript H refers to the Heisenberg picture. Operator $\hat{S} = e^{-\frac{\mathrm{i}}{\hbar}\hat{H}t}$ is considered to be the unitary operator making transformation from the Heisenberg pictures to the Schrödinger picture.

Let us examine whether the Heisenberg picture is general. If $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})$ does not satisfy the time-dependent Schrödinger equation, then this is sufficient to confirm that the Heisenberg picture is not general.

Substituting $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})$ into the Schrödinger equation, we get

$$\begin{aligned}\mathrm{i}\hbar\frac{\partial}{\partial t}\left[\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})\right] &= \left[\hat{H} + t\frac{\partial \hat{H}}{\partial t}\right]\Phi_S(\mathbf{q},t)\\ &\neq \hat{H}\Phi_S(\mathbf{q},t).\end{aligned} \tag{3}$$

Obviously, the formal solution of the Schrödinger equation, $\exp\left(-\frac{\mathrm{i}}{\hbar}\hat{H}t\right)\Phi_0(\mathbf{q})$ does not satisfy the wave equation in case the Hamiltonian is time dependent. Therefore, it is a matter of course to obtain the generalized formal solution of the Schrödinger equation in a rigorous way.

We aim to obtain a consistent time-evolution operator which satisfies all requirements for the scattering operator.

These requirements are as follows.

- The scattering operator should give the finite solution of the Schrödinger equation.
- It should satisfy the causality condition of time evolution.

From now on, we derive a consistent time-evolution operator alternative to the Dyson series in three ways.

As the first approach, we can derive a scattering operator in such a way of obtaining a formal solution of the Schrödinger equation. Let the wave function be set as

$$\Phi_S(\mathbf{q},t) = \hat{f}(\mathbf{q},t)\varphi(\mathbf{q}), \tag{4}$$

where $\hat{f}(\mathbf{q},t)$ is an operator dependent on time and position, and $\varphi(\mathbf{q})$ an arbitrary function. Inserting Eq. (4) into Eq. (1) yields

$$\mathrm{i}\hbar\frac{\partial\left[\hat{f}(\mathbf{q},t)\varphi(\mathbf{q})\right]}{\partial t} = \hat{H}(\mathbf{q},t)\left[\hat{f}(\mathbf{q},t)\varphi(\mathbf{q})\right]. \tag{5}$$

If there exists $\hat{f}(\mathbf{q},t)$ satisfying Eq. (5), then the solution of the Schrödinger equation, $\Phi_S(\mathbf{q},t)$ is to be determined. Since operator $\mathrm{i}\hbar\frac{\partial}{\partial t}$ is applied only to $\hat{f}(\mathbf{q},t)$, we have

$$\mathrm{i}\hbar\left(\frac{\partial \hat{f}(\mathbf{q},t)}{\partial t}\right)\varphi(\mathbf{q}) = \hat{H}(\mathbf{q},t)\hat{f}(\mathbf{q},t)\varphi(\mathbf{q}). \tag{6}$$

For Eq. (6) to hold for arbitrary $\varphi(\mathbf{q})$, the operator equation:

$$\mathrm{i}\hbar\frac{\partial \hat{f}(\mathbf{q},t)}{\partial t} = \hat{H}(\mathbf{q},t)\hat{f}(\mathbf{q},t) \tag{7}$$

should be an identical relation. On the other hand, the identical relation (7) is a sufficient condition for Eq. (6) to hold. Therefore, Eq. (7) is a necessary and sufficient condition for Eq. (6) to hold for arbitrary $\varphi(\mathbf{q})$. To obtain operator $\hat{f}(\mathbf{q},t)$ in the form of an algebraic expression, we assume it to be able to be operated algebraically. Thus, through the following algebraic operations beginning with Eq. (7):

$$\mathrm{i}\hbar\frac{\partial \hat{f}(\mathbf{q},t)}{\partial t}\cdot\frac{1}{\hat{f}(\mathbf{q},t)} = \hat{H}(\mathbf{q},t),$$

$$i\hbar\frac{\partial \ln \hat{f}(\mathbf{q},t)}{\partial t} = \hat{H}(\mathbf{q},t),$$

we immediately obtain

$$\hat{f}(\mathbf{q},t) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\hat{f}(\mathbf{q},t_0). \tag{8}$$

Eventually, the solution of the Schrödinger equation is represented as

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\hat{f}(\mathbf{q},t_0)\varphi(\mathbf{q}). \tag{9}$$

Clearly, for the initial condition $t = t_0$, we have

$$\Phi_S(\mathbf{q},t_0) = \hat{f}(\mathbf{q},t_0)\varphi(\mathbf{q}).$$

Accordingly, we write the exact formal solution as

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\Phi_S(\mathbf{q},t_0). \tag{10}$$

By taking the time-independent function:

$$\Phi_H(\mathbf{q}) = \Phi_S(\mathbf{q},t_0), \tag{11}$$

we may also write Eq. (10) as

$$\Phi_S(\mathbf{q},t) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\Phi_H(\mathbf{q}). \tag{12}$$

Finally, we adopt the time-evolution operator as

$$U(t,t_0) = \exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]. \tag{13}$$

We shall refer to $U(t,t_0)$ as the global time-evolution operator, since it gives an analytical representation of time evolution from an initial time to a final time, i.e., on the whole interval of time, $[t_0,t]$. For the scattering problem, the global time-evolution operator becomes the scattering operator. Thus, we write as

$$\Phi_S(\mathbf{q},t) = U(t,t_0)\Phi_S(\mathbf{q},t_0) = S(t,t_0)\Phi_S(\mathbf{q},t_0). \tag{14}$$

Evidently, Eq. (13) becomes the generalized time-evolution operator which comprises the Heisenberg picture as a special case. Thus, we confirm the improvement of the Heisenberg picture.

This operator is represented by the Maclaurin expansion as

$$S(t,t_0) = \sum_{n=0}^{\infty}\frac{1}{n!}\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'\right]^n. \tag{15}$$

The fact that the formal solution, Eq. 12 is exact is verified easily. Let us consider whether Eq. 12 satisfies the Schrödinger equation. Setting $\hat{x}(t) = -\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'$ and using formally the chain rule, $\frac{\partial}{\partial t} = \frac{\partial \hat{x}(t)}{\partial t}\frac{\partial}{\partial \hat{x}(t)}$, we calculate

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Phi_S(\mathbf{q},t) &= i\hbar\frac{\partial \hat{x}(t)}{\partial t}\frac{\partial}{\partial \hat{x}(t)}\Phi_S(\mathbf{q},t) \\ &= \hat{H}(\mathbf{q},t)\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]\Phi_H(\mathbf{q}) \\ &= \hat{H}(\mathbf{q},t)\Phi_S(\mathbf{q},t). \end{aligned}$$

Thus, we confirm

$$i\hbar\frac{\partial}{\partial t}\Phi_S(\mathbf{q},t) = \hat{H}(\mathbf{q},t)\Phi_S(\mathbf{q},t).$$

This is nothing but the Schrödinger equation. Therefore, we verify that the global time-evolution operator is exact. In the end, it is concluded that the generalized time-evolution operator should be $\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(\mathbf{q},t')dt'\right]$ instead of $\exp\left(-\frac{i}{\hbar}\hat{H}t\right)$.

As the second approach, we can obtain the scattering operator, Eq. (13) by using the method of power series expansion, . Suppose that the scattering operator is represented as a function of an unknown operator $\hat{x}(t)$, i.e.,

$$S(t,t_0) = S(\hat{x}(t,t_0)). \tag{16}$$

Then the scattering operator can be expanded into the Maclaurin series:

$$\begin{aligned} S(\hat{x}(t,t_0)) = S(0) + S'(0)\hat{x}(t,t_0) + \frac{1}{2!}S''(0)[\hat{x}(t,t_0)]^2 + \cdots \\ + \frac{1}{n!}S^{(n)}(0)[\hat{x}(t,t_0)]^n + \cdots. \end{aligned} \tag{17}$$

To determine Eq. (17), it is necessary to consider the solution of the Schrödinger equation in the first-order approximation:

$$|\Phi(t)\rangle \approx |\Phi(0)\rangle - \frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'|\Phi(0)\rangle. \tag{18}$$

Comparing Eq. (17) and Eq. (18), we immediately identify

$$S(0) = 1,\ \ S'(0) = 1,\ \ \hat{x}(t,t_0) = -\frac{i}{\hbar}\int_{t_0}^{t}\hat{H}(t')dt'.$$

Combining $S(0) = 1$ and $S'(0) = 1$, we can take

$$\left.\frac{\partial S(\hat{x}(t,t_0))}{\partial \hat{x}(t,t_0)}\frac{1}{S(\hat{x}(t,t_0))}\right|_{\hat{x}(t,t_0)=0} = 1. \tag{19}$$

This is supposed to be the equation for finding the form of the scattering operator $S(t, t_0)$. Eq. (19) is recast as

$$\left.\frac{\partial \ln S(\hat{x}(t, t_0))}{\partial \hat{x}(t, t_0)}\right|_{\hat{x}(t,t_0)=0} = 1. \tag{20}$$

Obviously, the sufficient condition for Eq. (20) to hold is

$$\frac{\partial \ln S(\hat{x}(t, t_0))}{\partial \hat{x}(t, t_0)} = 1. \tag{21}$$

Hence, we obtain

$$\begin{aligned} S(t, t_0) &= S(\hat{x}(t, t_0)) = \exp[\hat{x}(t, t_0)] \\ &= \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right]. \end{aligned} \tag{22}$$

Here, for the purpose of obtaining the scattering operator in the form of an algebraic formula, we dealt with $\hat{x}(t, t_0)$ like an algebraic quantity. In fact, if an obtained result needs only the sums and products of operators, such an algebraic operation for operators is valid. From Eq. (22), it is obvious that $S(0) = 1,\ \ S'(0) = 1,\ \ S''(0) = 1 \cdots, S^{(n)}(0) = 1, \cdots$. As a consequence, the scattering operator, Eq. (13) has been derived in another way.

As the third approach, we can obtain the scattering operator, Eq. (13) based on the causality of scattering. From the physical point of view, the scattering operator should satisfy the causal product relation:

$$S(t_N, t_0) = S(t_N, t_{N-1}) \cdots S(t_2, t_1) S(t_1, t_0). \tag{23}$$

Evidently, Eq. (13) is valid, since it satisfies the key causality axiom of scattering. Eq. (23) can be viewed as a functional equation for finding $S(t, t_0)$. It is possible to obtain the scattering operator, based on the causality axiom. Eq. (23) tells us that for time interval, the rule of sum must be satisfied, while for the scattering operator, the rule of product must be satisfied.

Hence, it is obvious that the scattering operator has to take the form of $S(t, t_0) = \exp(\hat{x}(t, t_0))$. Here, the condition:

$$\hat{x}(t_N, t_0) = \hat{x}(t_N, t_{N-1}) + \cdots + \hat{x}(t_2, t_1) + \hat{x}(t_1, t_0). \tag{24}$$

should be satisfied. Substituting the formal solution $\Phi(t) = S(t, t_0)\Phi(t_0)$ into the Schrödinger equation, we get

$$\mathrm{i}\hbar \frac{\partial S(t, t_0)}{\partial t} \Phi(t_0) = \hat{H}(t) S(t, t_0) \Phi(t_0).$$

From this, we immediately determine

$$\hat{x}(t, t_0) = -\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'.$$

Obviously, this result satisfies Eq. (24). Thus, we arrive at

$$S(t, t_0) = \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right).$$

In the end, we again confirm that Eq. (13) is valid from the point of view of causality.

The above result derived from the causality relation of scattering may be obtained by using the successive approximation. If one partitions the time interval equally and makes staged time evolution in every time interval instead of applying the successive approximation for the whole time interval, then the desired convergence can be achieved. We begin with the formal solution of the Scrödinger equation:

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')\Phi(t')dt'. \tag{25}$$

To write

$$|\Phi(t_1)\rangle \approx |\Phi(t_0)\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1} \hat{H}(t')dt'|\Phi(t_0)\rangle, \tag{26}$$

t_1 should be in a close proximity of t_0. Actually, only for sufficiently short integration interval, Eq. (26) is meaningful. The violation of this condition may lead to the divergence of solution. Let us partition the interval, $[t_0, t]$ into n equal elementary subintervals defined by a set of points $\{t_0, t_1, t_2, \cdots t_n\}$ such that $t_0 < t_1 < t_2, \cdots < t_n = t$. From Eq. (25), we take as the first time-evolution

$$|\Phi(t_1)\rangle \approx \left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_0}^{t_1} \hat{H}(t')dt'\right]|\Phi(t_0)\rangle, \tag{27}$$

and as the second time-evolution

$$\begin{aligned} |\Phi(t_2)\rangle &\approx |\Phi(t_1)\rangle - \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2} \hat{H}(t')dt'|\Phi(t_1)\rangle \\ &= \left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_1}^{t_2} \hat{H}(t')dt'\right]|\Phi(t_1)\rangle \\ &\approx \prod_{k=1}^{2}\left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_{k-1}}^{t_k} \hat{H}(t')dt'\right]|\Phi(t_0)\rangle. \end{aligned} \tag{28}$$

Then, the state function, in general, is written as

$$|\Phi(t_n)\rangle \approx \prod_{k=1}^{n}\left[1 - \frac{\mathrm{i}}{\hbar}\int_{t_{k-1}}^{t_k} \hat{H}(t')dt'\right]|\Phi(t_0)\rangle. \tag{29}$$

Here, $\int_{t_{k-1}}^{t_k} \hat{H}(t')dt'$ is termed the local time-evolution operator in the sense that it is represented by integrals in short-time intervals. Thus, the time-evolved state function is represented, using the local time-evolution operator, as

$$|\Phi(t)\rangle = \lim_{n\to\infty} \prod_{k=1}^{n} \left[1 - \frac{\mathrm{i}}{\hbar} \int_{t_{k-1}}^{t_k} \hat{H}(t')dt' \right] |\Phi(t_0)\rangle. \tag{30}$$

This procedure of using the local time-evolution is self-consistent. Thus, the time-evolution operator is represented as

$$S(t, t_0) = \lim_{n\to\infty} \prod_{k=1}^{n} \left[1 - \frac{\mathrm{i}}{\hbar} \int_{t_{k-1}}^{t_k} \hat{H}(t')dt' \right]. \tag{31}$$

Now, let us elucidate the relationship between Eq. (13) and Eq. (30). According to the theorem of mean value of integral calculus, we can write

$$\int_{t_{k-1}}^{t_k} \hat{H}(t')dt' = \hat{H}(\bar{t}_k)(t_k - t_{k-1}) = \hat{H}(\bar{t}_k)\triangle t, \tag{32}$$

where $\bar{t}_k$ is between $[t_{k-1}, t_k]$. From Eq. (30), the n-th order approximation of the time-evolution operator is taken as

$$S^{(n)}(t, t_0) = \prod_{k=1}^{n} \left[1 - \frac{\mathrm{i}}{\hbar} \int_{t_{k-1}}^{t_k} \hat{H}(t')dt' \right]. \tag{33}$$

By the mean value theorem of calculus, we have

$$\int_{t_{k-1}}^{t_k} \hat{H}(t')dt'|\Phi(t_0)\rangle = \triangle t \hat{H}(\bar{t}_k)|\Phi(t_0)\rangle = \triangle t \bar{E}_k|\Phi(t_0)\rangle,$$

namely,

$$\hat{H}(\bar{t}_k)|\Phi(t_0)\rangle = \bar{E}_k|\Phi(t_0)\rangle,$$

where $\triangle t = (t - t_0)/n$. To clarify this, it is necessary to take into consideration that for an element of the scattering matrix, it holds that

$$\begin{aligned}\langle\Phi(\bar{t}_k)|\hat{H}(\bar{t}_k)|\Phi(t_0)\rangle &= \langle\hat{H}(\bar{t}_k)\Phi(\bar{t}_k)|\Phi(t_0)\rangle \\ &= \bar{E}_k\langle\Phi(\bar{t}_k)|\Phi(t_0)\rangle,\end{aligned}$$

where we took into consideration that $\hat{H}(\bar{t}_k)$ is Hermitian and the following eigenvalue equation for it should hold:

$$\hat{H}(\bar{t}_k)\Phi(\bar{t}_k) = E(\bar{t}_k)\Phi(\bar{t}_k).$$

Therefore, we in the sense of the scattering matrix can set $\hat{H}(\bar{t}_k)|\Phi(t_0)\rangle = \bar{E}_k|\Phi(t_0)\rangle$. In the end, Eq. (33) yields

$$S^{(n)}(t, t_0)|\Phi(t_0)\rangle = \prod_{k=1}^{n} \left(1 - \frac{\mathrm{i}}{\hbar}\bar{E}_k\triangle t \right) |\Phi(t_0)\rangle. \tag{34}$$

By geometric average, there exists $\bar{E}$ that satisfies

$$\prod_{k=1}^{n} \left(1 - \frac{\mathrm{i}}{\hbar}\bar{E}_k\triangle t \right) = \left(1 - \frac{\mathrm{i}}{\hbar}\bar{E}\triangle t \right)^n. \tag{35}$$

Then setting $A = -\frac{\mathrm{i}}{\hbar}\bar{E}(t - t_0)$, we have

$$\left(1 - \frac{\mathrm{i}}{\hbar}\bar{E}\triangle t \right)^n = \left(1 + \frac{A}{n} \right)^n. \tag{36}$$

Putting $x = n/A$ in Eq. (36), we immediately obtain

$$\lim_{x\to\infty} \left(1 + \frac{1}{x} \right)^{xA} = e^A.$$

Taking into consideration

$$\begin{aligned}\int_{t_0}^{t} \hat{H}(t')dt'|\Phi(t_0)\rangle &= (t - t_0)\hat{H}(\bar{t})|\Phi(t_0)\rangle \\ &= (t - t_0)\bar{E}|\Phi(t_0)\rangle,\end{aligned}$$

we obtain

$$S(t, t_0) = \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} \hat{H}(t')dt' \right].$$

In the end, Eq. (30) is identical to Eq. (13). Thus, once again the scattering operator, Eq. (13) has been validated.

Let us consider whether the Dyson series satisfies the causality axiom or not. Evidently, the causality axiom of scattering is a criterion to ascertain whether an obtained scattering operator is correct or not. Let us start with the Dyson series

$$\begin{aligned}S(t, t_0) = \sum_{n=0}^{\infty} \left(-\frac{\mathrm{i}}{\hbar} \right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \cdots \int_{t_0}^{t_{n-1}} dt_n \cdot \\ \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n).\end{aligned}$$

Calculating its second-order term, we get

$$2\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}(t_1)\hat{H}(t_2)$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}(t_1)\hat{H}(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \hat{H}(t_2)\hat{H}(t_1)$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}(t_1)\hat{H}(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \hat{H}(t_1)\hat{H}(t_2)$$

$$- \int_{t_0}^{t} dt_2 \int_{t_0}^{t_1} dt_1 \hat{H}(t_1)\hat{H}(t_2) + \int_{t_0}^{t} dt_2 \int_{t_0}^{t_2} dt_1 \hat{H}(t_2)\hat{H}(t_1)$$

$$= \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \hat{H}(t_1)\hat{H}(t_2)$$

$$+ \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \left[\hat{H}(t_1)\hat{H}(t_2) - \hat{H}(t_2)\hat{H}(t_1)\right].$$

Thus, we finalize the above relation as

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \hat{H}(t_1)\hat{H}(t_2) = \frac{1}{2}\left[\int_{t_0}^{t} \hat{H}(t')dt'\right]^2 + R^{(2)}(t),$$

where $R^{(2)}(t)$ is the term relative to the noncommutativity of the Hamilton operators distinguished by time. Generalizing this result, we can write the Dyson series as

$$\begin{aligned} S(t, t_0) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}\right)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \cdots \int_{t_0}^{t} dt_n \\ &\quad \hat{H}(t_1)\hat{H}(t_2)\cdots\hat{H}(t_n) + \sum_{n=2}^{\infty} R^{(n)}(t) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right]^n + R^{(n)}(t) \\ &= \exp\left(-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(t')dt'\right) + \sum_{n=2}^{\infty} R^{(n)}(t). \end{aligned} \tag{37}$$

Obviously, the scattering operator in this form cannot satisfy the causal relation of scattering, $S(t_N, t_0) = \prod_{n=0}^{N-1} S(t_{n+1}, t_n)$. From this, it follows that the Dyson series satisfies the causality axiom of scattering only if $\sum_{n=2}^{\infty} R^{(n)}(t)$ which is related to the characteristic of approximate calculation can be ignored. Otherwise, the Dyson series is meaningless because it does not satisfy the axiom of causality as a main requirement of the scattering operator. This fact also explains the relationship between the global time-evolution operator and the Dyson series. In fact, if $\sum_{n=2}^{\infty} R^{(n)}(t)$ vanishes, the Dyson series coincides with the global time-evolution operator.

2.2 Convergence of scattering matrix

Let us consider whether the scattering matrix in terms of the global time-evolution operator is always convergent. Obviously, a finite order approximation of Eq. (15) is convergent. Now, let us examine whether the infinite series of the global time-evolution operator is convergent. According to the mean value theorem of integral calculus, we set

$$\int_{t_0}^{t} \hat{H}(\mathbf{q}, t')dt' = \hat{H}(\mathbf{q}, \bar{t})(t - t_0), \tag{38}$$

where $\bar{t} \in [t_0, t]$ becomes a parameter. Eq. (38) enables us to eliminate the integral symbol from the time-evolution operator to get

$$\begin{aligned} S(t, t_0) &= \exp\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q}, t')dt'\right] \\ &= \exp\left[-\frac{\mathrm{i}}{\hbar}(t - t_0)\hat{H}(\mathbf{q}, \bar{t})\right]. \end{aligned} \tag{39}$$

Next, Eq. (39) is expanded into the Maclaurin series:

$$\begin{aligned} S(t, t_0) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}\int_{t_0}^{t} \hat{H}(\mathbf{q}, t')dt'\right]^n \\ &= \sum_{n} \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}(t - t_0)\hat{H}(\mathbf{q}, \bar{t})\right)^n. \end{aligned} \tag{40}$$

The solution of the time-dependent Schrödinger equation is represented with the help of the time-evolution operator as

$$\Phi_S(\mathbf{q}, t) = \left[\sum_{n} \frac{1}{n!}\left(-\frac{\mathrm{i}}{\hbar}(t - t_0)\hat{H}(\mathbf{q}, \bar{t})\right)^n\right]\Phi_0(\mathbf{q}). \tag{41}$$

From the finiteness condition of the state function, $\Phi_S(\mathbf{q}, t)$, it is possible to suppose that there exists a definite number, E which satisfies

$$\begin{aligned} |\Phi_S(\mathbf{q}, t)| &= \left|\sum_{n} \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}(t - t_0)\right]^n \left[\hat{H}(\mathbf{q}, \bar{t})\right]^n \Phi_0(\mathbf{q})\right| \\ &\le \left|\sum_{n} \frac{1}{n!}\left[-\frac{\mathrm{i}}{\hbar}(t - t_0)E\right]^n \Phi_0(\mathbf{q})\right|. \end{aligned} \tag{42}$$

Then the transition probability becomes

$$W = |S(t, t_0)|^2 \leq \left| \sum_n \frac{1}{n!} \left[-\frac{\mathrm{i}}{\hbar}(t - t_0)E \right]^n \right|^2 = \left| \exp\left(-\frac{\mathrm{i}}{\hbar}(t - t_0)E \right) \right|^2 = 1. \tag{43}$$

This is in agreement with our common knowledge that the transition probability always should be less than one. Eventually, the scattering matrix as well as the scattering operator is convergent. Thus, it is proved that using the global time-evolution operator leads to the scattering matrix without divergence.

Let us consider the case where the Hamiltonian is expressed by the use of the Hamiltonian density $\mathcal{H}_I$ as $\hat{H}_I = \int \mathcal{H}_I(x)d^3x$. The wave equation in the Dirac interaction picture is

$$\mathrm{i}\hbar \frac{\partial \Phi_I}{\partial t} = \hat{H}_I \Phi_I. \tag{44}$$

Since Eq. (44) takes the same form as Eq. (1), the solution of Eq. (44) becomes

$$\Phi_I(t) = \exp\left[-\frac{\mathrm{i}}{\hbar} \int_{t_0}^{t} \hat{H}_I(t')dt' \right] \Phi_I(t_0). \tag{45}$$

As an example, in the case of the interaction between electron-positron field and electromagnetic field, the Hamiltonian density of interaction is written by the use of field operators as

$$\mathcal{H}_I = -j_\mu(x)A^\mu(x) = -\frac{e}{2}\left[\bar{\psi}\gamma_\mu\psi\right]A^\mu = eN\left(\bar{\psi}\gamma_\mu\psi\right)A^\mu. \tag{46}$$

By Eqs. (15) and (46), the scattering operator is given as

$$S(t, t_0) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\mathrm{i}}{\hbar} \right)^n \left[\int eN\left(\bar{\psi}\gamma_\mu\psi\right)A^\mu d^4x \right]^n. \tag{47}$$

Obviously, the first approximation $eN\int\left(\bar{\psi}\gamma_\mu\psi\right)A^\mu d^4x$, in general, is convergent. Therefore, the Maclaurin expansion, Eq. (47) is always convergent and thus the transition probability from an initial state to a final state:

$$W(t, t_0) = |S(t, t_0)|^2 \tag{48}$$

is the same too. Thus, the global time-evolution operator gives the scattering matrix without divergence. Hence, it is concluded that within the framework of our formulation, there is not the infinity problem.

3 Discussion

The main aim of our work has been to present an alternative mathematical formulation which enables us to avoid the divergence problem of the scattering matrix. A new time-evolution operator independent of the Dyson series has been derived in several ways and its convergence has been proved. Our time-evolution operator satisfies the conditions of finiteness and causality. The construction of a new formulation without the infinity problem indubitably would lead to verifying that the scattering theory without renormalization is possible as well.

Of course, it is well known that the time-evolution operator in the Schrödinger picture is the Dyson series. However, this fact does not mean that the Dyson series is a unique selection. Unfortunately, the Dyson series encounters the infinity problem due to the time-ordering operator. Moreover, the Dyson series does not satisfy the axiom of causality as a trivial reason: $S(t_2, t_0) = S(t_2, t_1)S(t_1, t_0)$. This shows that the Dyson series is not an exact scattering operator which fulfils the physical requirement. On the other hand, this means that even if we might obtain a convergent scattering matrix using a refined renormalization method, it would not be perfect because it does not satisfy the causality of scattering. In our view, the situation that the Dyson series does not satisfy indispensable physical requirements needs a new selection. In this work, we showed an alternative time-evolution operator without the time ordering and infinity. It is possible to find a new scattering operator able to avoid the time-ordering operator responsible for the infinity problem. In fact, it is obvious to be able to avoid the infinity problem by relying on our formulation.

As an essential problem, it is necessary to discuss whether it is inevitable that the infinity problem in the scattering matrix arises. We presuppose that the Schrödinger equation,

$$\mathrm{i}\hbar \frac{\partial \Phi_S}{\partial t} = \hat{H}\Phi_S \tag{49}$$

has a correct solution. It means that its solution,

$$\Phi_S(\mathbf{q}, t) = S(t, t_0)\Phi_S(\mathbf{q}, t_0) \tag{50}$$

represented in terms of the scattering operator should be finite.

Through a simple consideration, we can understand the truth of the infinity problem. Let us start from Eq. (25). By the mean value theorem of integral calculus:

$$\int_a^b f(x)g(x)dx = f(c)g(c)(b-a) \qquad (c \in [a, b]),$$

it is possible to take

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle - \frac{\mathrm{i}}{\hbar}(t - t_0)\hat{H}(\bar{t})|\Phi(\bar{t})\rangle, \tag{51}$$

where $\bar{t} \in [0, t]$.

For $\hat{H}(\bar{t})|\Phi(\bar{t})\rangle$, we can imagine an eigenvalue equation with parameter $\bar{t}$:

$$\hat{H}(\bar{t})|\Phi(\bar{t})\rangle = E(\bar{t})|\Phi(\bar{t})\rangle. \tag{52}$$

Then, we have

$$|\Phi(t)\rangle = |\Phi(t_0)\rangle - \frac{\mathrm{i}}{\hbar}(t - t_0)E(\bar{t})|\Phi(\bar{t})\rangle. \tag{53}$$

Since the Schrödinger equation presupposes the finiteness of solution, Eq. (52) is finite and thus Eq. (53) is finite as well.

In this case, in a formal manner, the scattering operator can be taken as

$$S(t, t_0) = 1 - \frac{\mathrm{i}}{\hbar}(t - t_0)E(\bar{t})\frac{\Phi(\bar{t})}{\Phi(t_0)}. \tag{54}$$

Of course, since $|\Phi(\bar{t})\rangle$ is unknown, the solution is formal but Eq. (54) is enough to verify that the scattering operator should be finite. In fact, according to the definition of the Schrödinger equation, $|\Phi(\bar{t})\rangle, |\Phi(t_0)\rangle$ and $E(\bar{t})$ should be finite and nonzero valued. Consequently, the scattering matrix is finite. Considering in this way, we can draw the conclusion that there is no infinity problem as far as $\hat{H}$ is defined correctly.

The infinity problem of scattering matrix, in a sense, is an instance demonstrative of the imperfection of the adopted mathematical language for quantum field theory. Purely from the point of view of mathematics, such a mathematical theory that one must separate a finite quantity from a given infinity cannot be justified. We can understand the truth of the infinity problem purely based on mathematical logic. Obviously, Eqs. (49) and (50) should be considered to be mathematically identical. If the scattering matrix is divergent, it means that the scattering operator is not exact. Therefore, in this case, we should ascribe the infinity problem to the scattering operator. This fact shows that it is necessary to review the validity of the adopted scattering operator. The new formalism we propose is not related to these problems, since it is free of infinity.

4 Conclusion

We have presented a new mathematical representation of scattering matrix in terms of the global time-evolution operators, based on the generalized Heisenberg picture. Using a mathematically rigorous method, the global time-evolution operator, Eq. (13) has been derived. The derived time-evolution operator satisfies all the requirements for the scattering operator, so it is consistent. Within the framework of our formulation using this time-evolution operator, there is no necessity of dealing with the Feynman diagram and thus the overlapping divergence problem of the scattering matrix. Thus, it is demonstrated that it is possible to formulate a consistent scattering theory avoiding overlapping divergences.

What is best is to obtain a time-evolution operator which in any case is irrelevant to infinity. With this aim, we conceived of a consistent time-evolution operator independent of the Dyson series, beginning with a new starting point. Our study has elucidated the imperfect aspects of the Dyson series by arguing the fact that the Dyson series does not satisfy the causality of scattering is fatal. The obtained time-evolution operator is the nonperturbative expression, whereas the Dyson series is a perturbative expansion. The Dyson series which is given by perturbative expansion is impossible to satisfy the causality condition of scattering. However, these two time-evolution operators are related as seen in Eq. (37), which apparently explains the relation between the two operators. The fact that our theory with a different starting point from the Dyson series satisfies all the requirement for the scattering operator shows that the new selection distinguished from the Dyson series is valid.

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