THE MAXWELL EQUATIONS WITH RADIATIVE CORRECTIONS

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Abstract

The one-loop radiative correction to the photon propagator can be graphically represented by the Feynman diagram of the second order. The physical meaning of this diagram is the process $\gamma \rightarrow (e^- + e^+) \rightarrow \gamma$, where γ is denotation for photon, and e^-, e^+ is the electron-positron pair. It means that photon can exist in the intermediate state with e^+, e^- virtual particles. The photon propagation function based on such process with electron-positron pair e^-, e^+ is derived. The modified Lagrangian of electromagnetic field is derived supposing the modified propagator of photon. The Schwinger source methods of quantum field theory is applied. Then, the modified Maxwell equations are derived from the new Lagrangian.

1 Introduction

Maxwell's Equations, formulated around 1861 by James Clerk Maxwell, describe the interrelation between electric and magnetic fields. They were a synthesis of what was known at the time about electricity and magnetism, particularly building on the work of Michael Faraday, Charles-Augustin Coulomb, Andre-Marie Ampere, and others. These equations predicted the existence of Electromagnetic waves, giving them properties that were recognized to be properties of light, leading to the (correct) realization that light is an electromagnetic wave. Other forms of electromagnetic waves, such as radio waves, were not known at the time, but were subsequently demonstrated by Heinrich Hertz in 1888. These equations are the most elegant edifices of mathematical physics.

They are usually formulated as four equations and are usually expressed in differential form, that is, as partial differential equations involving the divergence and curl operators. They can also be expressed with integrals.

Albert Einstein developed special and general relativity to accommodate the invariant speed of light, a consequence of Maxwell's equations, with the principle that only relative movement has physical consequences. The equations are the unification of a theory for previously separately described phenomena: magnetism, electricity, light, and associated radiation.

2 The modified Maxwell equations

The one-loop radiative correction to the photon propagator can be graphically represented by the Feynman diagram of the second order. The physical meaning of this diagram is the process $\gamma \rightarrow (e^- + e^+) \rightarrow \gamma$, where γ is denotation for photon, and e^-, e^+ is the electron-positron pair. It means that photon can exist in the intermediate state with e^+, e^- virtual particles. The photon propagation function based on such process with electron-positron pair e^-, e^+, was determined (Schwinger,1973; 2018) from the effective emission and absorption sources .

The photon propagator involving the two-particle exchange process obviously leads to the modified Maxwell equations which are based on the new propagator. Let us here try to derive the more realistic Maxwell equations which are adequate to the more realistic photon propagator.

The vacuum amplitude involving the electron and positron exchange has been derived in the form (Schwinger, 1970; 1973; 2018; Dittrich, 1978):

$$\langle 0_+|0_-\rangle = -e^2 \int dM^2 I(M^2) d\omega_k A_1^\mu(-k) \left(g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right) A_2^\nu(k) \tag{1}$$

with

$$I(M^2) = \frac{4}{3}\frac{1}{(4\pi)^2} M^2 \left(1 + \frac{2m^2}{M^2}\right)\left(1 - \frac{4m^2}{M^2}\right)^{1/2}; \; d\omega_k = \frac{d(\mathbf{k})}{2\pi^3}\frac{1}{2k^0}, \tag{2}$$

where $k^0 = +\sqrt{\mathbf{k}^2 + M^2}$.

Using the field strength in the momentum representation

$$F_{\mu\nu} = ik_\mu A_\nu(k) - ik_\nu A_\mu(k) \tag{3}$$

we get with $k = p + p'$

$$-\frac{1}{2} F_1^{\mu\nu}(-k) F_{2\mu\nu}(k) = M^2 A_1^\mu(-k) \left(g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right) A_2^\nu(k) \tag{4}$$

and after inserting of eq. (4) into eq. (1), we get

$$\langle 0_+|0_-\rangle = ie^2 \int \frac{dM^2}{M^2} I(M^2) \left(-\frac{1}{2}\right) F_1^{\mu\nu}(-k) i d\omega_k F_{2\mu\nu}(k), \tag{5}$$

which leads directly to a space-time extrapolation which we formulate as an action expression

$$W = \int dM^2 M^2 a(M^2) \left(-\frac{1}{4}\right) F^{\mu\nu}(x) \left[\Delta_+(x - x', M^2) + C.T.\right] F_{\mu\nu}(x'), \tag{6}$$

where C.T. is appropriate contact term and

$$M^2 a(M^2) = \frac{4\pi\alpha}{M^2} I(M^2) = \frac{\alpha}{3\pi}\left(1 + \frac{2m^2}{M^2}\right)\left(1 - \frac{4m^2}{M^2}\right)^{1/2}. \tag{7}$$

The contact term can be determined by the physical normalization condition that the action appropriate to the real photons ($k^2 = 0$) must not be altered. This can be achieved by the combination

$$\Delta_+(x - x', M^2) - \frac{1}{M^2}\delta(x - x') = \frac{1}{M^2}\partial^2\Delta_+(x - x', M^2) \tag{8}$$

which is transformed in the momentum space as

$$\frac{1}{k^2 + M^2 - i\varepsilon} - \frac{1}{M^2} = -\frac{k^2}{M^2}\frac{1}{k^2 + M^2 - i\varepsilon}. \tag{9}$$

Thus a more complete action for the electromagnetic field is given by the formula

$$W = \int (dx)\left[J^\mu(x)A_\mu(x) - \frac{1}{4}F^{\mu\nu}(x)F_{\mu\nu}(x) \quad - \right.$$

$$\left. \int dM^2 a(M^2)(-\frac{1}{4})\int (dx)(dx')\partial^\lambda F^{\mu\nu}(x)\Delta_+(x - x', M^2)\partial'_\lambda F_{\mu\nu}(x')\right. . \tag{10}$$

In the last action the locality is lost. However, if we consider fields that vary slowly over the interval $1/M < 1/2m$, we can simplify (10) by substituting x for x' in the field structure. Then, using

$$\int (dx')\Delta_+(x - x', M^2) = \Delta_+(k = 0, M^2) = \frac{1}{M^2} \tag{11}$$

together with

$$\int_{(2m)^2}^{\infty} \frac{dM^2}{M^2} a(M^2) = \int_{(2m)^2}^{\infty} \frac{\alpha}{3\pi}\frac{dM^2}{M^4}\left(1 + \frac{2m^2}{M^2}\right)\left(1 - \frac{4m^2}{M^2}\right)^{1/2} =$$

$$\frac{\alpha}{\pi}\frac{1}{(2m)^2}\int_0^1 dv\, v^2\left(1 - \frac{1}{3}v^2\right) = \frac{\alpha}{15\pi}\frac{1}{m^2}, \tag{12}$$

where we have used substitution

$$v = \left(1 - \frac{4m^2}{M^2}\right)^{1/2}, \tag{13}$$

we can replace the last term in W by

$$-\frac{\alpha}{15\pi}\frac{1}{m^2}\int (dx)(-\frac{1}{4})\partial^\lambda F^{\mu\nu}(x)\partial_\lambda F_{\mu\nu}(x), \tag{14}$$

and in the considered limit the appropriate Lagrange function is

$$\mathcal{L} = (-\frac{1}{4})\left[F^{\mu\nu}F_{\mu\nu} - \frac{\alpha}{15\pi}\frac{1}{m^2}\partial^\lambda F^{\mu\nu}\partial_\lambda F_{\mu\nu}\right]. \tag{15}$$

The final Lagrangian implies the modified Maxwell equations in the following form:

$$\left(1 + \frac{\alpha}{15\pi}\frac{1}{m^2}\partial^2\right)\partial_\nu F^{\mu\nu}(x) = J^\mu(x) \tag{16}$$

Since these equations are restricted to $\partial^2 \ll m^2$, the exact solution has no meaning. The approximate solution is as follows:

$$A_\mu(x) = \left(1 - \frac{\alpha}{15\pi}\frac{1}{m^2}\partial^2\right)\int(dx')D_+(x-x')J_\mu(x') \quad =$$

$$\int(dx')D_+(x-x')J_\mu + \frac{\alpha}{15\pi}\frac{1}{m^2}J_\mu(x). \tag{17}$$

apart from a gauge term. The explicit expression of action corresponding to the extended potential A_μ is now given by the formula:

$$W = \frac{1}{2}\int(dx)J^\mu(x)A_\mu(x) \quad =$$

$$\frac{1}{2}\int(dx)(dx')J^\mu(x)D_+(x-x')J_\mu(x') + \frac{\alpha}{15\pi}\frac{1}{m^2}\frac{1}{2}\int(dx)J^\mu(x)J_\mu(x) \tag{18}$$

and it implies the modification of the energy of two quasistatic charge-current distribution:

$$E_{int} = -\int(d\mathbf{x})(d\mathbf{x}')J_a^\mu(\mathbf{x})V(\mathbf{x}-\mathbf{x}')J_{b\mu}(\mathbf{x}') \quad -$$

$$\frac{\alpha}{15\pi}\frac{1}{m^2}\int(dx)J_a^\mu(\mathbf{x})J_{b\mu}(\mathbf{x}), \tag{19}$$

where

$$V(\mathbf{x}-\mathbf{x}') = \frac{1}{4\pi|\mathbf{x}-\mathbf{x}'|} \tag{20}$$

3 Discussion

So-called Maxwell equations is system of basic laws or "postulates" which plays the same part in electrodynamics as Newton's "axioms" do in classical mechanics. Particularly, the correctness of these main postulates of macroscopic electrodynamics (like the correctness of Newton's axioms) can be substantiated in the most convincing way not by the inductive method (which is the only one that can be used in finding fundamental laws, but which, however, cannot give an absolutely strict proof of their correctness), but by agreement with experimental results of the entire complex of corollaries following from the theory and covering all the laws of a macroscopic electromagnetic field (Tamm, 1979).

Let us remark some words to the so called contact terms in eq. (6). So, source couplings that are inferred through space-time extrapolations of causal arrangements can always be supplemented by contact interactions. Unless additional physical considerations can be adduced, the contact terms remain arbitrary and may be omitted. But, when fields replace sources such local interaction terms do have physical content; their existence must be recognized and related to the accompanying physical requirements. Since contact couplings in coordinate space appear as polynomial functions $F(\gamma p)$ of momenta in momentum space, the correct form of $F(\gamma p)$ supplements propagation function by a polynomial in $\gamma p + m$. Quadratic and higher powers of this convenient combination modify the propagation function by constant or polynomial functions of momenta (Schwinger, 1970; 2018).

The situation is different in the traditional quantum field theory. The widely known features of Quantum Field Theory is that it is plagued by divergences. This has traditionally been cited as the need for renormalization. However, renormalization is needed even if the theory is finite: divergences are not the cause for this procedure. Moreover, there are many unknowns at high energy because we have not yet explored all energies. Divergences are just one of these uncertainties. We have no way of knowing whether the divergences really occur or whether there is some compensating physics that makes a finite theory. But in the end, these unknowns do not matter. Because all unknown high energy physics is local when viewed at low energy, and because we measure the experimental values of the low energy constants as required by renormalization, all ultraviolet divergences are irrelevant for physics at ordinary energies (Donoghue et al., 2022).

References

Dittrich, W. (1978). Source methods in quantum field theory, Fortschritte der Physik **26**, 289.

Donoghue, J. and Sorbo, L. *A prelude to quantum field theory* (Princeton University Press, 2022)

Schwinger, J. *Particles, Sources and Fields*, Vol. I, (Addison-Wesley, Reading, Mass., 1970).(Or, Published 2018 by CRC Press Taylor & Francis Group 6000 Broken Sound Parkway NW, Suite 300 Boca Raton, FL 33487-2742).

Schwinger, J. *Particles, Sources and Fields* Vol. II, (Addison-Wesley, Reading, Mass., 1973). (Or, Published 2018 by CRC Press Taylor & Francis Group 6000 Broken Sound Parkway NW, Suite 300 Boca Raton, FL 33487-2742).

Tamm, I. E. *Fundamentals of the Theory of Electricity*
(Mir Publishers Moscow, 1979), Translated from the 1976 Russian edition by G. Leib.