A New Proof That The Reals Numbers ℝ Are Uncountable

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Abstract. We show that the reals are uncountable using Russell's Paradox.

A fairly recent paper[1] called, "Another Proof That The Real Number ℝ Are Uncountable" uses Cousin's lemma. Probably the most well-known proof of the uncountability of the real numbers is Georg Cantor's diagonalization argument.[2] Cantor's diagonalization is not a proof that relies upon Russell's paradox.[3,4,5] In the present paper, we show that the reals are uncountable using Russell's Paradox. To the author's knowledge, this is a new proof.

1. **PROOF THAT THE REAL NUMBERS ARE NOT COUNTABLE.** Let us assume, for the sake of later contradiction, that the real numbers between 0 and 1 (inclusive) are countable. Since *"…the continuum of numbers, or real numbers system … is the totality of infinite decimals,"* [6] then a listing of the real numbers between 0 and 1 in base 3 (a ternary numeral system) would have *all possible* sequences of the digits 0, 1, and 2 after the radix point.[7] For example:

<u>Base 3 List:</u>

1) 0.1210221101211001…
2) 0.1122222222222222…
3) 0.1111111111111111…
4) 0.0111111221002111…
5) 0.2222222222222222…
6) 0.1011122211111100…

⋮

From this list, create another countable list as follows. First, eliminate the leading 0 and radix point. Then replace every digit 2 with a space. Put the resulting binary number(s) in a set, where space(s) are replaced by a comma. We have a New List, with a set of binary numbers(s) for each row (finite numbers are shown in blue):

<u>New List:</u>

1) {**1**, 10, 1101, 11001…}
2) {11}
3) {111111111111111…}
4) {111111, **100**, 11…}
5) {}
6) {10111, 11111100…}

⋮

Now, some rows on the New List cannot include their finite row number in their set (*i.e.*, see row 3 which only has an infinite string of 1s; and see row 5 which is the null set). Other rows can possibly include their finite row number in their set (*i.e.*, row 1; and row 4 where $4_{10} = 100_2$).

The entire New List must consist of all possible base 2 numbers (finite and infinite). If not, then there was a binary string that was missing from the list, and the original Base 3 List did not include some real number (a contradiction).

Moreover, the New List must have sets of *all possible combinations* of finite binary numbers. Yet, we have already shown that some rows cannot list themselves. So, there must be a row *among all possible rows* that has a set of all the row numbers that do not list themselves.

The last statement is similar to Russell's Paradox.[8] Does this row that catalogues "all the rows that do not include themselves" include itself? If it does, then it must not list itself. If it doesn't, then it must include itself in the list. This is a contradiction.[9] Therefore, we must reject the initial assumption that a countable listing of all the real numbers between 0 and 1 can be complete. Accordingly, the real numbers cannot be countable.

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[1] José Gascón (2015) Another Proof That the Real Numbers R Are Uncountable, The American Mathematical Monthly, 122:6, 596-597, DOI: 10.4169/amer.math.monthly.122.6.596

[2] Georg Cantor, Ueber eine elementare Frage der Mannigfaltigkeitslehre, Jahresbericht der Deutschen Mathematiker-Vereinigung (1891) 75–78.

[3] John L. Bell, "Russell's Paradox and Diagonalization in a Constructive Context." In Dodehard Link (editor), One Hundred Years of Russell's Paradox, de gruyter series in logic and its applications (2004)

[4] "… the essential point of my paper is that, in constructive contexts, Russell's paradox and Cantor's diagonalization cannot be considered "equivalent' in proving the uncountability of the real numbers," – private email correspondence with John Bell on Dec. 17, 2022.

[5] The author could not find any reference in the literature which states that Russell's Paradox proves the reals are uncountable.

[6] Courant, Richard, and Herbert Robbins. 1996. *What Is Mathematics? : An Elementary Approach to Ideas and Methods*. Oxford: Oxford University Press. p.68.

[7] Note that base 3 real numbers are not all expressed uniquely on the list, since some numbers end in all 2s (i.e., since $0.1_3 = 0.0222\ldots_3$, which is $0.333\ldots_{10}$). However, a unique list in base 3 can be obtained by eliminating duplicates that end in all 2s, so that the real number 0.1_3 would only appear once on the base 3 list. This does not impact the proof later, since every 2 is later replaced - to separate base 2 numbers (*i.e.*, those consisting of digits 0 and 1) in the New List. In any case, the New List can always be made unique for each row by eliminating duplicate sets.

[8] Weisstein, Eric W. "Russell's Antimony." From *MathWorld*—A Wolfram Web Resource. https://mathworld.wolfram.com/RussellsAntinomy.html (accessed Dec. 18, 2021).

[9] This is reminiscent of Kurt Gödel's 1931 mathematically rigorous proof that:

> … we could not set out a complete set of rules for arithmetic. Gödel showed that we could use any set of possible rules to create sentences similar to the sentence, "this sentence is false." If it's true, then it's false. But if it's false, then it's true. Any attempt to create rules would either allow sentences like, "this sentence is unprovable" to be proven. And so, we would have sentences that can be proved but are false. We would have just proven the sentence that says it can't be proven. Alternatively, we could strengthen our rules to exclude these sentences. But then, because we could no longer prove the sentence, the sentence, "this sentence is unprovable" would be true. And so, we would have true sentences that we can't prove in our system, making our system incomplete.

Transcript of text @21 minutes from: Gimbel, Stephen. *The Great Courses*: *Redefining Reality: The Intellectual Implications of Modern Science*, Episode 3: *Mathematics in Crises*. 2015.