Nontrivial zeros of the Riemann zeta function

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ABSTRACT

The Riemann hypothesis, stating that all nontrivial zeros of the Riemann zeta function have real parts equal to $\frac{1}{2}$, is one of the most important conjectures in mathematics. In this paper we prove the Riemann hypothesis by solving an integral form of the zeta function for the real parts and showing that a ratio of divergent terms can only be finite and nonzero, as required, when the real parts are exactly $\frac{1}{2}$.

1. INTRODUCTION

In 1859 Bernhard Riemann published an article titled, *On the number of primes less than a given quantity*. In that work he speculated that all complex valued nontrivial zeros of the zeta function have a real part equal to $\frac{1}{2}$. And this became known as the Riemann hypothesis. Ever since then, mathematicians have endeavoured to prove it. In 1900 David Hilbert added this problem to his list of 23 most important problems of the twentieth century. And since 2000 it is still one of six remaining millennium problems.

The Riemann zeta function with real part, $\operatorname{Re} z > 1$ is traditionally defined by the infinite sum

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad z \in \mathbb{C} \cap \{\operatorname{Re} z > 1\}. \tag{1}$$

In this work we use a form of the zeta function in integral form, which is analytically continued to the imaginary axis but excludes the only pole at $z = 1$. This form is given by (Heymann, 2020, p8)

$$\zeta(z) = \frac{z}{z-1} - z\int_1^{\infty} \{x\} x^{-z-1} dx, \quad z \neq 1, \ \operatorname{Re} z > 0. \tag{2}$$

We further rely on Riemann's functional equation, which provides symmetry information on the positions of nontrivial zeros, and allows the zeta function to be analytically continued to the whole complex plane. This is given by (Riemann, 1859)

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z) \tag{3}$$

where

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx, \quad \operatorname{Re} z > 0$$

is the gamma function extending the factorial function to the complex plane. From equation (3) it is straightforward to determine the trivial zeros at $z = -2n,\ n \in \mathbb{N} - \{0\}$. All others are known to lie within the critical strip defined by $0 < \operatorname{Re} z < 1$. Apart from the trivial zeros it is known that there are no others outside of the critical strip (Heymann, 2020).

In this work we adopt the broad strategy of Tegetmeyer (2022), which has proven invaluable. Here we analyse the relationship between the integral form in equation (2) and its reflection about the critical line at $\operatorname{Re} z = \frac{1}{2}$. Once these forms are determined and simplified we solve for $\operatorname{Re} z$ and show that a ratio of divergent terms can only be finite and nonzero when $1 - 2\operatorname{Re} z = 0$, thereby proving the Riemann hypothesis. The main difference between Tegetmeyer's and our approach is that we keep our analysis in complex form right until the end of the proof. We also find that, in determining the integrals by parts, we take into account the discontinuities in the fractional part of integration variable when differentiating. This Tegetmeyer neglected and we find that this has significant consequences. In the following analysis we use the notation $z = \sigma + it$.

2. PROOF OF THE RIEMANN HYPOTHESIS

We begin, as Tegetmeyer did, with the following integral form of the zeta function

$$\zeta(z) = \frac{z}{z-1} - z\int_1^\infty \{x\} x^{-z-1}dx. \tag{2}$$

And by symmetry this can be expressed as

$$\zeta(1-z) = \frac{1-z}{-z} - (1-z)\int_1^\infty \{x\} x^{z-2}dx. \tag{4}$$

where the fractional part of *x* is given by $\{x\} = x - \lfloor x \rfloor$. From Riemann's (1859) functional equation we know that the nontrivial zeros are symmetric about the critical line, and this is expressed by $\zeta(z) = 0 \Leftrightarrow \zeta(1-z) = 0$. In what follows the region of interest is restricted to the critical strip, where $0 < \sigma < 1$. Setting both left hand sides in equations (2, 4) to zero, we have

$$\frac{1}{z-1} = \int_1^\infty \{x\} x^{-z-1}dx \tag{5}$$

$$-\frac{1}{z} = \int_1^\infty \{x\} x^{z-2}dx. \tag{6}$$

Next we evaluate the integrals by parts $\int udv = uv - \int vdu$, using $u = \{x\}$ and its derivative

$$u' = 1 - \sum_{n\in\mathbb{Z}} \delta(x-n).$$

Tegetmeyer, in his proof used $u' = 1$, which is true almost everywhere in $\{x : x \in \mathbb{R}\}$. However, he did not take account of the discontinuities in $\{x\}$ as we do here. And, as will become apparent, it turns out that this is inconsistent with the convergence of integrals in equations (5, 6).

Evaluating the integrals in equations (5, 6) gives

$$\begin{aligned}\frac{1}{z-1} &= \left[\{x\}\frac{x^{-z}}{-z}\right]_1^\infty - \int_1^\infty \left(1 - \sum_{n\in\mathbb{Z}}\delta(x-n)\right)\frac{x^{-z}}{-z}dx \\ &= \left[\{x\}\frac{x^{-z}}{-z}\right]_1^\infty - \left[\frac{x^{1-z}}{-z(1-z)}\right]_1^\infty + \sum_{n=1}^{\infty}\frac{n^{-z}}{-z} \\ &= \lim_{N_1\to\infty}\left[\{N_1\}\frac{N_1^{-z}}{-z} + \frac{N_1^{1-z}}{z(1-z)}\right] - \frac{1}{z(1-z)} - \frac{1}{z}\sum_{n=1}^{\infty}n^{-z}\end{aligned} \tag{7}$$

and

$$\begin{aligned}-\frac{1}{z} &= \left[\{x\}\frac{x^{z-1}}{z-1}\right]_1^\infty - \int_1^\infty \left(1 - \sum_{n\in\mathbb{Z}}\delta(x-n)\right)\frac{x^{z-1}}{z-1}dx \\ &= \left[\{x\}\frac{x^{z-1}}{z-1}\right]_1^\infty - \left[\frac{x^{z}}{z(z-1)}\right]_1^\infty + \sum_{n=1}^{\infty}\frac{n^{z-1}}{z-1} \\ &= \lim_{N_2\to\infty}\left[\{N_2\}\frac{N_2^{z-1}}{z-1} + \frac{N_2^{z}}{z(1-z)}\right] - \frac{1}{z(1-z)} - \frac{1}{1-z}\sum_{n=1}^{\infty}n^{z-1}\end{aligned} \tag{8}$$

where we assume $N_{1,2}$ diverge independently. In both cases we arrive at a right hand side consisting of four terms. The first term constitutes a power, with negative real part, of a divergent quantity. Therefore, these terms must vanish. The left hand sides and the third terms are finite. The last terms are finite multiples of the zeta function, in the well-known series form of equation (1), as defined for $\sigma > 1$. But these cannot be considered convergent in the critical strip. However, because the integrals they are derived from are known to be convergent due to $0 \le \{x\} < 1$, then the algebraic sum of the second and last terms must converge. The next stage is to simplify equations (7, 8) and this is followed by a procedure to isolate the diverging terms in the infinite sums.

For $N_{1,2} \to \infty$ equations (7, 8) can be written as

$$\frac{1}{z-1}=\frac{N_1^{1-z}}{z(1-z)}-\frac{1}{z(1-z)}-\frac{1}{z}\sum_{n=1}^{\infty}n^{-z} \quad \text{and}$$

$$-\frac{1}{z}=\frac{N_2^{z}}{z(1-z)}-\frac{1}{z(1-z)}-\frac{1}{1-z}\sum_{n=1}^{\infty}n^{z-1}$$

Rearranging to get all the divergent terms on one side in both cases gives

$$\frac{z}{z-1}+\frac{1}{1-z}=\frac{N_1^{1-z}}{1-z}-\sum_{n=1}^{\infty}n^{-z}\Rightarrow$$

$$\frac{N_1^{1-z}}{1-z}-\sum_{n=1}^{\infty}n^{-z}=1$$

and

$$-\frac{1-z}{z}+\frac{1}{z}=\frac{N_2^{z}}{z}-\sum_{n=1}^{\infty}n^{z-1}\Rightarrow$$

$$\frac{N_2^{z}}{z}-\sum_{n=1}^{\infty}n^{z-1}=1.$$

Another strategy adopted by Tegetmeyer was to divide the results from equations (2, 4) by each other. Also multiplying top and bottom by $z(1-z)$ gives

$$\frac{zN_1^{1-z}-z(1-z)\sum_{n=1}^{\infty}n^{-z}}{(1-z)N_2^{z}-z(1-z)\sum_{n=1}^{\infty}n^{z-1}}=1. \tag{9}$$

Now considering the infinite sum in the numerator (The following is adapted from https://math.stackexchange.com/questions/3861533/riemann-zeta-function-in-the-critical-strip Last accessed 10/04/2023)

$$\begin{aligned}\sum_{n=1}^{\infty}n^{-z}&=\sum_{n=1}^{1}\frac{n}{n^z}+\sum_{n=2}^{\infty}\frac{1}{n^z}=\sum_{n=1}^{1}\frac{n}{n^z}+\sum_{n=2}^{\infty}\frac{n-(n-1)}{n^z}\\&=\sum_{n=1}^{1}\frac{n}{n^z}+\sum_{n=2}^{\infty}\frac{n}{n^z}-\sum_{n=2}^{\infty}\frac{n-1}{n^z}=\sum_{n=1}^{\infty}\frac{n}{n^z}-\sum_{n=2}^{\infty}\frac{n-1}{n^z}\\&=\sum_{n=1}^{\infty}\frac{n}{n^z}-\sum_{n=1}^{\infty}\frac{n}{(n+1)^z}=\sum_{n=1}^{\infty}n\left[\frac{1}{n^z}-\frac{1}{(n+1)^z}\right]\\&=z\sum_{n=1}^{\infty}n\int_n^{n+1}x^{-z-1}dx\end{aligned}$$

$$\sum_{n=1}^{\infty} n^{-z} = z\sum_{n=1}^{\infty}\int_{n}^{n+1}\lfloor x\rfloor x^{-z-1}dx = z\int_{1}^{\infty}\lfloor x\rfloor x^{-z-1}dx$$

$$= z\int_{1}^{\infty} x^{-z}dx - z\int_{1}^{\infty}\{x\}x^{-z-1}dx = z\left[\frac{x^{1-z}}{1-z}\right]_{1}^{\infty} - z\int_{1}^{\infty}\{x\}x^{-z-1}dx$$

$$= \frac{z}{1-z}N_1^{1-z} - \frac{z}{1-z} - z\int_{1}^{\infty}\{x\}x^{-z-1}dx, \quad N_1 \to \infty.$$

And similarly for the infinite sum in the denominator

$$\sum_{n=1}^{\infty} n^{z-1} = \frac{1-z}{z}N_2^{z} - \frac{1-z}{z} - (1-z)\int_{1}^{\infty}\{x\}x^{z-2}dx, \quad N_2 \to \infty\,.$$

Substituting these in, equation (9) can now be expressed as

$$\frac{zN_1^{1-z} - z^2N_1^{1-z} - z^2 - z^2(1-z)\int_{1}^{\infty}\{x\}x^{-z-1}dx}{(1-z)N_2^{z} - (1-z)^2N_2^{z} - (1-z)^2 - z(1-z)^2\int_{1}^{\infty}\{x\}x^{z-2}dx} = 1\,. \tag{10}$$

Because the terms containing positive real powers of $N_{1,2}$ dominate the numerator and the denominator respectively, and from the observation that the integrals converge, we can make the following simplification by removing the convergent terms. Moreover, from equations (5, 6) it is seen that the convergent terms actually cancel out.

$$\frac{z(1-z)N_1^{1-z}}{(1-z)N_2^{z} - (1-z)^2N_2^{z}} = 1 \Rightarrow$$

$$\frac{zN_1^{1-z}}{N_2^{z} - (1-z)N_2^{z}} = 1 \Rightarrow$$

$$\frac{N_1^{1-z}}{N_2^{z}} = 1.$$

Multiplying top and bottom by N_2^{-z} gives

$$N_1^{1-z}N_2^{-z} = 1, \quad N_{1,2} \to \infty\,. \tag{11}$$

Next we need to address the assumed independence of the divergent terms as seen in equation (10). The divergent terms in the numerator/denominator are generated by the top limit of the integral in equation (5)/(6). Therefore, in this regard, we assume independent divergence between the numerator and denominator only. And because these come from limits on integrals, they are assumed constant. Therefore, without loss of generality we can write N_2 in terms of N_1 as

$$N_2 = N_1^w, \quad w \in \mathbb{C}.$$

Letting $N_1 = N$, we may write equation (11) as

$$N^{1-z} N^{-zw} = 1, \quad N \to \infty .$$

Decomposing the exponents into real and imaginary parts, we get

$$\begin{aligned} 1 - z(1+w) &= 1 - \sigma - it - (\sigma + it)(w_r + iw_i) \\ &= 1 - \sigma - it - \sigma w_r + tw_i - i(tw_r + \sigma w_i) \\ &= 1 - \sigma(1 + w_r) + tw_i - i(t + tw_r + \sigma w_i). \end{aligned}$$

Because the real part contains the term, tw_i, where t can potentially take any value, this term must vanish, i.e. $w_i = 0$. Therefore, w is effectively real and the complete exponent becomes

$$1 - \sigma(1 + w_r) + it(1 + w_r) .$$

By letting the real part equal zero, as required, we obtain a general equation for σ given by

$$1 - \sigma(1 + w_r) = 0 . \qquad (12)$$

However, Riemann's functional equation also requires

$$\begin{aligned} &1 - (1-\sigma)(1 + w_r) = 0 \Rightarrow \\ &\quad w_r - \sigma(1 + w_r) = 0. \end{aligned}$$

Since equation (12) is linear in σ it can only have one solution. Therefore, the only way that the functional equation can be satisfied is for $w_r = 1$, and so equation (11) becomes

$$N^{1-2z} = 1, \quad N \to \infty$$

Decomposing the exponent and writing the factor generated from the imaginary part in trigonometric form, we obtain

$$N^{1-2\sigma}\left(\cos(2t \ln N) - i \sin(2t \ln N)\right) = 1, \quad N \to \infty . \qquad (13)$$

The only way that this equation can be satisfied is for $1 - 2\sigma = 0$ as this sets the magnitude to unity. With $1 - 2\sigma > 0$ the left hand side diverges, while it vanishes for $1 - 2\sigma < 0$. Because $\ln N \to \infty$, only infinitesimal changes in t are required to set the phase factor equal to unity.

Therefore, this says nothing about the positions and number of zeros on the critical line. This concludes the proof of the Riemann hypothesis.

3. CONCLUSION

In this work we have shown that the Riemann hypothesis is true and that the real parts of all nontrivial zeros is $\frac{1}{2}$. By setting the zeta function to zero and solving for the real part of z, it was found that a ratio of diverging factors could only be finite and nonzero, as required, when the real part of z is exactly $\frac{1}{2}$.

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