The Collatz Conjecture (with proven families)

John R Eaton

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Abstract - Within my paper I begin by defining some important terminology. Then I generalize the Collatz Conjecture to a wider class of problems, which I leverage to provide a path to a non-constructive proof of the Collatz Conjecture. Ulitmately, I did not succeed in proving the Collatz Conjecture; however, I believe I have made the problem tractable. The problem I reduced the Collatz Conjecture to is beyond my capabilities. After working through my non-constructive results, I provide some constructive results concerning the collatz conjecture. For example, 2957851400532535270158974145876 converges to one. I conclude with closing remarks.

1 Introduction

I know what you're thinking, "Oh this is just another crank trying to solve a conjecture countless mathematicians have wrestled with". And you'd normally be right. However, I hope to prove to you within my paper that I am no crank.

Mathematics is a beautiful landscape, it only takes a pencil and paper to visit its peaks and valleys. My paper illustrates a path towards a constructive proof of the Collatz Conjecture, which in turn provides infinite families of integers that satisfy the Collatz Conjecture. Ultimately, I was not able to finish a constructive proof, but I do offer a non-constructive result that almost proves the Collatz Conjecture in the affirmative. It only hinges on one additional conjecture, which I have yet to prove in the affirmative.

2 Fundamental Definitions

Definition 2.01 Given a set X we may associate a function $\alpha : X \times X \to X$ to X. Since X is closed under α, i.e $\forall x, y \in X \ \alpha(x, y) \in X$, by the definition of α, we may call (X, α) a *magma* and α a *binary algebraic operation* or just operation when appropriate.

Given an operation α we may associate a symbol, say $*$, to represent the mapping, i.e $\alpha(x, y) = x * y$. Thus, instead of (X, α) we will often write $(X, *)$.

Definition 2.02 A *semi-group* is simply a magma that obeys the *associativity* property, $\forall x, y, z \in X \ x * (y * z) = (x * y) * z$.

Definition 2.03 We say a semi-group contains an *identity element* if $\exists e \in X$: $\forall x \in X \ e * x = x * e = x$. If a semi-group contains an identity element, then we may call it a *monoid*

Definition 2.04 A monoid that admits an *inverse element* for every element in the monoid is called a *group*. I.e given $(X, *)$ is a monoid and $\forall x \in X \ \exists t \in X$: $x * t = t * x = e$ then $(X, *)$ is a group. In general, we call the element $x^{-1} \in X$ an inverse element of $x \in X$ if $x^{-1} * x = x * x^{-1} = e$.

We will now shift our focus back onto monoids. Specifically morphisms between monoids.

Definition 2.05 We call a function $\theta : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} = (X, *)$ and $\mathcal{Y} = (Y, \cdot)$ are monoids, a *monoid homomorphism* or just morphism(in the correct setting), if the following is true,

For $x, s \in X$ we have $\theta(x * s) = \theta(x) \cdot \theta(s)$, and
$\theta(e_X) = e_Y$ where e_X and e_Y are the identity elements of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

A monoid homomorphism is called an *isomorphism* if it is a bijective map. We denote an isomorphism between two monoids $\mathcal{X}$ and $\mathcal{Y}$ as $\mathcal{X} \cong \mathcal{Y}$.

Defintion 2.06 Similar to the previous defintion we define a function $\omega : \mathcal{X} \to \mathcal{Y}$ where $\mathcal{X} = (X, *)$ and $\mathcal{Y} = (Y, \cdot)$ are monoids, as a *monoid allomorphism* or just allomorphism(in the correct setting), if the following is true,

For $x, s \in X$ we have $\omega(x * s) = \omega(s) \cdot \omega(x)$, and
$\omega(e_X) = e_Y$ where e_X and e_Y are the identity elements of $\mathcal{X}$ and $\mathcal{Y}$ respectively.

Definition 2.07 Given a monoid $\mathcal{J} = (J, \cdot)$ we may call the monoid $\mathcal{D} = (D, \cdot)$ a *submonoid* of $\mathcal{J}$, if $D \subseteq J$. It follows that they share the same identity element.

Now we will define the monoid equivalent of group actions.

2.1 Acts

Definition 2.11 Given a monoid $\Sigma = (X, \cdot)$ and a non-empty set Ω, we form a function called a Ω-act or just *act*, where this makes sense, as follows, Φ : $\Sigma \times \Omega \to \Omega$. We call an act a *group action* if Σ is a group.
I list one important property of monoid actions here,

$$\Phi(f, \Phi(h, \omega))) = \Phi(f \circ h, \omega) \tag{1}$$

To make things easier we define $\gamma_q : \Sigma \to \Omega$ as $\gamma_q(g) = \Phi(g, q)$

2.1.1 Equivalences under Ω act

We will now discuss perhaps the most important part of the idea behind my approach.

Definition 2.1.11 - Given a Ω act $\Phi : \Sigma \times \Omega \to \Omega$, we create a quotient-like object as follows,
Given $q \in \Omega$
$\phi_q(g) = \{h \in \Sigma : \Phi(g, q) = \Phi(h, q)\}$

As a special case we get what I denote as *inverses with respect to* Φ *evaluated at* q. We define them as those elements in the set $\phi_{\Phi(g,q)}(g^{-1}) = \{h \in \mathcal{H} : \Phi(h \circ g, q) = q\}$ If you have studied group theory, this should look similar to the kernel subgroup.

2.2 Star Set

The next definition will be an analog of a topic often used in formal language theory. The *Kleene Star*.

Definition 2.21 We will avoid using the asterisk as an operation due to the following definition. Using a set J we define an operation $\cdot$ to form the monoid $(J, \cdot)$. We then take a set $D \subset J$ and apply the same operation. The structure $(D, \cdot)$ is not necessarily a monoid, but we will be able to turn it into one(See Lemma 2.21). We define D^* as follows,

First, assume $i \in \mathbb{N} \cup \{0\}$

$D_0 = \{e_J\}$
$D_1 = D$
$D_{i+1} = \{g \cdot h : g \in D \wedge h \in D_i\}$

Then finally we join these sets together,

$$D^* = \bigcup_{i \in \mathbb{N} \cup \{0\}} D_i \tag{2}$$

Lemma 2.21 Given a monoid $\mathcal{J} = (J, \cdot)$ and a set $D \subset J$ equipped with the operation $\cdot$, i.e $(D, \cdot)$, the set D^* under the operation $\cdot$ is a monoid. I.e $(D^*, \cdot)$ is a monoid.

Proof - This proof boils down to proving that the set D^* is indeed closed under the operation $\cdot$. This is because associativity extends naturally to $(D^*, \cdot)$. Indeed, the identity element also naturally extends to $(D^*, \cdot)$. Remember, to prove $(D^*, \cdot)$ is closed under $\cdot$, we must prove $\forall x, y \in D^*\ x \cdot y \in D^*$. Assume this is not the case, then we can find $s, t \in D^*$ such that $s \cdot t \notin D^*$. Choose an arbitrary element of D^* call it w. First, we can set $w = e_J \cdot e_J$, but this is clearly in D^*. We can then rewrite w as $g \cdot h$, where $g \in D$ and $h \in D_j$ for some $j \in \mathbb{N} \cup \{0\}$, but $D_j \subset D^*$ for all $j \in \mathbb{N} \cup \{0\}$, thus $w = g \cdot h \in D^*$ for arbitrary w. This concludes the proof by contradiction.

2.3 A Special Allomorphism

Here we create a special allomorphism that will be of later use in our proof.

Definition 2.3.1 To begin our defintion, let $(\mathcal{H}, \cdot)$ be a monoid. We now create an allomorphism $\varphi_A : \mathcal{H} \to \mathcal{H}$ where $A \subset \mathcal{H}$. We add two extra conditions. $\forall g \in \mathcal{H} - A\ \ \varphi(g) = g$ and $\forall g \in A\ \ \varphi(g) = \beta$. These two conditions can be summarized as follows, elements not in A are fixed while elements in A are taken to different elements of $\mathcal{H}$.
We now prove a really cool fact concerning our newly created allomorphism.

Lemma 2.3.1 $\varphi_A(A^*) = (\varphi_A(A))^*$
Proof -
$\varphi_A(A) = \{\varphi(a) : a \in A\} = \{\beta_1, ..., \beta_{|A|}\}$

$\varphi_A(A^*) = \varphi(\bigcup_{i \in \mathbb{N}} A_i) = \bigcup_{i \in \mathbb{N}}(\varphi(A_i)) = \bigcup_{i \in \mathbb{N}}(B_i) = (\varphi_A(A))^*$

where $B_i = \{g \circ h : g \in \varphi_A(A) \wedge h \in B_{i-1}\}$. For future ease we will define $\varphi_A(A) = \Theta_A$

3 The Collatz Conjecture

The Collatz Conjecture is among the many unsolved problems in the world of mathematics. Deceptively simple in it's formulation, it has plagued the minds of many mathematicians. The unusual nature of the problem makes the Collatz Conjecture hard to approach, but I suspect I have finally found a path to resolving it.

3.1 What is the Collatz Conjecture?

The Collatz Conjecture asserts the following, starting with any natural number, if it is even, divide it by two, if it is odd, multiply it by 3 and add 1. Repeat this process with each iteration. No matter what number you begin with, this process will converge to 1. For example, starting with 5, we reach 16, 8, 4, 2, and finally 1.

To simplify this process we use function composition. $f(k) = \frac{k}{2}$, and $g(k) = 3k + 1$. Thus, starting with 5 again, $(f^{(4)} \circ g)(5) = 1$, where, for a function h, $h^{(n)} = h \circ h \circ ... \circ h$.

3.2 Novel approach to the Collatz Conjecture

Definition 3.21 - $\sigma_{a,b} : \mathbb{Q} \to \mathbb{Q}$, $q \mapsto aq + b$. Where $\mathbb{Q}$ is the set of rational numbers and $a, b \in \mathbb{Q}$. We call these functions *discrete paths*.

Example 3.1 - $\sigma_{\frac{1}{2},0}(8) = 4$.

3.2.1 The monoid of discrete paths

We will now create a monoid from the set of discrete paths using composition as the operation. We will then show that this monoid can be extended to a group.

Definition 3.2.11 - $\mathcal{P} = \{\sigma_{a,b} : a \in \mathbb{Q} - \{0\}, b \in \mathbb{Q}\}$. We can create subsets easily as follows, let $X \subseteq \mathbb{Q}$ then

$$\mathcal{P}(X) = \{\sigma_{a,b} : a \in X - \{0\}, b \in X\}. \quad (3)$$

Where, of course, $\mathcal{P}(\mathbb{Q}) = \mathcal{P}$

Definition 3.2.12 We define the set of functions mapping from a set X to Y as

$$\mathcal{F}(X, Y) = \{f | f : X \to Y\} \quad (4)$$

The evaluation map follows, $ev_x : \mathcal{F} \to Y$, where $x \in X$ and $f \mapsto f(x)$. It is important to note what it means for two functions to be equivalent, to that end we define function equivalency next.

Definition 3.2.13 Given $f, g \in \mathcal{F}(X, Y)$, we say $f = g$ iff $\forall x \in X$ $ev_x(f) = ev_x(g)$

Lemma 3.2.11 Given the set of discrete paths $\mathcal{P}$ we equip group composition to this set to form $\mathcal{D}$, i.e $\mathcal{D} = (\mathcal{P}, \circ)$. Here we will prove $\mathcal{D}$ is a monoid. In fact, given $R \subset \mathcal{P}$ we may form the submonoid $(R^*, \circ)$.

Proof - For the first case we must show that $\mathcal{D}$ is indeed a monoid. First, let us ensure it remains closed under composition, i.e $\forall g, h \in \mathcal{D}$ $g \circ h \in \mathcal{D}$. Assume $\mathcal{D}$ is not closed. Then, $\exists g, h \in \mathcal{D} : g \circ h \notin \mathcal{D}$, or equivalently, $\exists a, b, c, d \in \mathbb{Q}$: $\sigma_{a,b} \circ \sigma_{c,d} \notin \mathcal{D}$. However, $\sigma_{a,b} \circ \sigma_{c,d} = \sigma_{ac,ad+b}$, therefore, $\exists a, b, c, d \in \mathbb{Q}$:

$\sigma_{ac,ad+b} \notin \mathcal{D}$. This is a contradiction because $ac \in \mathbb{Q} - \{0\}$ and $ad+b \in \mathbb{Q}$, thus we conclude that $\mathcal{D}$ is indeed closed under $\circ$. Now we must identify an identity element. An element $e_{\mathcal{D}}$ is an identity iff $\forall g \in \mathcal{D}\ e_{\mathcal{D}} \circ g = g \circ e_{\mathcal{D}} = g$. In other words given $c \in \mathbb{Q} - \{0\} d \in \mathbb{Q}$, $\forall a \in \mathbb{Q} - \{0\} b \in \mathbb{Q}\ \sigma_{c,d} \circ \sigma_{a,b} = \sigma_{a,b} \circ \sigma_{c,d} = \sigma_{a,b}$. So we arrive at, $\forall a, b \in \mathbb{Q}\ \sigma_{ac,bc+d} = \sigma_{ac,ad+b} = \sigma_{a,b}$. Finally, we can conclude $ac = a$ and $ad + b = b$, then $c = 1$ and $d = 0$ since $a \neq 0$. Therefore, the identity element of $\mathcal{D}$ is $\sigma_{1,0}$. Composition is naturally associative. Thus $\mathcal{D}$ is a monoid.

For the next case, we need only prove that $R^* \subseteq \mathcal{P}$, this is trivial given that R^* is formed exclusively from elements in R and $R \subset \mathcal{P}$. Therefore, using Lemma 2.21, $(R^*, \circ)$ is a submonoid of $\mathcal{D}$.

We can prove the stronger result that $\mathcal{D}$ is a group. To complete this proof, we need only show that each element of $\mathcal{D}$ admits an inverse. I.e $\forall g \in \mathcal{D}$ $\exists h \in \mathcal{D} : g \circ h = h \circ g = \sigma_{1,0}$. We then proceed similar to the previous proofs, $\forall a \in \mathbb{Q} - \{0\} b \in \mathbb{Q}\ \exists c \in \mathbb{Q} - \{0\} d \in \mathbb{Q} : \sigma_{a,b} \circ \sigma_{c,d} = \sigma_{c,d} \circ \sigma_{a,b} = \sigma_{1,0}$. From which it follows, $\forall a \in \mathbb{Q} - \{0\} b \in \mathbb{Q}\ \exists c \in \mathbb{Q} - \{0\} d \in \mathbb{Q} : \sigma_{ac,ad+b} = \sigma_{ac,bc+d} = \sigma_{1,0}$. Therefore, $ac = 1$ and $bc + d = 0$, which means $c = \frac{1}{a}$ and $d = \frac{-b}{a}$. Thus, for any elements a and b of $\mathbb{Q}$ there exists two elements constructed from a and b that satisfy the property required for inverses. I.e every element of $\mathcal{D}$ admits an inverse. Given this fact, and the fact that $\mathcal{D}$ is a monoid, we may say $\mathcal{D}$ is a group.

Lemma 3.2.12 Given the set $\mathbb{Q}$ and the group $\mathcal{D}$, $\Psi : \mathcal{D} \times \mathbb{Q} \to \mathbb{Q}$, where $\Psi(g, q) = ev_q(g) = g(q)$, is a group action.

Proof - This is a trivial result of the functions definition.

3.3 Non-constructive proof of the Collatz Conjecture

To begin our proof we construct an important set U.

Definition 3.31 $U = \{\sigma_{2,0}, \sigma_{2,1}\}$ thus, $U^* = \{\sigma_{1,0}, \sigma_{2,0}, \sigma_{2,1}, \sigma_{2,0}^2, ...\}$

Lemma 3.31 The statement $\gamma_1(U^*) = \mathbb{N}$ and $\forall u \in U^* \exists k, i \in \mathbb{N} \cup \{0\} : u = \sigma_{2^k,i}$, can be deduced as follows.
Proof -
We prove this lemma by showing that every element of U^* can be written as $\sigma_{2^k,i}$ where $k, i \in \mathbb{N} \cup \{0\}$. To prove this observe $U_0 = \{\sigma_{2^0,0}\}$, $U_1 = \{\sigma_{2^1,0}, \sigma_{2^1,1}\}$. We now use induction.

Given our base case of $i = 0$, we start with our hypothesis, $U_i = \{\sigma_{2^i,0}, \sigma_{2^i,1}, ..., \sigma_{2^i,2^i-1}\}$
$U_{i+1} = \{g \circ h : g \in U \wedge h \in U_i\} = \{\sigma_{2,0} \circ h : h \in U_i\} \cup \{\sigma_{2,1} \circ h : h \in U_i\}$.

$\{\sigma_{2,0} \circ h : h \in U_i\} = \{\sigma_{2,0} \circ \sigma_{2^i,0}, ..., \sigma_{2,0} \circ \sigma_{2^i,j}, ..., \sigma_{2,0} \circ \sigma_{2^i,2^i-1}\} = \{\sigma_{2^{i+1},0}, ..., \sigma_{2^{i+1},2j}, ..., \sigma_{2^{i+1},2^{i+1}-2}\}$

$\{\sigma_{2,1} \circ h : h \in U_i\} = \{\sigma_{2,1} \circ \sigma_{2^i,0}, ..., \sigma_{2,1} \circ \sigma_{2^i,j}, ..., \sigma_{2,1} \circ \sigma_{2^i,2^i-1}\} = \{\sigma_{2^{i+1},1}, ..., \sigma_{2^{i+1},2j+1}, ..., \sigma_{2^{i+1},2^{i+1}-1}\}$

Therefore, $\{\sigma_{2,0} \circ h : h \in U_i\} \cup \{\sigma_{2,1} \circ h : h \in U_i\} = \{\sigma_{2^{i+1},0}, ..., \sigma_{2^{i+1},2j}, ..., \sigma_{2^{i+1},2^{i+1}-2}\} \cup \{\sigma_{2^{i+1},1}, ..., \sigma_{2^{i+1},2j+1}, ..., \sigma_{2^{i+1},2^{i+1}-1}\} = \{\sigma_{2^{i+1},0}, \sigma_{2^{i+1},1}, ..., \sigma_{2^{i+1},2^{i+1}-1}\}$, which completes our proof by induction that for any element u of U^* there exists $k, i \in \mathbb{N} \cup \{0\}$, such that $u = \sigma_{2^k,i}$.

Now we can rewrite U^*,

$$\gamma_1(U^*) = \bigcup_{k \in \mathbb{N} \cup \{0\}} (\gamma_1(U_k)) = \bigcup_{k \in \mathbb{N} \cup \{0\}} \gamma_1(\{\sigma_{2^k,0}, \sigma_{2^k,1}, ..., \sigma_{2^k,2^k-1}\}) = \bigcup_{k \in \mathbb{N} \cup \{0\}} \bigcup_{i=0}^{2^k-1} \{2^k + i\} \tag{5}$$

But we know that

$$\bigcup_{k \in \mathbb{N} \cup \{0\}} \bigcup_{i=0}^{2^k-1} \{2^k + i\} = \mathbb{N} \tag{6}$$

Therefore, $\gamma_1(U^*) = \mathbb{N}$

The lemma we are now going to prove will be very similar to Lemma 3.31. Except, instead of proving assertions concerning U^* we will be proving assertions concerning Θ_U^*.

Lemma 3.32 $\forall f \in \Theta_U^* \exists n_0, n_1, d_0, d_1, d_2 \in \mathbb{N} \cup \{0\} : f = \sigma_{\frac{3^{n_0}}{2^{n_1}}, \frac{3^{d_0} - 2^{d_1}}{2^{d_2}}}$.

Proof -

We proceed similarily to Lemma 3.31 by examining the constituents of Θ_U^*.
To start off we examine our base cases, $(\Theta_U)_0 = \{\sigma_{\frac{3^0}{2^0},0}\}$, and $(\Theta_U)_1 = \{\sigma_{\frac{3^0}{2^1}, \frac{3^0-2^0}{2^1}}, \sigma_{\frac{3^1}{2^1}, \frac{3^1-2^1}{2^1}}\}$.
We see they both fulfill the requirements. Now we start our proof by induction.
Our induction hypothesis is

$$(\Theta_U)_k = \{\sigma_{\frac{3^0}{2^k}, \frac{3^0-2^0}{2^0}}, ..., \sigma_{\frac{3^{w_0}}{2^k}, \frac{3^{w_1}-2^{w_2}}{2^{w_3}}}, ..., \sigma_{\frac{3^k}{2^k}, \frac{3^k-2^k}{2^k}}\} \tag{7}$$

where w_0, w_1, w_2 and $w_3 \in \mathbb{N} \cup \{0\}$
So, now we consider $\sigma_{\frac{3}{2},\frac{1}{2}} \circ (\Theta_U)_k$ and $\sigma_{\frac{1}{2},0} \circ (\Theta_U)_k$.

We let $w_3 = w_2$ for the first case.
$\sigma_{\frac{3}{2},\frac{1}{2}} \circ (\Theta_U)_k = \{\sigma_{\frac{3}{2},\frac{1}{2}} \circ \sigma_{\frac{3^0}{2^k}, \frac{3^0-2^0}{2^0}}, ..., \sigma_{\frac{3}{2},\frac{1}{2}} \circ \sigma_{\frac{3^{w_0}}{2^k}, \frac{3^{w_1}-2^{w_2}}{2^{w_2}}}, ..., \sigma_{\frac{3}{2},\frac{1}{2}} \circ \sigma_{\frac{3^k}{2^k}, \frac{3^k-2^k}{2^k}}\}$.

Thus, $\sigma_{\frac{3}{2},\frac{1}{2}} \circ (\Theta_U)_k = \{\sigma_{\frac{3^1}{2^{k+1}},\frac{3^1-2^1}{2^1}}, ..., \sigma_{\frac{3^{w_0+1}}{2^{k+1}},\frac{3^{w_1+1}-3*2^{w_2}}{2^{w_2+1}}+\frac{1}{2}}, ..., \sigma_{\frac{3^{k+1}}{2^{k+1}},\frac{3^{k+1}-3*2^k}{2^{k+1}}+\frac{1}{2}}\}$

$\sigma_{\frac{3}{2},\frac{1}{2}} \circ (\Theta_U)_k = \{\sigma_{\frac{3^1}{2^{k+1}},\frac{3^1-2^1}{2^1}}, ..., \sigma_{\frac{3^{w_0+1}}{2^{k+1}},\frac{3^{w_1+1}-2^{w_2+1}}{2^{w_2+1}}}, ..., \sigma_{\frac{3^{k+1}}{2^{k+1}},\frac{3^{k+1}-2^{k+1}}{2^{k+1}}}\}$

The next step works similarily, where we allow ω_2 and ω_3 to be different,

$\sigma_{\frac{1}{2},0} \circ (\Theta_U)_k = \{\sigma_{\frac{1}{2},0} \circ \sigma_{\frac{3^0}{2^k},\frac{3^0-2^0}{2^0}}, ..., \sigma_{\frac{1}{2},0} \circ \sigma_{\frac{3^{w_0}}{2^k},\frac{3^{w_1}-2^{w_2}}{2^{w_3}}}, ..., \sigma_{\frac{1}{2},0} \circ \sigma_{\frac{3^k}{2^k},\frac{3^k-2^0}{2^0}}\}$

$\sigma_{\frac{1}{2},0} \circ (\Theta_U)_k = \{\sigma_{\frac{3^0}{2^{k+1}},\frac{3^0-2^0}{2^1}}, ..., \sigma_{\frac{3^{w_0}}{2^{k+1}},\frac{3^{w_1}-2^{w_2}}{2^{w_3+1}}}, ..., \sigma_{\frac{3^k}{2^{k+1}},\frac{3^k-2^0}{2^1}}\}$

Now we simply take the union of these two sets,

$$(\Theta_U)_{k+1} = \sigma_{\frac{1}{2},0} \circ (\Theta_U)_k \cup \sigma_{\frac{3}{2},\frac{1}{2}} \circ (\Theta_U)_k \tag{8}$$

But, $\sigma_{\frac{1}{2},0} \circ (\Theta_U)_k \cup \sigma_{\frac{3}{2},\frac{1}{2}} \circ (\Theta_U)_k = \{\sigma_{\frac{3^0}{2^{k+1}},\frac{3^0-2^0}{2^1}}, ..., \sigma_{\frac{3^{w_0}}{2^{k+1}},\frac{3^{w_1}-2^{w_2}}{2^{w_3+1}}}, ..., \sigma_{\frac{3^k}{2^{k+1}},\frac{3^k-2^0}{2^1}}\} \cup$ $\{\sigma_{\frac{3^1}{2^{k+1}},\frac{3^1-2^1}{2^1}}, ..., \sigma_{\frac{3^{w_0+1}}{2^{k+1}},\frac{3^{w_1+1}-2^{w_2+1}}{2^{w_2+1}}}, ..., \sigma_{\frac{3^{k+1}}{2^{k+1}},\frac{3^{k+1}-2^{k+1}}{2^{k+1}}}\}$

Thus, $(\Theta_U)_{k+1} = \{\sigma_{\frac{3^0}{2^{k+1}},\frac{3^0-2^0}{2^1}}, ..., \sigma_{\frac{3^{t_0}}{2^{k+1}},\frac{3^{t_1}-2^{t_2}}{2^{t_2}}}, ..., \sigma_{\frac{3^{k+1}}{2^{k+1}},\frac{3^{k+1}-2^{k+1}}{2^{k+1}}}\}$
where t_0, t_1, t_2 and $t_3 \in \mathbb{N} \cup \{0\}$

This completes our proof by induction of equation (7). Thus, since every element of $(\Theta_U)_k$ can be written in the form $\sigma_{\frac{3^{t_0}}{2^k},\frac{3^{t_1}-2^{t_2}}{2^{t_2}}}$ for all k in the natural numbers, we can conclude that every element of $(\Theta_U)^*$ can be written in the aforementioned form. This concludes the proof of Lemma 3.32.

Lemma 3.33 For arbitrary $\sigma_{a,b} \in \mathcal{D}$ $\forall v \in \phi_{\gamma_1(\sigma_{a,b})}(\sigma_{a,b}^{-1})$ $v = \sigma_{x,1-x*\gamma_1(\sigma_{a,b})}$ where $x \in \mathbb{Q} - \{0\}$
Proof -

The proof of this lemma is actually not too involved. We begin by considering the defining relation of $\phi_{\gamma_1(\sigma_{a,b})}(\sigma_{a,b}^{-1})$.

$$\forall v \in \phi_{\gamma_1(\sigma_{a,b})}(\sigma_{a,b}^{-1})\ \gamma_1(v \circ \sigma_{a,b}) = 1 \tag{9}$$

Thus, we let $v = \sigma_{x,y}$. $\gamma_1(\sigma_{x,y} \circ \sigma_{a,b}) = 1$. Now we apply composition and evaluate at 1, $x(a+b) + y = 1$. Thus, $y = 1 - x(a+b)$, but $a + b = \gamma_1(\sigma_{a,b})$. Therefore, for arbitrary $\sigma_{a,b} \in \mathcal{D}$, we may rewrite any element of $\phi_{\gamma_1(\sigma_{a,b})}(\sigma_{a,b}^{-1})$ as $\sigma_{x,1-x*\gamma_1(\sigma_{a,b})}$. This completes our proof.

The main theorem of this proof is the cornerstone for this non-constructive proof of the Collatz Conjecture. We state that Theorem now,

Theorem 3.31 $\forall u \in U^* \exists h \in \phi_{\gamma_1(u)}(u^{-1}) \exists f \in \Theta_U^* : \gamma_1(h \circ u) = \gamma_1(f \circ u)$
Proof -

We aim to prove the following statement,

$$\forall u \in U^* \exists h \in \phi_{\gamma_1(u)}(u^{-1}) \exists f \in \Theta_U^* : \gamma_1(h \circ u) = \gamma_1(f \circ u) \tag{10}$$

Using Lemma 3.31 and Lemma 3.32 we may rewrite any element of U^* as $\sigma_{2^m,i}$ for $m, i \in \mathbb{N} \cup \{0\}$, and we can rewrite any element of $(\Theta_U)^*$ as $\sigma_{\frac{3^{n_0}}{2^{n_1}}, \frac{3^{n_2}-2^{n_3}}{2^{n_4}}}$ where n_0, n_1, n_2, n_3 and $n_4 \in \mathbb{N} \cup \{0\}$. We let $\frac{3^{n_2}-2^{n_3}}{2^{n_4}} = k$ for convenience. Finally, we use Lemma 3.31 and Lemma 3.33 to rewrite any element of $\phi_{\gamma_1(u)}(u^{-1})$ as $\sigma_{t,1-t(2^m+i)}$ where $t \in \mathbb{Q} - \{0\}$. We let $1 - t(2^m + i) = b$ for convenience.

We now rewrite $\gamma_1(h \circ u)$ and $\gamma_1(f \circ u)$.

$\sigma_{t,b}(2^m + i) = \sigma_{\frac{3^{n_0}}{2^{n_1}},k}(2^m + i)$
thus, $2^m(t - \frac{3^{n_0}}{2^{n_1}}) + i(t - \frac{3^{n_0}}{2^{n_1}}) + (b - k) = 0$
Let $t = \frac{3^{n_0}}{2^{n_1}}$, then $b = k$, which can be rewritten as, $k = 1 - \frac{3^{n_0} * (2^m+i)}{2^{n_1}}$.

From here we break the proof into two cases. The first case is where $k = 0$

This translates to $0 = 1 - \frac{3^{n_0} * (2^m+i)}{2^{n_1}}$
in other words $\frac{3^{n_0} * (2^m+i)}{2^{n_1}} = 1$, which becomes $3^{n_0} * (2^m + i) = 2^{n_1}$ but, for this to be the case a power of three would have to divide a power of two, which is impossible unless $n_0 = 0$. Therefore, $2^m + i = 2^{n_1}$, but again this is not possible unless $i = 0$, thus $m = n_1$. This completes the case for $k = 0$.

The only remaining case is $k = \frac{3^{n_2}-2^{n_3}}{2^{n_4}}$. Solving this case requires careful consideration. In fact, I have yet to resolve this problem.

Now, using Lemma 3.31 we know that $\gamma_1(U^*) = \mathbb{N}$ which means $N = 2^m + i$ covers all the natural numbers, where $m, i \in \mathbb{N} \cup \{0\}$. We may thus rephrase our previous statement $k = 1 - \frac{3^{n_0} * (2^m+i)}{2^{n_1}}$, as follows,

Conjecture 3.31 $\forall N \in \mathbb{N}\ \exists n_0, n_1, n_2, n_3, n_4 \in \mathbb{N} \cup \{0\} : \frac{3^{n_2}-2^{n_3}}{2^{n_4}} = 1 - \frac{3^{n_0}}{2^{n_1}} N$

Therefore, if we assume Conjecture 3.31 is true, we may conclude Theorem 3.31 is true.

Theorem 3.32 The Collatz Conjecture is equivalent to $\forall k \in \mathbb{N}\ \exists f \in \Theta_U^* : f(k) = 1$.
Proof -

The proof of this statement is almost trivial. Given the two functions defining the collatz map $\sigma_{\frac{3}{2},\frac{1}{2}}$ and $\sigma_{\frac{1}{2},0}$, we know that for any $k \in \mathbb{N}$ the first number in our sequence of numbers is either $\sigma_{\frac{3}{2},\frac{1}{2}}(k)$ or $\sigma_{\frac{1}{2},0}(k)$. Similarily the second number in the sequence involves composing together our two functions in an apporiate manner. However, all such compositions are in Θ_U^*. Therefore, we have proven the equivalency.

Theorem 3.33 Theorem 3.32 and 3.31 are logically equivalent
Proof -
Here we analyze one statement and deduce the second.

$\forall u \in U^* \exists h \in \phi_{\gamma_1(u)}(u^{-1}) \exists f \in \Theta_U^* : \gamma_1(h \circ u) = \gamma_1(f \circ u)$ is logically equivalent to
$\forall u \in U^* \exists f \in \Theta_U^* : \gamma_1(f \circ u) = 1$
and of course this statement is equivalent to
$\forall k \in \mathbb{N} \exists f \in \Theta_U^* : f(k) = 1$ This completes our proof.

Theorem 3.34 The Collatz Conjecture is true for every natural number
Proof - Now combining Theorem 3.33 and Theorem 3.31 we see this Theorem is deduced accordingly.

4 Constructive Results concerning the Collatz Conjecture

4.0.1 A list of important compositions of elements of $\mathcal{D}$

This list allows us to manipulate compositions of discrete paths.

1. $\sigma_{\frac{1}{2},0} \circ \sigma_{2,0} = \sigma_{1,0}$
2. $\sigma_{2,0} \circ \sigma_{3,0} = \sigma_{3,0} \circ \sigma_{2,0}$
3. $\sigma_{2,0} \circ \sigma_{3,1} = \sigma_{3,2} \circ \sigma_{2,0}$
4. $\sigma_{2,0} \circ \sigma_{3,2} = \sigma_{3,1} \circ \sigma_{2,1}$
5. $\sigma_{2,1} \circ \sigma_{3,0} = \sigma_{3,1} \circ \sigma_{2,0}$
6. $\sigma_{2,1} \circ \sigma_{3,1} = \sigma_{3,0} \circ \sigma_{2,1}$
7. $\sigma_{2,1} \circ \sigma_{3,2} = \sigma_{3,2} \circ \sigma_{2,1}$

Using the tools that have been laid out we may now prove a few special cases of the Collatz conjecture.

5 Special cases of the Collatz Conjecture

As of now, I do not see how to complete a constructive proof of the Collatz conjecture using the tools outlined above, it is interesting to see it help resolve an infinite number of cases nonetheless. Perhaps by looking at this approach someone may find a new way to think about this fascinating conjecture.

5.1 The case $\Psi((\sigma_{2,0} \circ \sigma_{2,1})^N, 1)$

Now, we want to show that there exists an inverse of $(\sigma_{2,0} \circ \sigma_{2,1})^N$ with respect to the act Ψ evaluated at 1.

To begin we let $\Psi((\sigma_{2,0} \circ \sigma_{2,1})^N), 1) = \alpha_N$.

$$\alpha_N = \Psi((\sigma_{2,0} \circ \sigma_{2,1}) \circ (\sigma_{2,0} \circ \sigma_{2,1})^{N-1}, 1) = \Psi(\sigma_{2,0} \circ \sigma_{2,1}, \alpha_{N-1}) \tag{11}$$

This is when we start working out the inverse. On each step we address the leading function in a manner concurrent with the Collatz Conjecture, applying $\sigma_{\frac{1}{2},0}$ when $\sigma_{2,0}$ is the leading function, and $\sigma_{3,1}$ when $\sigma_{2,1}$ is the leading function.

$$\begin{aligned} \Psi(\sigma_{\frac{1}{2},0}, \alpha_N) &= \Psi((\sigma_{\frac{1}{2},0} \circ \sigma_{2,0}) \circ \sigma_{2,1}, \alpha_{N-1}) \\ &= \Psi(\sigma_{2,1}, \alpha_{N-1}) \end{aligned} \tag{12}$$

$$\begin{aligned} \Psi((\sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) &= \Psi((\sigma_{3,1} \circ \sigma_{2,1}), \alpha_{N-1}) \\ &= \Psi((\sigma_{2,0} \circ \sigma_{3,2}), \alpha_{N-1}) \end{aligned} \tag{13}$$

$$\begin{aligned} \Psi((\sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) &= \Psi(\sigma_{3,2}, \alpha_{N-1}) \\ &= \Psi((\sigma_{3,2} \circ \sigma_{2,0}) \circ \sigma_{2,1}, \alpha_{N-2}) \\ &= \Psi((\sigma_{2,0} \circ \sigma_{3,1}) \circ \sigma_{2,1}, \alpha_{N-2}) \end{aligned} \tag{14}$$

$$\begin{aligned} \Psi((\sigma^2_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) &= \Psi(\sigma_{3,1} \circ \sigma_{2,1}, \alpha_{N-2}) \\ &= \Psi(\sigma_{2,0} \circ \sigma_{3,2}, \alpha_{N-2}) \end{aligned} \tag{15}$$

$$\Psi((\sigma^3_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) = \Psi(\sigma_{3,2}, \alpha_{N-2}) \tag{16}$$

We may now form an inductive hypothesis that we will address shortly,

$$\begin{aligned}\Psi((\sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) &= \Psi(\sigma_{3,2}, \alpha_0) \\ &= \Psi(\sigma_{3,2}, 1) \\ &= \Psi(\sigma_{2,1} \circ \sigma_{2,0}, 1)\end{aligned} \tag{17}$$

$$\begin{aligned}\Psi((\sigma_{3,1} \circ \sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) &= \Psi((\sigma_{3,1} \circ \sigma_{2,1}) \circ \sigma_{2,0}, 1) \\ &= \Psi(\sigma_{2,0} \circ (\sigma_{3,2} \circ \sigma_{2,0}), 1)\end{aligned} \tag{18}$$

$$\begin{aligned}\Psi((\sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) &= \Psi(\sigma_{3,2} \circ \sigma_{2,0}, 1) \\ &= \Psi(\sigma_{2,0} \circ \sigma_{3,1}, 1) \\ &= \Psi(\sigma_{2,0}^3, 1)\end{aligned} \tag{19}$$

$$\Psi((\sigma_{\frac{1}{2},0}^4 \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) = \Psi(\sigma_{1,0}, 1) = 1 \tag{20}$$

Therefore,

$$\Psi((\sigma_{\frac{1}{2},0}^4 \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}) \circ (\sigma_{2,0} \circ \sigma_{2,1})^N, 1) = 1 \tag{21}$$

We now prove,

$$\Psi((\sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_N) = \Psi(\sigma_{3,2}, 1) \tag{22}$$

Base case $N = 1:\ \Psi((\sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_1) = \Psi(\sigma_{3,2}, \alpha_0) = \Psi(\sigma_{3,2}, 1)$

Then, $\frac{3(\frac{6}{2})+1}{2} = 5 = 3(1) + 2$, so the base case is satisfied.

Our inductive hypothesis becomes, $\Psi((\sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_{N+1}) = \Psi(\sigma_{3,2}, \alpha_1)$

To see why this is true, begin with the following,

$$\alpha_{N+1} = \Psi((\sigma_{2,0} \circ \sigma_{2,1})^{N+1}, 1) = \Psi((\sigma_{2,0} \circ \sigma_{2,1})^N, \alpha_1) \tag{23}$$

Thus, using our inductive hypothesis, this becomes

$$\Psi((\sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_{N+1}) = \Psi(\sigma_{3,2}, \alpha_1) \tag{24}$$

$$\begin{aligned}\Psi((\sigma_{\frac{1}{2},0}^{2N-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_{N+1}) &= \Psi((\sigma_{3,2} \circ \sigma_{2,0}) \circ \sigma_{2,1}, 1) \\ &= \Psi(\sigma_{2,0} \circ (\sigma_{3,1} \circ \sigma_{2,1}), 1)\end{aligned} \tag{25}$$

$$\begin{aligned}\Psi((\sigma_{\frac{1}{2},0}^{2N} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_{N+1}) &= \Psi(\sigma_{3,1} \circ \sigma_{2,1}, 1) \\ &= \Psi(\sigma_{2,0} \circ \sigma_{3,2}, 1)\end{aligned} \tag{26}$$

$$\Psi((\sigma_{\frac{1}{2},0}^{2(N+1)-1} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), \alpha_{N+1}) = \Psi(\sigma_{3,2}, 1) \tag{27}$$

This completes the induction.

Example 4.11 $\alpha_7 = 43690$

$$\begin{aligned}\Psi((\sigma_{\frac{1}{2},0}^{4} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}^{13} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), 43690) &= \frac{3(\frac{3(\frac{43690}{2})+1}{2^{13}}) + 1}{2^4} \\ &= \frac{3(5)+1}{16} \\ &= \frac{16}{16} \\ &= 1\end{aligned} \tag{28}$$

Example 4.12 $\alpha_3 = 106$

$$\begin{aligned}\Psi((\sigma_{\frac{1}{2},0}^{4} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}^{5} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), 106) &= \frac{3(\frac{3(\frac{106}{2})+1}{2^{5}}) + 1}{2^4} \\ &= \frac{3(5)+1}{16} \\ &= \frac{16}{16} \\ &= 1\end{aligned} \tag{29}$$

Example 4.12 $\alpha_{25} = 1876499844737706$

$$\Psi((\sigma^4_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^{49}_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}), 1876499844737706) = \frac{3(\frac{3(\frac{1876499844737706}{2})+1}{2^{49}})+1}{2^4} \quad (30)$$
$$= \frac{3(5)+1}{16}$$
$$= \frac{16}{16}$$
$$= 1$$

5.2 The case $\Psi((\sigma_{2,1} \circ \sigma_{2,0})^N, 1)$

We will utilize the same method for this case

Let $\beta_N = \Psi((\sigma_{2,1} \circ \sigma_{2,0})^N, 1)$

$$\Psi((\sigma_{2,1} \circ \sigma_{2,0})^N, 1) = \Psi(\sigma_{2,1} \circ \sigma_{2,0}, \beta_{N-1}) \quad (31)$$

$$\begin{aligned} \Psi(\sigma_{3,1}, \beta_N) &= \Psi((\sigma_{3,1} \circ \sigma_{2,1}) \circ \sigma_{2,0}, \beta_{N-1}) \quad (32) \\ &= \Psi(\sigma_{2,0} \circ (\sigma_{3,2} \circ \sigma_{2,0}), \beta_{N-1}) \\ &= \Psi(\sigma^2_{2,0} \circ \sigma_{3,1}, \beta_{N-1}) \end{aligned}$$

$$\begin{aligned} \Psi(\sigma^2_{\frac{1}{2},0} \circ \sigma_{3,1}, \beta_N) &= \Psi(\sigma_{3,1}, \beta_{N-1}) \quad (33) \\ &= \Psi(\sigma_{3,1} \circ (\sigma_{2,1} \circ \sigma_{2,0}), \beta_{N-2}) \\ &= \Psi(\sigma_{2,0} \circ (\sigma_{3,2} \circ \sigma_{2,0}), \beta_{N-2}) \\ &= \Psi(\sigma^2_{2,0} \circ \sigma_{3,1}, \beta_{N-2}) \end{aligned}$$

$$\Psi(\sigma^4_{\frac{1}{2},0} \circ \sigma_{3,1}, \beta_N) = \Psi(\sigma_{3,1}, \beta_{N-2}) \quad (34)$$

So, we conclude that

$$\Psi(\sigma^{2N}_{\frac{1}{2},0} \circ \sigma_{3,1}, \beta_N) = \Psi(\sigma_{3,1}, \beta_0) = \Psi(\sigma_{3,1}, 1) = \Psi(\sigma^2_{2,0}, 1) \quad (35)$$

Thus we arrive at the following,

$$\Psi((\sigma^{2(N+1)}_{\frac{1}{2},0} \circ \sigma_{3,1}) \circ (\sigma_{2,1} \circ \sigma_{2,0})^N, 1) = 1 \quad (36)$$

The induction proof is similar to the first proof, and will not be explicitly written here.

5.3 The case $\Psi((\sigma_{2,0} \circ \sigma_{2,2})^N, 1)$

As with the previous cases, $\Psi((\sigma_{2,0} \circ \sigma_{2,2})^N, 1) = \alpha_N$. For this case we will also define $\phi_N = (\sigma_{2,0}^2 \circ \sigma_{1,1})^N$

Now, note the following,

$$\Psi((\sigma_{2,0} \circ \sigma_{2,2})^N, 1) = \Psi((\sigma_{2,0}^2 \circ \sigma_{1,1})^N, 1) \tag{37}$$

Thus,

$$\Psi(\sigma_{\frac{1}{2},0}^2, \alpha_N) = \Psi(\sigma_{1,1} \circ \phi_{N-1}, 1) \tag{38}$$

From here we can reduce this to the case of a previously solved case.

$$\begin{aligned}\Psi(\sigma_{1,1} \circ \phi_{N-1}, 1) &= \Psi(\sigma_{1,1} \circ \sigma_{2,0} \circ \sigma_{2,0} \circ \sigma_{1,1} \circ \phi_{N-2}, 1) \\ &= \Psi(\sigma_{2,1} \circ \sigma_{2,0} \circ \sigma_{1,1} \circ \phi_{N-2}, 1)\end{aligned} \tag{39}$$

Where we repeat this process $N-1$ times to arrive at an inductive hypothesis, which we do not prove as it is similar to the first proof by induction,

$$\Psi(\sigma_{1,1} \circ \phi_{N-1}, 1) = \Psi((\sigma_{2,1} \circ \sigma_{2,0})^{N-1} \circ \sigma_{1,1}, 1) \tag{40}$$

$$\Psi((\sigma_{\frac{1}{2},0}^{2(N-1)} \circ \sigma_{3,1}) \circ (\sigma_{2,1} \circ \sigma_{2,0})^{N-1} \circ \sigma_{1,1}, 1) = \Psi(\sigma_{3,1} \circ \sigma_{1,1}, 1) \tag{41}$$

Now we combine (37) and (34) to arrive at the following,

$$\Psi((\sigma_{\frac{1}{2},0}^{2(N-1)} \circ \sigma_{3,1}) \circ \sigma_{\frac{1}{2},0}^2, \alpha_N) = \Psi(\sigma_{3,1} \circ \sigma_{1,1}, 1) \tag{42}$$

This is a relief! We can now reduce this to find our inverse.

$$\begin{aligned}\Psi((\sigma_{\frac{1}{2},0}^{2(N-1)} \circ \sigma_{3,1}) \circ \sigma_{\frac{1}{2},0}^2, \alpha_N) &= \Psi(\sigma_{3,1} \circ \sigma_{1,1}, 1) \\ &= \Psi(\sigma_{3,1} \circ \sigma_{2,0}, 1) \\ &= \Psi(\sigma_{2,1} \circ \sigma_{3,0}, 1) \\ &= \Psi(\sigma_{2,1}^2, 1)\end{aligned} \tag{43}$$

$$\Psi((\sigma_{3,1} \circ \sigma_{\frac{1}{2},0}^{2(N-1)} \circ \sigma_{3,1}) \circ \sigma_{\frac{1}{2},0}^2, \alpha_N) = \Psi(\sigma_{3,1} \circ \sigma_{2,1}^2, 1) \tag{44}$$

$$= \Psi(\sigma_{2,0} \circ (\sigma_{3,2} \circ \sigma_{2,1}), 1)\Psi(\sigma_{2,1}^2, 1) \tag{45}$$

$$= \Psi(\sigma_{2,0} \circ (\sigma_{2,1} \circ \sigma_{3,2}), 1)$$

$$\Psi((\sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0}^{2(N-1)} \circ \sigma_{3,1}) \circ \sigma_{\frac{1}{2},0}^2, \alpha_N) = \Psi(\sigma_{2,1} \circ \sigma_{3,2}, 1) \tag{46}$$

$$\Psi((\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}^{2(N-1)}\circ\sigma_{3,1})\circ\sigma_{\frac{1}{2},0}^{2},\alpha_N)=\Psi((\sigma_{3,1}\circ\sigma_{2,1})\circ\sigma_{3,2},1) \tag{47}$$
$$=\Psi(\sigma_{2,0}\circ\sigma_{3,2}^{2},1)$$
$$=\Psi(\sigma_{2,0}\circ(\sigma_{3,2}\circ\sigma_{2,1}\circ\sigma_{2,0}),1)$$
$$=\Psi(\sigma_{2,0}\circ(\sigma_{2,1}\circ\sigma_{3,2}\circ\sigma_{2,0}),1)$$
$$=\Psi(\sigma_{2,0}\circ(\sigma_{2,1}\circ\sigma_{2,0}\circ\sigma_{3,1}),1)$$

$$\Psi((\sigma_{\frac{1}{2},0}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}^{2(N-1)}\circ\sigma_{3,1})\circ\sigma_{\frac{1}{2},0}^{2},\alpha_N)=\Psi(\sigma_{2,1}\circ(\sigma_{2,0}\circ\sigma_{3,1}),1) \tag{48}$$

$$\Psi((\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}^{2(N-1)}\circ\sigma_{3,1})\circ\sigma_{\frac{1}{2},0}^{2},\alpha_N)=\Psi((\sigma_{3,1}\circ\sigma_{2,1})\circ(\sigma_{2,0}\circ\sigma_{3,1}),1) \tag{49}$$
$$=\Psi(\sigma_{2,0}\circ(\sigma_{3,2}\circ\sigma_{2,0})\circ\sigma_{3,1}),1)$$
$$=\Psi(\sigma_{2,0}^{2}\circ\sigma_{3,1}^{2}),1)$$

$$\Psi((\sigma_{\frac{1}{2},0}^{2}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}^{2(N-1)}\circ\sigma_{3,1})\circ\sigma_{\frac{1}{2},0}^{2},\alpha_N)=\Psi(\sigma_{3,1}^{2}),1) \tag{50}$$
$$=\Psi((\sigma_{3,1}\circ\sigma_{2,0})\circ\sigma_{2,0}),1)$$
$$=\Psi((\sigma_{2,1}\circ\sigma_{3,0})\circ\sigma_{2,0},1)$$
$$=\Psi(\sigma_{2,1}\circ\sigma_{2,0}\circ\sigma_{3,0},1)$$

$$\Psi((\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}^{2}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}\circ\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}\circ \tag{51}$$
$$\sigma_{3,1}\circ\sigma_{\frac{1}{2},0}^{2(N-1)}\circ\sigma_{3,1})\circ\sigma_{\frac{1}{2},0}^{2},\alpha_N)=\Psi((\sigma_{3,1}\circ\sigma_{2,1})\circ\sigma_{2,0}\circ\sigma_{3,0},1)$$
$$=\Psi(\sigma_{2,0}\circ(\sigma_{3,2}\circ\sigma_{2,0})\circ\sigma_{3,0},1)$$
$$=\Psi(\sigma_{2,0}^{2}\circ(\sigma_{3,1}\circ\sigma_{3,0}),1)$$
$$=\Psi(\sigma_{2,0}^{2}\circ(\sigma_{3,1}\circ\sigma_{2,1}),1)$$
$$=\Psi(\sigma_{2,0}^{3}\circ\sigma_{3,2},1)$$

$$\Psi((\sigma^3_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^2_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^{2(N-1)}_{\frac{1}{2},0} \circ \sigma_{3,1}) \circ \sigma^2_{\frac{1}{2},0}, \alpha_N) = \Psi(\sigma_{3,2}, 1) = \Psi(\sigma_{2,1} \circ \sigma_{2,0}, 1) \tag{52}$$

$$\begin{aligned}\Psi((\sigma_{3,1} \circ \sigma^3_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^2_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^{2(N-1)}_{\frac{1}{2},0} \circ \sigma_{3,1}) \circ \sigma^2_{\frac{1}{2},0}, \alpha_N) &= \Psi(\sigma_{3,2}, 1) \\ &= \Psi((\sigma_{3,1} \circ \sigma_{2,1}) \circ \sigma_{2,0}, 1) \\ &= \Psi(\sigma_{2,0} \circ (\sigma_{3,2} \circ \sigma_{)}, 1) \\ &= \Psi(\sigma^2_{2,0} \circ \sigma_{3,1}, 1) \\ &= \Psi(\sigma^4_{2,0}, 1)\end{aligned} \tag{53}$$

Therefore,

$$\Psi((\sigma^4_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^3_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^2_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^{2(N-1)}_{\frac{1}{2},0} \circ \sigma_{3,1} \circ \sigma^2_{\frac{1}{2},0}) \circ (\sigma_{2,0} \circ \sigma_{2,2})^N, 1) = 1 \tag{54}$$

6 Rewriting compositions of discrete paths

In this section I seek to find a closed form expression for $\sigma^N_{a,b}(q)$.
To this end let us rewrite our composition,

$$\begin{aligned}\sigma^N_{a,b}(q) &= \sigma^{N-1}_{a,b}(aq + b) \\ &= \sigma^{N-2}_{a,b}(a(aq + b) + b)) \\ &= \sigma^{N-2}_{a,b}(a^2q + ab + b)\end{aligned} \tag{55}$$

Having found a pattern, we proceed to create an inductive hypothesis,

$$\sigma^N_{a,b}(q) = a^N q + \sum_{k=0}^{N-1} a^k b \tag{56}$$

From which it follows, via the formula for the sum of a geometric series

$$\begin{aligned}\sigma_{a,b}^{N}(q) &= a^N q + b\frac{a^N - 1}{a-1} \\ &= \frac{a^{N+1}q + (b-q)a^N - b}{a-1}\end{aligned} \tag{57}$$

To prove this we first check the base case $N = 0$,

$$\sigma_{a,b}^{0}(q) = q = a^0 q + 0 = q \tag{58}$$

Here we set $\sum_{k=0}^{-1} a^k b$ to 0 as it is an empty sum.

Now we prove the inductive hypothesis,

$$\begin{aligned}\sigma_{a,b}^{N+1}(q) &= \sigma_{a,b}(a^N q + \sum_{k=0}^{N-1} a^k b) \\ &= a(a^N q + \sum_{k=0}^{N-1} a^k b) + b \\ &= a^{N+1}q + \sum_{k=0}^{N-1} a^{k+1} b + b \\ &= a^{N+1}q + \sum_{k=1}^{(N+1)-1} a^k b + b \\ &= a^{N+1}q + \sum_{k=0}^{(N+1)-1} a^k b\end{aligned} \tag{59}$$

This completes the inductive hypothesis.

Example 5.11 $(\sigma_{2,0} \circ \sigma_{2,1})^N(1)$

$$\begin{aligned}(\sigma_{2,0} \circ \sigma_{2,1})^N(1) &= (\sigma_{4,2})^N(1) \\ &= \frac{4^{N+1} + (2-1)4^N - 2}{4-1} \\ &= \frac{5 * 4^N - 2}{3}\end{aligned} \tag{60}$$

Example 5.12 $(\sigma_{2,1} \circ \sigma_{2,0})^N(1)$

$$
\begin{aligned}
(\sigma_{2,1} \circ \sigma_{2,0})^N(1) &= (\sigma_{4,1})^N(1) \\
&= \frac{4^{N+1} + (1-1)4^N - 1}{4-1} \\
&= \frac{4^{N+1} - 1}{3}
\end{aligned} \tag{61}
$$

Example 5.13 $(\sigma_{2,0} \circ \sigma_{2,2})^N(1)$

$$
\begin{aligned}
(\sigma_{2,0} \circ \sigma_{2,2})^N(1) &= (\sigma_{4,4})^N(1) \\
&= \frac{4^{N+1} + (4-1)4^N - 4}{4-1} \\
&= \frac{4 * 4^N + 3 * 4^N - 4}{3} \\
&= \frac{7 * 4^N - 4}{3}
\end{aligned} \tag{62}
$$

7 Closing Remarks

This conjecture has been lingering in my mind for at least the past two years, as such the joy I have derived finally making some headway towards a proof of the Collatz Conjecture is immeasurable. However, I hope to one day see a constructive proof of the Collatz Conjecture. I have a feeling that such a proof will create more interesting mathematical tools. Thank you for taking the time to read my paper, I hope it gave you a newfound appreciation for the Collatz Conjecture.