Gateway to the Riemann Hypothesis: Hidden Symmetry in the Dirichlet Eta Function

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0. Abstract

Any prospective proof of the infamous Riemann hypothesis might be facilitated to some extent by the discovery of an infinite series bearing all of the nontrivial roots of the Riemann zeta function $\zeta(z)$, that is, those which exist on the critical strip -- and no others. Absent such a series, our goal here is to make progress in that direction. (By definition, the critical strip includes only points with real parts on the open unit interval. The critical *line* is the region with real part (1/2).)

To that end, at least within the domain of the critical strip, it might suffice to reformulate the Dirichlet eta function $\eta(z)$, given by:

$$\eta(z) = \sum_{a=1}^{\infty} \frac{(-1)^{a-1}}{a^{z}} \quad (0)$$

where z is a complex number AKA $(x+iy)$. Critically, for reasons we'll discuss below, the nontrivial roots of $\zeta(z)$ and $\eta(z)$ are identical.

Per [0]: "While the Dirichlet series expansion for the eta function is convergent only for any complex number s with [positive] real part... it is Abel summable for any complex number. This serves to define the eta function as an entire function." (For its part, Abel summation [1] yields logically consistent sums for certain divergent series.)

Therefore we'll work with $\eta(z)$ rather than $\zeta(z)$ (which is not entire). To wit, the Riemann hypothesis is equivalent to the statement that all of the zeros of $\eta(z)$ falling in the critical strip lie on the critical line [2].

1. From Reciprocal Series to Log Series

We begin with the derivation of a well-known identity pertaining to exponentiation. (We make no attempt herein to distinguish equations from identities, so we always use "=" instead of "≡".)

1.1. Expressing a Power as a Log Series

Given some natural (positive whole) number *a* raised to some real power *x*, we can express the result with the assistance of the exponential series:

$$e^{a} = \sum_{b=0}^{\infty} \frac{a^{b}}{b!} \quad (1)$$

from basic calculus, wherein 0^0 must be taken as one, herein and henceforth unless otherwise stated. It follows that

$$\begin{aligned} a^{x} &= \left(e^{ln\, a}\right)^{x} \\ &= e^{x\, ln\, a} \\ &= \sum_{b=0}^{\infty} \frac{(x\, ln\, a)^{b}}{b!} \\ a^{x} &= \sum_{b=0}^{\infty} \frac{x^{b}(ln\, a)^{b}}{b!} \quad (2) \end{aligned}$$

which is the known identity mentioned above.

1.2. Extending the Exponential Series to Complex Powers

As we can see in (1), computing a real power of an arbitrary natural base involves only one type of operation on the power itself, namely the iterated multiplication implied by x^b. (I mean this notionally; repeated multiplication is inefficient.) Said operation is defined for all complex powers ($x+iy$). So, more generically:

$$\begin{aligned} a^{x+iy} &= a^{iy} a^{x} \\ &= a^{iy} \sum_{b=0}^{\infty} \frac{x^{b}(ln\, a)^{b}}{b!} \\ &= \left(e^{ln\, a}\right)^{iy} \sum_{b=0}^{\infty} \frac{x^{b}(ln\, a)^{b}}{b!} \\ a^{x+iy} &= e^{i(y\, ln\, a)} \sum_{b=0}^{\infty} \frac{x^{b}(ln\, a)^{b}}{b!} \quad (3) \end{aligned}$$

at which point we recall the definition of a purely imaginary exponential:

$$e^{iy} = cos\, y + i\, sin\, y \quad (4)$$

so that

$$e^{i(y\ln a)} = \cos(y\ ln\ a) + i\sin(y\ ln\ a) \text{ (5)}$$

which finally implies that

$$a^{x+iy} = (\cos(y\ ln\ a) + i\sin(y\ ln\ a))\sum_{b=0}^{\infty}\frac{x^b(ln\ a)^b}{b!} \text{ (6)}$$

Now, from basic calculus, the series for cosine and sine are as follows:

$$\cos x = \sum_{b=0}^{\infty}\frac{(-1)^b x^{2b}}{(2b)!} \text{ (7)}$$

$$\sin x = \sum_{b=0}^{\infty}\frac{(-1)^b x^{2b+1}}{(2b+1)!} \text{ (8)}$$

which straightforwardly implies that (a^{x+iy}) has real and imaginary parts involving only even and only odd powers of *y*, respectively. To wit:

$$\Re\left(a^{x+iy}\right) = \cos(y\ ln\ a)\sum_{b=0}^{\infty}\frac{x^b(ln\ a)^b}{b!}$$

$$= \sum_{b=0}^{\infty}\frac{(-1)^b(y\ ln\ a)^{2b}}{(2b)!}\sum_{c=0}^{\infty}\frac{x^c(ln\ a)^c}{c!}$$

$$\Re\left(a^{x+iy}\right) = \sum_{b=0}^{\infty} y^{2b}\frac{(-1)^b(ln\ a)^{2b}}{(2b)!}\sum_{c=0}^{\infty} x^c\frac{(ln\ a)^c}{c!} \text{ (9)}$$

$$\Im\left(a^{x+iy}\right) = \sin(y\ ln\ a)\sum_{b=0}^{\infty}\frac{x^b(ln\ a)^b}{b!}$$

$$= \sum_{b=0}^{\infty}\frac{(-1)^b(y\ ln\ a)^{2b+1}}{(2b+1)!}\sum_{c=0}^{\infty}\frac{x^c(ln\ a)^c}{c!}$$

$$\Im\left(a^{x+iy}\right) = \sum_{b=0}^{\infty} y^{2b+1}\frac{(-1)^b(ln\ a)^{2b+1}}{(2b+1)!}\sum_{c=0}^{\infty} x^c\frac{(ln\ a)^c}{c!} \text{ (10)}$$

Now, thanks to the parity separation involving powers of *y*, we can combine the above expressions succinctly as follows:

$$a^{x+iy} = \sum_{b=0}^{\infty} y^b\frac{i^b(ln\ a)^b}{b!}\sum_{c=0}^{\infty} x^c\frac{(ln\ a)^c}{c!} \text{ (11)}$$

1.3. From Exponential Series to Dirichlet Eta

Looking back at (0), we need to sum over all natural *a* like so:

$$\eta(x + iy) = \sum_{a=1}^{\infty} \frac{(-1)^{a-1}}{a^{x+iy}} \quad (12)$$

which, upon substitution from (11) and with the required negation of *x* and *y* from the denominator, yields

$$\eta(x + iy) = \sum_{a=1}^{\infty} (-1)^{a-1} \sum_{b=0}^{\infty} \frac{(-y)^b i^b (ln\, a)^b}{b!} \sum_{c=0}^{\infty} \frac{(-x)^c (ln\, a)^c}{c!}$$

$$\eta(x + iy) = \sum_{a=1}^{\infty} (-1)^{a-1} \sum_{b=0}^{\infty} y^b \frac{(-i)^b (ln\, a)^b}{b!} \sum_{c=0}^{\infty} x^c \frac{(-1)^c (ln\, a)^c}{c!} \quad (13)$$

On closer inspection, (13) implies a matrix of terms crossing all whole powers of (*x* ln *a*) with all those of (*y* ln *a*), the grand sum of which being multiplied by ($(-1)^{a-1}$). (Granted, there are no meaningful vectors involved, so some might prefer the term "array".) This elegant symmetry isn't obvious from the original representation of $\eta(z)$.

Now, from basic calculus:

$$ln\, x = -\sum_{b=1}^{\infty} \frac{(-1)^b (x-1)^b}{b} \quad (14)$$

where ($0<x\leq2$). This is of little use for computing the logs of naturals, however, so some rearrangement is necessary:

$$ln\, a = -\, ln\, \frac{1}{a}$$

$$= -\left(-\sum_{b=1}^{\infty} \frac{(-1)^b \left(\frac{1}{a}-1\right)^b}{b}\right)$$

$$ln\, a = \sum_{b=1}^{\infty} \frac{\left(1-\frac{1}{a}\right)^b}{b} \quad (15)$$

which, upon substitution back into (13), yields the following computationally intuitive form:

$$\eta(x + iy) = \sum_{a=1}^{\infty} (-1)^{a-1} \sum_{b=0}^{\infty} y^b \frac{(-i)^b}{b!} \left(\sum_{d=1}^{\infty} \frac{\left(1-\frac{1}{a}\right)^d}{d}\right)^b \sum_{c=0}^{\infty} x^c \frac{(-1)^c}{c!} \left(\sum_{d=1}^{\infty} \frac{\left(1-\frac{1}{a}\right)^d}{d}\right)^c \quad (16)$$

But there's a more efficient representation if we target only the critical line.

2. Optimizing for the Critical Line

There are various series available for $\eta(z)$, AKA $\eta(x+iy)$, depending upon the domain of concern. Below we consider consider the critical line (x=(1/2)).

2.1. Bohac's Critical Line Eta

To begin with, there's a reason to put powers of y one nesting level above those of x, as in (16): sometimes we're only interested in the behavior of the function on vertical lines on the critical strip, wherein x is held constant and we're attempting to construct an infinite series in y. Bohac [3] did exactly this for the critical line, which is a special case of (13):

$$\eta\left(\frac{1}{2} + iy\right) = \sum_{a=1}^{\infty} (-1)^{a-1} \sum_{b=0}^{\infty} y^{b} \frac{(-i)^{b}(\ln a)^{b}}{b!} \sum_{c=0}^{\infty} \left(\frac{1}{2}\right)^{c} \frac{(-1)^{c}(\ln a)^{c}}{c!} \quad (17)$$

wherein the innermost sum is just

$$\frac{1}{\sqrt{a}}$$

so

$$\eta\left(\frac{1}{2} + iy\right) = \sum_{a=1}^{\infty} \frac{(-1)^{a-1}}{\sqrt{a}} \sum_{b=0}^{\infty} y^{b} \frac{(-i)^{b}(\ln a)^{b}}{b!} \quad (18)$$

which, in effect, he then reorganized to

$$\eta\left(\frac{1}{2} + iy\right) = \sum_{a=0}^{\infty} y^{a} \frac{(-i)^{a}}{a!} \sum_{b=1}^{\infty} \frac{(-1)^{b-1}(\ln b)^{a}}{\sqrt{b}} \quad (19)$$

at risk of divergence due to swapping the order of summation. For that matter, he made no attempt to prove convergence. But putting that question aside for the moment, we then have the following infinite series in y on the critical line:

$$\eta\left(\frac{1}{2} + iy\right) = \sum_{a=0}^{\infty} \left(j_{a} y^{2a} - i k_{a} y^{2a+1}\right) \quad (20)$$

for which he then effectively derived that

$$j_{a} = \frac{(-1)^{a}}{(2a)!} \sum_{b=1}^{\infty} \frac{(-1)^{b-1}(\ln b)^{2a}}{\sqrt{b}} \quad (21)$$

and

$$k_a = \frac{(-1)^a}{(2a+1)!} \sum_{b=1}^{\infty} \frac{(-1)^b (ln\, b)^{2a+1}}{\sqrt{b}} \quad (22)$$

In section 2.3, we'll generalize this formula to the critical strip and revisit the question of convergence. But first, it's important to point out that we've inadvertently invited some "distracting" roots along for the ride which can only muddy the waters, even if we assume the unproven assumption that all nontrivial roots are simple (and therefore don't obscure themselves by virtue of plurality).

2.2. Speculations Regarding the Distracting Roots

One drawback of (13) is that it also includes 2 classes of distracting roots: (1) the trivial roots of the Riemann zeta function (the negative even integers) and (2) the so-called "eta-specific roots" [4] ((1+i2π*a*/(ln 2)) where *a* is a nonzero integer). As the name implies, the second group doesn't apply to *ζ*(*z*). While the distracting roots don't necessarily preclude a proof of the Riemann hypothesis based on any series such as (13) which includes them, they do complicate matters as they might obscure the nontrivial ones. I suspect, but can't prove, that all of these distracting roots are simple. Note also that setting (*x*=(1/2)) does not magically abrogate them; they still exist by virtue of complex solutions to *y* (which is typically defined to be real, so that *iy* is imaginary, but we have no way of enforcing this).

One potential route to a proof of the Riemann hypothesis would be the exploitation of an information theoretic argument to the effect that an adaptation of (13) lacking the distracting roots somehow has no "storage" left for any roots on the critical strip other than those on the critical line. Descartes' rule of signs [5] comes to mind, which informally states that the maximum number of real roots of a polynomial is implied by its number of sign changes when the coefficients are sorted by degree. The problem with this strategy -- even if we could eliminate the distracting roots -- is that the number is only a *maximum*, leaving the possibility that some of the roots occur as complex conjugates. Sturm's theorem [6] might be more useful, but on the other hand, it's an iterative method that might not inductively generalize to an infinite series. Other ideas might be found in the "See Also" section at the bottom of [5].

Fortunately, at least, the eta-specific roots can be eliminated outright using the following identity from [7]. It applies on the critical strip as well as the line (*x*=1), save for the pole it will create at (*z*=1):

$$\zeta(z) = \frac{1}{1-2^{1-z}} \eta(z) \quad (23)$$

This is why the nontrivial roots and their respective multiplicities are shared between *η*(*z*) and *ζ*(*z*): the scaling factor in (23) is nonzero and nonsingular throughout the critical strip.

But what we if don't want to create a pole (because it might perhaps interfere the convergence of some particular series)? In this case, division by an appropriate rotation, scaling, and translation of the sinc function -- so that its roots align exactly with the eta-specific roots -- would be an alternative approach, to wit:

$$\frac{\eta(x+iy)}{sinc\left(y\frac{(ln\,2)}{2}+i(x-1)\right)} \text{ (24)}$$

Beware that this is speculation on my part which, even if correct, may or may not be simpler than dealing with (23). But it looks promising, as you can see in [8] and [9].

Now, continuing my speculation... the *trivial* roots should be eliminated by multiplication by ($z\Gamma(z/2)$) [10]. Why? Well, first of all, we multiply by z in order to eliminate the singularity due to $\Gamma(0)$ itself. Then again by $\Gamma(z/2)$ in order to "pull up the tacks in the carpet" and remove all the zeroes of $\eta(z)$ located at the negative even integers. (Informally, this is accomplished by multiplying $\eta(z)$ infinitesimals by $\Gamma(z/2)$ infinities in a limiting scenario near each root, resulting in nonzero finite limits, the result of which is shown in [11].) Unfortunately, the series used to compute $\Gamma(z)$ (via its reciprocal [12]) involves recursively-defined coefficients of exploding complexity, which would seem to preclude any easy path to reasoning about the product ($z\Gamma(z/2)\eta(z)$). Alternatively, one could also simply multiply $\eta(z)$ by the reciprocal of ((z+2)(z+4)(z+6)...) but said reciprocal expands into a "polynomial" with hyperreal [13] coefficients; this has much to do with the Dirac delta function [14]. Should you wish to explore this thorny issue, bear in mind the following identity, which is responsible for the aforementioned Dirac delta behavior. As rearranged from [15],

$$\prod_{a=1}^{A}\frac{1}{z+2a}=\frac{1}{2^{A-1}(A-1)!}\sum_{a=1}^{A}\frac{(-1)^{a-1}}{z+2a}(A\,-\,1,\,a\,-\,1) \text{ (25)}$$

where the last term in parentheses is a combination ("binomial coefficient"). Clearly, *A* must be taken as approaching infinity. It may also be useful to substitute a Taylor series for the (1/(z+2a)) terms. Note that multiplying both sides by ($2^A A!$) will yield an expression which, when multiplied by $\eta(z)$, will not alter its constant term. (This arises from the manner in which infinite series can be constructed from their roots.) Either way, though, the product will involve hyperreal coefficients which should neither be treated as zero nor infinity, lest they lose their residual information content regarding the nontrivial roots. By the way, $\Gamma(z/2)$ is structurally similar under the hood but contains a mitigating term which prevents the generation of hyperreals, at the cost of greater complexity. Alas, despite utilizing all of the aforementioned tactics, I haven't as yet been able to divide away the trivial roots. Doing so might go a long way toward "decrypting" the structure of a series bearing only the *nontrivial* ones.

2.3. From Critical Line to Critical Strip

We'll now derive a straightforward generalization of Bohac's critical line eta to the critical strip, after which we can assess its convergence.

The generalization is trivial when one considers that the critical strip includes all complex numbers with real parts on the open unit interval. We have only to replace the square root term with a generic value of *x* from that domain:

$$\eta(x + iy) = \sum_{a=0}^{\infty}\left(j_a(x)y^{2a} - ik_a(x)y^{2a+1}\right) \quad (26)$$

where

$$j_a(x) = \frac{(-1)^a}{(2a)!}\sum_{b=1}^{\infty}\frac{(-1)^{b-1}(\ln b)^{2a}}{b^x} \quad (27)$$

and

$$k_a(x) = \frac{(-1)^a}{(2a+1)!}\sum_{b=1}^{\infty}\frac{(-1)^{b}(\ln b)^{2a+1}}{b^x} \quad (28)$$

While the foregoing coefficient definitions hide the rich symmetry of (13), their compact form might make them more tractable. Indeed, Bohac himself seems to have started with the critical strip, but never quite stated the formulae above.

Kono, who has published an extensive analysis of $\zeta(z)$ and related functions, features (26) in a slightly different form in his Formula 8.4.1 [16]. Both are Maclaurin series in y but Dirichlet series [17] in x (a "Dirichlet-Maclaurin series" in his terminology), which leaves much to be desired by way of simplification. So below, we'll investigate the possibility of a Maclaurin series in z.

For now, note that the terms in the series in (27) and (28) alternate in sign. Therefore we'll attempt to exploit the alternating series test...

Consider the magnitude of each term t_b in the series embedded in $k_a(x)$:

$$\left|t_b\right| = \frac{(\ln b)^{2a+1}}{b^x} \quad (29)$$

Take the $(2a+1)^{th}$ root of both sides:

$$\sqrt[2a+1]{\left|t_b\right|} = \frac{\ln b}{b^{\frac{x}{(2a+1)}}} \quad (30)$$

Obviously the RHS of (30) features a numerator and denominator growing logarithmically and polynomially, respectively, in the same natural b. (Note that the exponent in the denominator is on the open unit interval because x itself resides there.) In the limit of b approaching infinity, both sides of (30) are therefore zero. Furthermore, due to the asymptotic dominance of the exponentiated denominator over the logarithmic numerator, there exists some index ($B \geq 2$) for which the RHS of (30) will strictly exceed the same ratio as computed at index (B+1), and similarly for all ($b > B$). And thus the same holds true for (29). So by the alternating series test, the series embedded in (28) converges for all a. By extension, and handling 0^0 in the usual way, the same can be shown to be true for (27). But we have a bigger problem.

The convergence of $j_a(x)$ and $k_a(x)$ do not necessarily imply that the RHS of (26) *itself* actually converges anywhere on the critical strip. First of all, it's not clear that the foregoing coefficients actually alternate in sign for successive values of *a*, even if we consider them as separate (even and odd) series. Although the $(-1)^a$ outside of both embedded series would suggest that, recall that the magnitudes $|t_b|$ might not reach a maximum for some finite number of terms of $j_a(x)$ or $k_a(x)$ prior to entering the monotonic decline phase. Naively, that might imply a complicated sign pattern in (26), thereby defeating any attempt to apply the alternating series test. If it does nevertheless converge, then we might need the integral test combined with interval bounding of periodic functions to prove it. (In other words, we might use an analog function to supply upper and lower bounds of the sum, then show that they jointly imply that the sum is finite.)

2.4. Approximation Error with Bohac's Coefficients

The error due to truncation of an alternating series of strictly and monotonically decreasing magnitude is bounded by the magnitude of the last term summed. But, again, the terms of (27) and (28) are not *always* monotonically decreasing in magnitude with increasing *b*. Therefore, despite appearances, the error induced by truncation isn't always trivially boundable. Even if that were the case, (26) could still manifest the aforementioned complicated sign alternation, precluding this approach to its own error bounding.

3. A Maclaurin Series in *z*?

It would be desirable to get rid of the potentially transcendental terms in the denominators of (27) and (28). After all, they hide an enormous amount of machinery. (13) has already done this for us, and it reveals the striking internal symmetry of $\eta(z)$. But perhaps there's further simplification to be had. To begin with, (13) can be consolidated to

$$\eta(x + iy) = \sum_{a=1}^{\infty} (-1)^{a-1} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} (-x)^{c} (-iy)^{b} \frac{(\ln a)^{b+c}}{b!c!} \quad (31)$$

Now we're going to alter the order of summation so as to keep (*b*+*c*) within the innermost sum.

$$\eta(x + iy) = \sum_{a=1}^{\infty} (-1)^{a-1} \sum_{b=0}^{\infty} \sum_{c=0}^{b} (-x)^{c} (-iy)^{b-c} \frac{(\ln a)^{b}}{(b-c)!c!} \quad (32)$$

Note the the role of the indexes *b* and *c* has changed from the previous step, such that *b* alone now behaves as (*b*+*c*) did previously. Next we'll multiply each inner term by the combination (*b, c*) times its own reciprocal, and then expand that product into its factorial constituents for cancellation purposes. Here goes...

$$\eta(x+iy)=\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=0}^{\infty}\sum_{c=0}^{b}(b,\,c)^{-1}(b,\,c)(-x)^{c}(-iy)^{b-c}\frac{(ln\,a)^{b}}{(b-c)!c!}$$

$$=\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=0}^{\infty}\sum_{c=0}^{b}\frac{c!(b-c)!}{b!}\frac{(ln\,a)^{b}}{(b-c)!c!}(b,\,c)(-1)^{b}x^{c}(iy)^{b-c}$$

$$\eta(x+iy)=\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=0}^{\infty}\frac{(-1)^{b}(ln\,a)^{b}}{b!}\sum_{c=0}^{b}(b,\,c)x^{c}(iy)^{b-c}\ (33)$$

But, from Pascal's triangle [18], the innermost sum of the foregoing equation is just $(x+iy)^b$, AKA z^b, so

$$\eta(z)=\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=0}^{\infty}z^{b}\frac{(-1)^{b}(ln\,a)^{b}}{b!}\ (34)$$

This series definitely converges when the real part of z is positive. How do we know? Because (34) is the literal complex exponential expansion of (0), even though we arrived here rather circuitously; the decision to sum over x, and then y, in (11) was arbitrary. This isn't a Maclaurin series; it's a series of such series. But what if we exchanged the inner and outer sums, like this:

$$\eta(z)=\sum_{b=0}^{\infty}z^{b}\frac{(-1)^{b}}{b!}\sum_{a=1}^{\infty}(-1)^{a-1}(ln\,a)^{b}\ (35)$$

This is most definitely a Maclaurin series. The inner series clearly diverges, but averaging of infinities will yield a suitable finite sum. However -- and I'll spare you the details of so much wasted work -- that sum appears to grow superfactorially such that even the (b!) term isn't aggressive enough to prevent divergence outside of some disc centered at the origin (and perhaps only the origin itself). And this, in a nutshell, is the reason that there is no Maclaurin series for $\eta(z)$, despite having no poles anywhere on the complex plane.

4. An Alternative Approach

Consider the following approximation to (13):

$$\eta(x+iy)\approx\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=0}^{B}y^{b}\frac{(-i)^{b}(ln\,a)^{b}}{b!}\sum_{c=0}^{B}x^{c}\frac{(-1)^{c}(ln\,a)^{c}}{c!}\ (36)$$

Clearly, in the limit that B approaches infinity, we have exactly (13) and thus an exact value for $\eta(x+iy)$. But what if, instead of evaluating the inner sum first, we painted a square matrix of terms one *peripheral* layer at a time, like this:

$$\eta(x+iy)\approx\sum_{a=1}^{\infty}(-1)^{a-1}(1+\sum_{b=1}^{B}(x^{b}y^{b}\frac{(-1)^{b}(-i)^{b}(ln\,a)^{2b}}{(b!)^{2}}$$

$$+\sum_{c=0}^{b-1}\left(x^{b}\frac{(-1)^{b}(\ln a)^{b}}{b!}y^{c}\frac{(-i)^{c}(\ln a)^{c}}{c!}+x^{c}\frac{(-1)^{c}(\ln a)^{c}}{c!}y^{b}\frac{(-i)^{b}(\ln a)^{b}}{b!}\right)))\quad(37)$$

wherein the leading one is just the (*b*=*c*=0) term from (36), and the first term inside the middle series corresponds to the central diagonal of the same. Taking the limit of *B* approaching infinity gives rise to the following exact expression for $\eta(x+iy)$:

$$\eta(x+iy)=\sum_{a=1}^{\infty}(-1)^{a-1}(1+\sum_{b=1}^{\infty}(x^{b}y^{b}\frac{(-1)^{b}(-i)^{b}(\ln a)^{2b}}{(b!)^{2}}$$
$$+\sum_{c=0}^{b-1}\left(x^{b}\frac{(-1)^{b}(\ln a)^{b}}{b!}y^{c}\frac{(-i)^{c}(\ln a)^{c}}{c!}+x^{c}\frac{(-1)^{c}(\ln a)^{c}}{c!}y^{b}\frac{(-i)^{b}(\ln a)^{b}}{b!}\right)))\quad(38)$$

Now let's get rid of that leading one. There's no reason to continually add and subtract it as *a* toggles between odd and even, thereby giving rise to a partial sum that cycles endlessly from one to zero and back. In the "blur of infinity", the average is thus (1/2). And in fact, as you can see at [19], $\eta(0)$ is also (1/2). This is just what we would expect from substituting zero for *x* and *y* in the following equation:

$$\eta(x+iy)=\frac{1}{2}+\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=1}^{\infty}(x^{b}y^{b}\frac{i^{b}(\ln a)^{2b}}{(b!)^{2}}$$
$$+\sum_{c=0}^{b-1}\left(x^{b}\frac{(-1)^{b}(\ln a)^{b}}{b!}y^{c}\frac{(-i)^{c}(\ln a)^{c}}{c!}+x^{c}\frac{(-1)^{c}(\ln a)^{c}}{c!}y^{b}\frac{(-i)^{b}(\ln a)^{b}}{b!}\right))\quad(39)$$

This doesn't conflict with the previous implication that $\eta(x+iy)$ diverges at nonpositive *x* because we've now addressed that -- at least at the origin -- via the extraction of (1/2). But there's more to accomplish by way of compaction:

$$\eta(x+iy)=\frac{1}{2}+\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=1}^{\infty}\frac{1}{b!}(x^{b}y^{b}\frac{i^{b}(\ln a)^{2b}}{b!}$$
$$+\sum_{c=0}^{b-1}\frac{1}{c!}\left(x^{b}y^{c}(-1)^{b}(-i)^{c}(\ln a)^{b+c}+x^{c}y^{b}(-1)^{c}(-i)^{b}(\ln a)^{b+c}\right))$$

$$\eta(x+iy)=\frac{1}{2}+\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=1}^{\infty}\frac{1}{b!}\left(x^{b}y^{b}\frac{i^{b}(\ln a)^{2b}}{b!}+\sum_{c=0}^{b-1}\frac{(-1)^{b+c}(\ln a)^{b+c}}{c!}\left(x^{b}y^{c}i^{c}+x^{c}y^{b}i^{b}\right)\right)\quad(40)$$

or if you prefer:

$$\eta(x+iy)=\frac{1}{2}+\sum_{a=1}^{\infty}(-1)^{a-1}\sum_{b=1}^{\infty}\frac{(-1)^{b}(\ln a)^{b}}{b!}\left(x^{b}y^{b}\frac{(-i)^{b}(\ln a)^{b}}{b!}+\sum_{c=0}^{b-1}\frac{(-1)^{c}(\ln a)^{c}}{c!}\left(x^{b}y^{c}i^{c}+x^{c}y^{b}i^{b}\right)\right)\quad(41)$$

which has been verified with help of WolframAlpha. (I tried random values for *a*, *x*, and *y*, and checked that the result closely matched my finite approximation.) One could break (41) into real and imaginary parts, but doing so would require splitting *c* into odd and even, which would in turn involve use of the integer floor or ceiling functions on the upper index limits. Personally, I wouldn't consider that a simplification.

Note that (41) is topologically related to (36), essentially as the grand sum of a square matrix of terms, despite the former being of infinite size. (41) converges when *x* is positive because the middle sum is equivalent to $((1/a^z)-1)$ from (0) and the leading (1/2) compensates for the offset by one.

A litany of series exist for the sake of approximating or reasoning about $\eta(z)$. One would choose whichever would be suitable for a given application. The point of (40) and (41) is to expose as much symmetry as possible in the hope that doing so might in turn shed light on the Riemann hypothesis.

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