A Probabilistic Proof of Goldbach's Conjecture, Part 2[1]

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This paper is designed to clarify points made in our previous paper, "A Probabilistic Proof of Goldbach's Conjecture".[3] *We also present a proof of 'twin prime conjecture'. We also examine some other aspects that relate to prime number distribution.*

Prologue

Hello everyone, thank you for your kind and generous readership //:-D This is rather a serious research paper, but I will keep it as entertaining as possible. Please enjoy-

1. Stories and Histories in Mathematics

Once upon a time, for centuries, mathematicians struggled to find generic solution formulas for polynomials of fifth order or higher. Then there came two bright young mathematicians, Evariste Galois[4] and Niels Abel[5], who independently proved that the generic solution for polynomials of fifth degree or higher cannot be expressed with then-known algebra.[6] The way they proved it was too ahead of time and too obscure in their time. Their proofs were

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[1] This paper is dedicated to the People in the world who support this author's 2024 US Presidential campaign: his social media and internet Friends (in DailyMotion, Medium, YouTube, Twitter, Facebook, Instagram, SSRN, and VIXRA and other websites), his past and current in-person Friends, and his Family in Korea. Started being written on 4/1/2023. He's a secular-religious, politically independent, and a private academic. The author is running for the US President in 2024 as an independent thinker.

[2] A lawyer by trade, a scientist by hobby, a humanologist by mission, a U.S. Army veteran by record, a former computer programmer, a prior PhD candidate in computational biology (withdrawn after 2 years without a degree), a former actor/writer/director/indie-filmmaker/background-music-composer. Born in the USA, 1978. Grew up in Seoul, South Korea as a child and a teenager. Returned to America as a college student. Still growing up in America as a person //!-)

[3] See https://vixra.org/abs/2303.0153 .

[4] See https://en.wikipedia.org/wiki/%C3%89variste_Galois .

[5] See https://en.wikipedia.org/wiki/Niels_Henrik_Abel .

[6] See https://en.wikipedia.org/wiki/Galois_theory ; https://en.wikipedia.org/wiki/Abel%E2%80%93Ruffini_theorem ; https://en.wikipedia.org/wiki/Degree_of_a_polynomial ; https://en.wikipedia.org/wiki/Fundamental_theorem_of_algebra .

not neat. They were handwritten notes that have smudges too much, bad handwriting, scribbles. It took a century for other more mature and more professional mathematicians to discover, decode, decrypt, realize, and appreciate the importance and the value of the two young amateur mathematicians' discoveries, now know and the group theory,[7] and the rest is history.

This author is also a young amateur mathematician. This author proved the Goldbach's conjecture in the previous paper. At this time, no professional mathematician takes this author seriously. That is 'a' okay. We shall give them a couple of centuries, if not decades //!-)

Also, let us learn from our beloved German mathematician, Mr. Professor David Hilbert,[8] who famously said,

> *Every correct hypothesis has a correct proof.*

Of course, this author is paraphrasing the professor's words somewhat, but what is important a lesson here is the optimism in mathematics, and everywhere else. We must have faith, hopes and yes, dreams.

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[7] See https://en.wikipedia.org/wiki/Group_theory .

[8] See https://en.wikipedia.org/wiki/David_Hilbert .

2. Goldbach Pair Counting Function

Let's say N is an even number. The question is, how many pairs of odd primes add up to N? We'll call this 'goldbach pair counting function', gamma(N).[9] Let's get down to earth, and make some concrete examples:

$2 = 1 + 1$	$\gamma(2) = 1$	
$4 = 1 + 3$	$\gamma(4) = 1$	
$6 = 1 + 5 = 3 + 3$	$\gamma(6) = 2$	← increased
$8 = 1 + 7 = 3 + 5$	$\gamma(8) = 2$	
$10 = 3 + 7 = 5 + 5$	$\gamma(10) = 2$	← stayed the same
$12 = 1 + 11 = 5 + 7$	$\gamma(12) = 2$	
$14 = 1 + 13 = 3 + 11 = 7 + 7$	$\gamma(14) = 3$	
$16 = 3 + 13 = 5 + 11$	$\gamma(16) = 2$	← decreased
$18 = 1 + 17 = 5 + 13 = 7 + 11$	$\gamma(18) = 3$	

As we can see, the gamma function sometimes does decrease, but it tends to slowly increase, more or less like some kind of logarithmic function fashion.

3. Prime Counting Difference Function

Prime counting function is known as pi(N).[10] Let us go ahead and define a delta function, which is the difference between lower half pie and upper half pie:

[9] The set of Goldbach pairs is a subset of the additive binary partition set. See https://en.wikipedia.org/wiki/Partition_function_(number_theory) ; https://en.wikipedia.org/wiki/Sigma-additive_set_function ; https://en.wikipedia.org/wiki/Subadditive_set_function ; https://en.wikipedia.org/wiki/Set_function .

[10] See https://en.wikipedia.org/wiki/Prime-counting_function .

$$\delta(N) = \pi(1, N/2) - \pi(N/2, N) \approx \frac{\frac{N}{2}}{\ln\frac{N}{2}} - \left(\frac{N}{\ln N} - \frac{\frac{N}{2}}{ln\frac{N}{2}}\right)$$

$$= \frac{N}{\ln\frac{N}{2}} - \frac{N}{ln\,N} = N(\frac{1}{\ln N - \mathrm{l}} - \frac{1}{\ln N})$$

$$= N(\frac{\ln 2}{(\ln N - \mathrm{l} \quad)\ln N})$$

Now, let us send N to infinity. And apply L'Hopital's rule, as it is infinity over infinity situation. Shall we?

$$\lim_{N\to\infty} \delta(N) = \lim_{n\to\infty} \frac{N\,ln\,2}{(ln\,N - ln\,2)\,ln\,N}$$

$$= \lim_{n\to\infty} \frac{\ln 2}{\frac{\ln N}{N} + \frac{\ln N - \ln 2}{N}} = \lim_{n\to\infty} \frac{N \ln 2}{2\ln N - \ln 2} = \lim_{n\to\infty} \frac{ln\,2}{\frac{2}{N}}$$

$$= \lim_{n\to\infty} \frac{N\ln 2}{2} = \infty$$

But, as of 4/1/2023 today, a Wikipedia article contains an error:[11]

[11] See https://en.wikipedia.org/wiki/Bertrand%27s_postulate .

Prime number theorem

The prime number theorem (PNT) implies that the number of primes up to x is roughly $x/\ln(x)$, so if we replace x with $2x$ then we see the number of primes up to $2x$ is asymptotically twice the number of primes up to x (the terms $\ln(2x)$ and $\ln(x)$ are asymptotically equivalent). Therefore, the number of primes between n and $2n$ is roughly $n/\ln(n)$ when n is large, and so in particular there are many more primes in this interval than are guaranteed by Bertrand's postulate. So Bertrand's postulate is comparatively weaker than the PNT. But PNT is a deep theorem, while Bertrand's Postulate can be stated more memorably and proved more easily, and also makes precise claims about what happens for small values of n. (In addition, Chebyshev's theorem was proved before the PNT and so has historical interest.)

As we can see, the world of prime numbers is still very poorly understood, even among professional mathematicians who made errors above, and we just corrected those errors in this section. We are that good //:-)

4. Inclusion-Exclusion Principle[12]

Well, this may look like a horror movie, but, let us do some binary encoding of the inclusion-exclusion principle, shall we? //xD

$$P(\varnothing) = 0 \qquad \leftarrow 2^0$$

$$P(A) = 0 + P(A) \qquad \leftarrow 2^1$$

$$P(A \cup B) = 0 + P(A) + P(B) - P(A \cap B) \qquad \leftarrow 2^2$$

$$P(A\cup B\cup C)=0+P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C) \leftarrow 2^3$$

It looks like:

0

01

00 01 10 11

000 001 010 100 011 101 110 111

As we can see, the number of terms increase exponentially. But, luckily for us, we were able to bypass the complexity of the inclusion-exclusion principle in out proof of Goldbach's conjecture proof in our previous Goldbach Conjecture Proof paper - part 1, which we will hereinafter refer to as Gold 1.[13]

Later in this paper, we will briefly mention the Exclusion-Inclusion-Summation function, that will call "*EIS(N)*".

[12] See https://en.wikipedia.org/wiki/Inclusion%E2%80%93exclusion_principle .

[13] See https://vixra.org/abs/2303.0153 . This author indeed did attempt to advertise the gold 1 paper in math dot stack exchange dot come but they didn't allow it //xD

5. <u>'Bigger EIS(N)' Lemma</u>

A probability is a number between 0 and 1, inclusive. So, a probability 'p' can be expressed as a fraction:

$$p = a / b \text{ , where } b > a$$

Then,

$$P = 1 / (b / a) \quad \text{, where } b/a > 1$$

Now, let us look at the EIS where there are two probabilities:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

We are going to prove:

$$P(A \cup B) > P(B)$$

Let's go:

proof)

$P(A) = 1/a$, where $a > 1$

$P(B) = 1/b$, where $b > 1$

$1 / ab < 1/a$

$0 < 1/a - 1/ab$

$1/b < 1/a + 1/b - 1/ab$

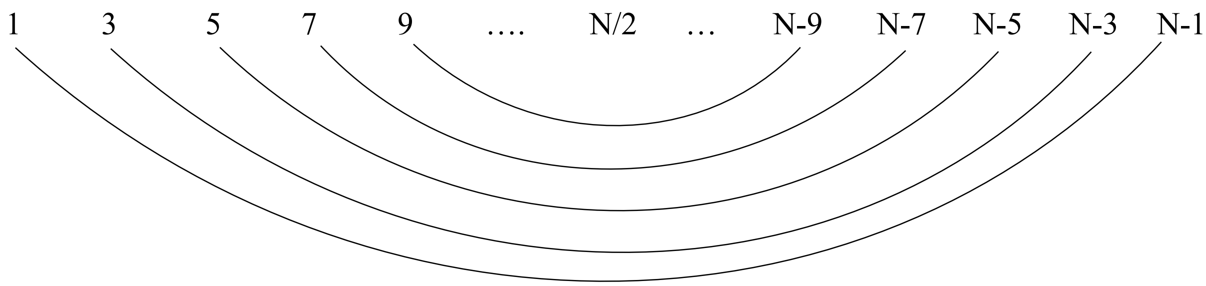
$P(B) < P(A \cup B)$

Q.E.D.

Of course, we assumed independence between the two probabilities and we only proved the case where we have only two probabilities. But, no matter what the case is, combined probability is always larger than an individual probability, because the unionized set is always larger than (or equal to) an individual subset.[14]

6. <u>POP(d) Function Revisited</u>

Let us assume that N is an even number like Gold Paper 1, and look at the symmetric pairs of odd numbers each of which sums to N:



As we can see, N-1 is the largest odd number under N. For each odd number ‘d’ above, we can calculate the Probability of Odd number being a Prime, i.e., the POP(d) function. On page 11 of Gold Paper 1, we proved ‘decreasing POP(d)’ lemma, which states that as d increases, POP(d) decreases. In other words, as d decreases, POP(d) increases. Therefore,

POP(1) > POP(3) > POP(5) > … > POP(N-5) > POP(N-3) > POP(N-1)

From page 24 to page 30 of Gold Paper 1, we proved that:

$$\lim_{N\to\infty} POP(N-1) = \lim_{x\to\infty} \frac{2(x^2-2x+4)}{(x-2)^2\, ln(x-2)+x^2\, ln\, x-2x(x-2)}$$

$$= \lim_{x\to\infty} \frac{2}{\ln(x-2)\ +\ln x-2} = 0$$

[14] See https://en.wikipedia.org/wiki/Union_(set_theory)

The result above sounds like a bad news but actually it is a good news, because POP(N-1) infinitely approaches zero from the positive side, but never becomes zero, but it is always a small positive number. Let us call this result as "positive POP(d) lemma". Since POP(N-1) is positive, all other POP(d)'s are positive as well, because they are all bigger than POP(N-1), thanks to 'decreasing POP(d) lemma'.

7. POPS(d) Function Revisited

Again, d is a positive odd number under the even number N. There are N/2 odd numbers. If we symmetrically pair two odd numbers as diagrammatically illustrated in the previous section, there are N/4 symmetric odd pairs each of which sums to N. Now, in an odd pair that sums to N, the Probability of both Odd numbers in the pair being Primes that sum to N is what we call POPSN(d, N - d):

$$POPSN(d, N\text{-}d) = POP(d) * POP(N - d)$$

Now, since both POP(d) and POP(N - d) are positive thanks to 'positive POP(d) lemma', their product is positive also. Therefore,

$$POPSN(d, N\text{-}d) > 0$$

Let us call the result above, 'positive POPSN(d, N-d) lemma'.

8. Probability of Existence of at Least One Goldbach Pair Function, PEOGOLD(N)

Let us put everything together. We will define a function that calculates the Probability of Existence of at Least One Goldbach Pair Function, PEOGOLD(N).

The function would be the exclusion-inclusion-summation of all probabilities of the form POPSN(d, N-d), for all odd numbers d under N.

$$d = 2k - 1$$

And k ranges from 1 to 4/N. We will call exclusion-inclusion-summation operator as EISigma. Then, the function would look like:

$$PEOGOLD(N) = EI\sum_{k=1}^{\frac{N}{4}}(POP(2k-1) * POP(N-2k+1))$$

Thanks to 'bigger EIS(N) lemma' we know that the combined probability above is bigger than each individual probability term inside the exclusion-inclusion-sigma. And by the 'positive POPS(d, N-d) lemma', we know the individual product term inside the exclusion-inclusion-sigma is positive. Therefore, PEOGOLD(N) is larger than the positive individual product term inside the exclusion-inclusion-sigma. Therefore,

$$PEOGOLD(N) > 0$$

This means that the probability of finding at least one Goldbach pair under N is positive, meaning that at least one Goldbach pair under N exists for all N. This concludes our Goldbach's conjecture proof.

Q.E.D.

9. A Proof of Twin Prime Conjecture[15]

On page 13 of Gold Paper 1, we proved 'independent POP(d) lemma', which safely enables us to multiply two probabilities to get the intersectional probability of two odd numbers being both prime numbers.

Twin prime conjecture states that there are infinitely many twin prime pairs, where the two primes are adjacent, consecutive odd numbers with distance of two betwixt them. Then, all we need to prove is that the probability of the two consecutive odd numbers being both primes is larger than zero. Thanks to 'positive POP(d) lemma',

[15] For the historical and contemporary background of the problem, see https://en.wikipedia.org/wiki/Twin_prime .

$$POP(d) > 0$$

$$POP(d+2) > 0$$

Therefore,

$$POP(d) * POP(d+2) > 0$$

Q.E.D.

Twin conjecture theorem is a corollary[16] of Goldbach's theorem. We expect that many conjectures regarding prime numbers will be proven in the near future using our Goldbach's theorem.

10. A Mathematical Essay

It is not known when the result of this and previous paper would be embraced by the mainstream mathematical community. It was crucial that this author does not belong to an academic institution so as to allow him the maximum freedom of thoughts and approaches, without the pressure or burden to follow established approaches.

There is an article about Professor Pogorzelski[17] who spent decades in the study of Goldbach's conjecture. The article basically stated that a mathematician's job is two-folds. First, a mathematician should find the proof. Second, the mathematicians should make the proof understandable to other mathematicians. We achieved the double goals of discovery and education via Goldbach Paper 1 and Goldbach Paper 2. And the rest is history.

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16 See https://en.wikipedia.org/wiki/Corollary .

17 See http://researchinstituteformathematics.edu/pogo.html ; https://en.wikipedia.org/wiki/Henry_Pogorzelski .

Epilogue[18]

Hello everyone, thank you for your kind and generous readership //:-D We hope you enjoyed the show. Our next article to write and publish will be titled, "Recent Development in Humanology". There, we'll introduce some interesting concepts in science and religion and anything in between.[19]

Thank you for your time and see you later, kind and generous ladies and gentlemen //:-)

[18] This paper was started being written on 4/1/2023. It was finished being written on 4/3/2023 //:-)
[19] See https://en.wikipedia.org/wiki/The_Road_Not_Taken .