ZFC Inconsistency Explained and Defended

By Jim Rock

Abstract: There is a class of sets that can be constructed within ZFC that both have and do not have a largest element. Two contradictory arguments creating these sets are explained and defended.

Introduction: For all rational numbers $\boldsymbol{a}$ in the closed interval $[0, 1]$ let the collection of all R_a sets be $\{ y$ is a rational number $| \ 0 \le y \le a \}$

The entire collection of R_a sets form a hierarchy of sets.

Each set contains all the elements in sets below it in the set hierarchy.

Each set contains a single element that is not in any set below it in the set hierarchy.

We take the largest element out of each set in the entire collection.

The set containing zero becomes the null set.

The largest element is now missing from every R_a set. However, all R_a sets remain in the same relative position in the set hierarchy as $\{ y$ is a rational number $| \ 0 \le y < a \}$

Argument #1: Each R_a **contains a largest element.**

1) Each R_a contains the former largest elements of the subsets below it in the set hierarchy.
2) Each R_a contains elements not in the subsets below it in the set hierarchy.
3) Let $\boldsymbol{c}$ and $\boldsymbol{d}$ be two elements of a single R_a set with $\boldsymbol{c} > \boldsymbol{d}$.
4) $\boldsymbol{d}$ is an element of R_c, which is a proper subset of R_a.
5) for any two elements in R_a the smaller element is contained in a proper subset of R_a.
6) From step 2) and step 5) each R_a set contains a largest element not in any set below it in the set hierarchy.

Argument #2: No R_a **contains a largest element.**

1) Suppose there is a largest element a' in some individual R_a.
2) $a' < (a' + a)/2 < a$.
3) Let $b = (a' + a)/2$.
4) Then b is in R_a and $a' < b$.

No attempt is made to specify a largest element in Argument #1. However, each step within Argument #1 is either a true statement or a logical conclusion from true statements.

When a largest element is assumed in Argument #2, it leads to a contradiction; so there is no largest element. That is a valid proof by contradiction.

Objection: For each R_a set it is claimed that every element is in some proper subset below R_a in the hierarchy. Step 2) in Argument #1 is claimed to be false. In Argument #2 a' is in R_b, which is a proper subset below R_a. However, R_b is missing $b' = (a+b)/2$. So it is with every subset R_x of R_a. They are all missing $x' = (a+x)/2$.

Since all the sets below each R_a are in a nested hierarchy with R_a at the top, there is one largest R_a set element missing from all subsets of each R_a. All R_a sets have a largest element or the entire hierarchy would collapse.

If you wish, email comments to Jim Rock at **collatz3106@gmail.com**.