Natural sets and the axiom of comprehension.

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Abstract.

A natural set never contains itself. The existence or non-existence of a natural set is decided using a modified "Axiom of Comprehension". This modified axiom keeps everything that contains itself out of set formation. This also includes the property of not containing itself, although every natural set has this property. If it were possible to form a set with this property, then the antinomy “contains itself” and “does not contain itself”, named after Bertrand Russell, would apply to this set. While this is paradoxical, it is still a corollary when trying to form a set with the property "does not contain itself". The modified “Axiom of Comprehension” takes this fact into account and decides on the non-existence of a set to be formed with the property “does not contain itself” because it would have the property “contains itself”. There can therefore be no antinomy like that of Bertrand Russell in the case of natural sets. The modified "Axiom of Comprehension" states that a natural set does not contain

itself. This means that the set to be formed cannot have the property or condition required for the set to be formed. This, together with the elimination of Russell's antinomy, provides an existence criterion for natural sets. With this, essential statements about the natural sets can be proved.

1 Selection object and set.

1.1 The essence of a set.

Sets are mental summaries of things in our being. Each set is generated by a selection rule. This selection rule clearly determines for each thing of our being whether it belongs to the set or not to the set. We call the things of our being objects. We call the objects selected by a selection rule elements of the associated set.

1.2 The essence of a collection.

A collection consists of an imaginary container that contains all the elements of a set. The container itself can therefore never be an element of the set. It is also irrelevant what type of container is used to contain the elements of the set. The collection always stays the same. It can only change if the set changes.

1.3 Representation of selection objects and sets.

Let x be an object of our being and P a property, then we say: $P(x)$ is true if and only if x the property P has. We represent the selection of all objects with the property P in the form $[x:P(x)]$. If x is a selected object, then we write $x\in[x:P(x)]$ for it. We replace square brackets with curly brackets when we want to make it clear that the selected object is a set.

2 The axiom of comprehension in set theory.

In the Oxford Reference, (**Axiom of comprehension - Oxford Reference**), we can find the following text:

„The unrestricted axiom of comprehension in set theory states that to every condition there corresponds a set of things meeting the condition: (∃*y*) (*y*={*x*: F*x*}). The axiom needs restriction, since Russell's paradox shows that in this form it will lead to contradiction. For the classical repair see separation, axiom of.“

Let's stick to the agreements in Section 1.3 (Representation of selection objects and sets.), then the axiom can be represented in the form $\exists y(y=\{x:P(x)\})$.

3 The restriction of the "axiom of comprehension".

3.1 Is a restriction necessary?

Definitely yes. If one does not take care to exclude Russell's antinomy $x \notin x$ when forming the set, then the selection object $R = [x : x \notin x]$ leads to the antinomy (Source 9.2).

3.2 How can it be restricted?

As shown in Section 3.1, the axiom of comprehension is grounded in a logical contradiction. There are requirements to avoid this contradiction.

3.2.1 Restriction by additional axioms.

One can avoid the contradiction in subordinate systems by additional requirements or axioms. Basically, you are dealing with a changed axiom of comprehension. The change cannot be read directly from the axiom.

3.2.2 Direct modification of the axiom.

One can reformulate the axiom of comprehension directly so that the change can be read from the axiom and causes the exclusion of Russell's antinomy in the formation of sets. This option is used by the natural sets.

3.3 The modified axiom of comprehension.

The modified axiom admits a selection object only if it does not contain itself.
The old form of the axiom is: $\exists y \ (y=\{x:P(x)\})$
The modified axiom is: $\exists y \ (\nexists y \in (y=\{x:P(x)\}))$

4 The exclusion of Russell's antinony.

As discussed in Section 3.1, the selection object $R = [x:x \notin x]$ leads to the antinomy $R \in R \Leftrightarrow R \notin R$. In this antinomy exists $R \in R$. The reformulated axiom of comprehension is $\exists y \ (\nexists y \in (y=\{x:P(x)\}))$. Because of the existence of property $R \in R$, part $\nexists y \in (y=\{x:E(x)\})$ of the axiom is not observed. The selection object $R = [x:x \notin x]$ is therefore not available for set formation.

5 The essence of natural sets.

The natural sets are subject to the dictates of the rephrased “Axiom of comprehension” in Section 3.3
This axiom excludes from the formation of sets all those properties that violate the essence $[x:P(x)] \notin [x:P(x)] \Leftrightarrow \exists \{x:P(x)\}$ of natural sets. The following therefore always applies to natural sets $[x:P(x)] \notin [x:P(x)] \Leftrightarrow \exists \{x:P(x)\}$.

The consequence of this is that the resulting selection object cannot have the selecting property. This corresponds to the essence of a collection, see section 1.2 (The essence of a collection.).

6 The existence criterion of natural sets.

Because of Section 5 (The essence of natural sets.), the selection object can only be a set if $P([x:P(x)])$ is false. Therefore $\neg P([x:P(x)]) \Leftarrow \exists\{x:P(x)\}$. However, the statement $\neg P([x:P(x)]) \Rightarrow \exists\{x:P(x)\}$ is false, because if we choose $P := x \notin x$, then $\neg P([x:P(x)]) \Rightarrow [x:P(x)] \in [x:P(x)] \Rightarrow \neg\exists\{x:P(x)\}$. The expression $\neg p([x:p(x)])$ is therefore not an existence criterion for the set $\{x:p(x)\}$. However, Section 5 (The essence of natural sets.) also states that the formation of natural sets is linked to the specifications of a collection, see Section 1.2 (The essence of a collection.).

That $\{x:p(x)\} \notin \{x:p(x)\}$ is true for all the properties p that can constitute a natural set. The statement $\neg P([x:P(x)]) \Rightarrow [x:P(x)] \in [x:P(x)]$ is only possible if $P = x \notin x$. The existence criterion for natural sets therefore is, $(P \neq (x \notin x) \wedge \neg P[x:P(x)]) \Leftrightarrow \exists\{x:P(x)\}$.

7 Consequences of reformulating the axiom.

The existence criterion of natural sets, see Section 6 (The existence criterion of natural sets.), provides insights into the nature of natural sets.

7.1 The set of all sets does not exist.

Proof: Let be $P \equiv 'Is\, a\, set'$. If $[x:P(x)]$ were a set, then $P([x:P(x)])$. Therefore, the selection object $[x:P(x)]$ cannot be a set.

7.2 The set of all objects does not exist.

Proof: Let be $P \equiv 'is\, an\, object'$. Since $[x:P(x)]$ is also an object, $p([x:p(x)])$.

Therefore, the selection object $[x:P(x)]$ cannot be a set.

7.3 The set of all selection objects does not exist.

Proof: Let be $P \equiv 'Is\, a\, selection\, object'$. Since $[x:P(x)]$ is also a selection object, $p([x:p(x)])$. Therefore, the selection object $[x:P(x)]$ cannot be a set.

7.4 The set of all identities does not exist.

Proof: Let be $P \equiv (x=x)$. It is valid $[x:P(x)] = [x:P(x)]$ and therefore also $p([x:p(x)])$. Therefore, the selection object $[x:P(x)]$ cannot be a set.

7.5 All properties that are always true cannot form a set.

Proof: If a property P is always true, then it is also $p([x:p(x)])$ true. Therefore, all of these properties cannot form a set.

7.6 All properties that are always false are sets-forming.

Proof: If a property P is always false, then it is $p([x:p(x)])$ also false. Therefore $\neg p([x:p(x)])$. These properties can therefore all form a set. But since there is no object x, so $P(x)$ is true, all selection objects $[x:P(x)]$ contain no elements and all lead to the empty set $\emptyset$.

7.7 There is an "empty set".

Proof: let be $P \equiv (x \neq x)$. It is valid $[x:p(x)] = [x:P(x)]$ and therefore $\neg P([x:P(x)])$. The selection object $[x:P(x)]$ is therefore a set. Since there can be no objects x with the property $x \neq x$, this set contains no elements. It is the empty set $\emptyset$.

8 Requirements for defining natural sets.

The natural sets can be defined with the following requirements.

8.1 Properties of an object.

$P(x)$ is true if and only if x the property P has .

8.2 Object selection.

The selection object $[x:P(x)]$ contains x if and only if x has the property P .

8.3 Set.

The selection object $[x:P(x)]$ represents a set if and only if it satisfies the requirement $\exists y \; (\nexists y \in (y = \{x:P(x)\}))$ of the reformulated „Axiom of comprehension“.

9 References.

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