Cardano’s formula

Edgar Valdebenito

March 15, 2023

ABSTRACT: Girolamo Cardano (1501-1576) was an Italian mathematician who is famed for his work Ars Magna which was the first Latin treatise devoted solely to algebra. In it he gave the methods of solution of the cubic equations.

1. Introduction

A general cubic equation is of the form:

$$x^3 + a\,x^2 + b\,x + c\,x = 0 \tag{1}$$

Solutions:

$$x_1 = S + T - \frac{a}{3} \tag{2}$$

$$x_2 = -\frac{1}{2}(S+T) - \frac{a}{3} + \frac{\sqrt{3}}{2}(S-T)\,i \tag{3}$$

$$x_3 = -\frac{1}{2}(S+T) - \frac{a}{3} - \frac{\sqrt{3}}{2}(S-T)\,i \tag{4}$$

where

$$S = \sqrt[3]{r + \sqrt{q^3 + r^2}} \tag{5}$$

$$T = \sqrt[3]{r - \sqrt{q^3 + r^2}} \tag{6}$$

$$q = \frac{3\,b - a^2}{9} \tag{7}$$

$$r = \frac{9\,a\,b - 27\,c - 2\,a^3}{54} \tag{8}$$

$$i = \sqrt{-1} \tag{9}$$

If a, b, c are real and if $D = q^3 + r^2$ is the discriminant, then

(i) one root is real and two are complex conjugate if $D > 0$

(ii) all roots are real and at least two are equal if $D = 0$

(iii) all roots are real and unequal if $D < 0$

If $D < 0$, computation is simplified by use of trigonometry:

Solutions:

$$x_1 = -\frac{a}{3} + 2\sqrt{-q}\cos\left(\frac{\theta}{3}\right) \tag{10}$$

$$x_2 = -\frac{a}{3} + 2\sqrt{-q}\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) \tag{11}$$

$$x_3 = -\frac{a}{3} + 2\sqrt{-q}\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) \tag{12}$$

where

$$\cos\theta = \frac{r}{\sqrt{-q^3}} \tag{13}$$

2. Elementary examples and pi formulas

Entry 1.

$$\sqrt{3}\,x^3 + \sqrt{3}\,x - 1 = 0 \tag{14}$$

Solution (real):

$$x_1 = \sqrt[3]{\sqrt{\frac{13}{108}} + \frac{1}{2\sqrt{3}}} - \sqrt[3]{\sqrt{\frac{13}{108}} - \frac{1}{2\sqrt{3}}} \tag{15}$$

Pi formula:

$$\pi = 6\sum_{n=0}^{\infty}(-1)^n\left(\sqrt[3]{\sqrt{\frac{13}{108}} + \frac{1}{2\sqrt{3}}} - \sqrt[3]{\sqrt{\frac{13}{108}} - \frac{1}{2\sqrt{3}}}\right)^{2n+1} c_n \tag{16}$$

where

$$c_n = \sum_{k=0}^{\left[\frac{2n+1}{3}\right]} \frac{(-1)^k}{2n-2k+1}\binom{2n-2k+1}{k}, \; n = 0, 1, 2, 3, \ldots \tag{17}$$

$$c_n = \frac{1}{2n+1}\,{}_3F_2\left(\begin{matrix}-\frac{1+2n}{3}, \frac{1-2n}{3}, -\frac{2n}{3} \\ \frac{1}{2} - n, -n\end{matrix} \;\middle|\; \frac{27}{4}\right), \; n = 0, 1, 2, 3, \ldots \tag{18}$$

Remark: ${}_3F_2$ is the generalized hypergeometric function.

Entry 2.

$$x^3 - x - \sqrt{3} = 0 \tag{19}$$

Solution (real):

$$x_1 = \sqrt[3]{\frac{\sqrt{3}}{2} + \sqrt{\frac{77}{108}}} + \sqrt[3]{\frac{\sqrt{3}}{2} - \sqrt{\frac{77}{108}}} \tag{20}$$

Pi formula:

$$\pi = 6 \sum_{n=0}^{\infty} \left(\sqrt[3]{\frac{\sqrt{3}}{2} + \sqrt{\frac{77}{108}}} + \sqrt[3]{\frac{\sqrt{3}}{2} - \sqrt{\frac{77}{108}}} \right)^{-2n-3} c_n \tag{21}$$

where

$$c_n = \sum_{k=0}^{[n/3]} \frac{(-1)^k}{2k+1} \binom{n-k}{n-3k}, \; n = 0, 1, 2, 3, \ldots \tag{22}$$

$$c_n = {}_3F_2\left(\begin{matrix} \frac{1-n}{3}, \frac{2-n}{3}, -\frac{n}{3} \\ \frac{3}{2}, -n \end{matrix} \middle| -\frac{27}{4} \right), \; n = 0, 1, 2, 3, \ldots \tag{23}$$

Remark: ${}_3F_2$ is the generalized hypergeometric function.

Entry 3.

$$3x^3 + 3x - 1 = 0 \tag{24}$$

Solution (real):

$$x_1 = \sqrt[3]{\sqrt{\frac{7}{108}} + \frac{1}{6}} - \sqrt[3]{\sqrt{\frac{7}{108}} - \frac{1}{6}} \tag{25}$$

Pi formula:

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \left(\sqrt[3]{\sqrt{\frac{7}{108}} + \frac{1}{6}} - \sqrt[3]{\sqrt{\frac{7}{108}} - \frac{1}{6}} \right)^{n} c_n \tag{26}$$

where

$$c_n = \sum_{k=0}^{[n/3]} \frac{1}{2n-4k+1} \binom{n-2k}{k}, \; n = 0, 1, 2, 3, \ldots \tag{27}$$

$$c_{3n} = \frac{1}{6n+1} {}_4F_3\left(\begin{matrix} -\frac{1}{4} - \frac{3n}{2}, \frac{1}{3} - n, \frac{2}{3} - n, -n \\ \frac{1}{2} - \frac{3n}{2}, \frac{3}{4} - \frac{3n}{2}, -\frac{3n}{2} \end{matrix} \middle| -\frac{27}{4} \right), \; n = 0, 1, 2, 3, \ldots \tag{28}$$

$$c_{3n+1} = \frac{1}{6n+3} {}_4F_3\left(\begin{matrix} -\frac{3}{4} - \frac{3n}{2}, -\frac{1}{3} - n, \frac{1}{3} - n, -n \\ -\frac{1}{2} - \frac{3n}{2}, \frac{1}{4} - \frac{3n}{2}, -\frac{3n}{2} \end{matrix} \middle| -\frac{27}{4} \right), \; n = 0, 1, 2, 3, \ldots \tag{29}$$

$$c_{3n+2} = \frac{1}{6n+5} {}_4F_3\left(\begin{matrix} -\frac{5}{4} - \frac{3n}{2}, -\frac{2}{3} - n, -\frac{1}{3} - n, -n \\ -1 - \frac{3n}{2}, -\frac{1}{2} - \frac{3n}{2}, -\frac{1}{4} - \frac{3n}{2} \end{matrix} \middle| -\frac{27}{4} \right), \; n = 0, 1, 2, 3, \ldots \tag{30}$$

Remark: ${}_4F_3$ is the generalized hypergeometric function.

Entry 4.

$$\sqrt{3}\, x^3 + x - 1 = 0 \tag{31}$$

Solution (real):

$$x_1 = \sqrt[3]{\sqrt{\frac{7}{12} + \frac{1}{81\sqrt{3}}} + \frac{1}{2\sqrt{3}}} - \sqrt[3]{\sqrt{\frac{7}{12} + \frac{1}{81\sqrt{3}}} - \frac{1}{2\sqrt{3}}} \tag{32}$$

Pi formula:

$$\pi = 6 \sum_{n=0}^{\infty} \left(\sqrt[3]{\sqrt{\frac{7}{12} + \frac{1}{81\sqrt{3}}} + \frac{1}{2\sqrt{3}}} - \sqrt[3]{\sqrt{\frac{7}{12} + \frac{1}{81\sqrt{3}}} - \frac{1}{2\sqrt{3}}} \right)^{n+3} c_n \tag{33}$$

where

$$c_n = \sum_{k=0}^{[n/6]} \frac{(-1)^k}{2k+1} \binom{n-4k}{n-6k}, \quad n = 0, 1, 2, 3, \ldots \tag{34}$$

$$c_n = {}_6F_5\left(\begin{matrix} \frac{1}{6} - \frac{n}{6}, \frac{1}{3} - \frac{n}{6}, \frac{1}{2} - \frac{n}{6}, \frac{2}{3} - \frac{n}{6}, \frac{5}{6} - \frac{n}{6}, -\frac{n}{6} \\ \frac{3}{2}, \frac{1}{4} - \frac{n}{4}, \frac{1}{2} - \frac{n}{4}, \frac{3}{4} - \frac{n}{4}, -\frac{n}{4} \end{matrix} \middle| -\frac{729}{16} \right), \quad n = 0, 1, 2, 3, \ldots \tag{35}$$

Remark: ${}_6F_5$ is the generalized hypergeometric function.

Entry 5.

$$x^3 - 3x + 1 = 0 \tag{36}$$

Solutions:

$$x_1 = 2\sin\left(\frac{\pi}{18}\right) = \frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \ldots\right)^3\right)^3 \tag{37}$$

$$x_2 = 2\sin\left(\frac{5\pi}{18}\right) = \sqrt{3 - \frac{1}{\sqrt{3 - \frac{1}{\sqrt{3 - \ldots}}}}} \tag{38}$$

$$x_3 = -2\sin\left(\frac{7\pi}{18}\right) = -\sqrt[3]{1 + 3\sqrt[3]{1 + 3\sqrt[3]{1 + \ldots}}} \tag{39}$$

Pi formulas:

$$\pi = 6\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \left(2\sin\left(\frac{\pi}{18}\right)\right)^{n+1} c_n \tag{40}$$

where

$$c_n = \sum_{k=0}^{[n/3]} \frac{1}{2n - 6k + 1} \binom{n-2k}{k}, \quad n = 0, 1, 2, 3, \ldots \tag{41}$$

$$c_{3n} = \frac{1}{6n+1} {}_4F_3\left(\begin{matrix} -\frac{1}{6} - n, \frac{1}{3} - n, \frac{2}{3} - n, -n \\ \frac{1}{2} - \frac{3n}{2}, \frac{5}{6} - n, -\frac{3n}{2} \end{matrix} \middle| -\frac{27}{4} \right), \quad n = 0, 1, 2, 3, \ldots \tag{42}$$

$$c_{3n+1} = \frac{1}{6n+3} {}_4F_3\left(\begin{matrix} -\frac{1}{2} - n, -\frac{1}{3} - n, \frac{1}{3} - n, -n \\ -\frac{1}{2} - \frac{3n}{2}, \frac{1}{2} - n, -\frac{3n}{2} \end{matrix} \middle| -\frac{27}{4} \right), \quad n = 0, 1, 2, 3, \ldots \tag{43}$$

$$c_{3n+2} = \frac{1}{6n+5}\, {}_4F_3\left(\begin{matrix} -\frac{5}{6}-n, -\frac{2}{3}-n, -\frac{1}{3}-n, -n \\ -1-\frac{3n}{2}, -\frac{1}{2}-\frac{3n}{2}, \frac{1}{6}-n \end{matrix} \;\middle|\; -\frac{27}{4}\right), \; n = 0, 1, 2, 3, \ldots \tag{44}$$

Remark: ${}_4F_3$ is the generalized hypergeometric function.

$$\pi = 6\sqrt{3}\sum_{n=0}^{\infty}(-1)^n \left(2\sin\left(\frac{5\pi}{18}\right)\right)^{-n} 3^{-n} c_n = 6\sqrt{3}\sum_{n=0}^{\infty}\left(2\sin\left(\frac{7\pi}{18}\right)\right)^{-n} 3^{-n} c_n \tag{45}$$

where

$$c_n = \sum_{k=[n/3]}^{[n/2]} \frac{(-1)^k 3^{2k}}{2k+1}\binom{k}{n-2k}, \; n = 0, 1, 2, 3, \ldots \tag{46}$$

3. Endnote

Entry 6.

$$\pi = 6\sum_{n=0}^{\infty}(-1)^n s^{2n+1}\left(\frac{1}{2n+1} + \frac{3s^2}{2n+3}\right) c_n \tag{47}$$

where

$$s = \sqrt[3]{\sqrt{\frac{13}{108}} + \frac{1}{2\sqrt{3}}} - \sqrt[3]{\sqrt{\frac{13}{108}} - \frac{1}{2\sqrt{3}}} \tag{48}$$

$$c_n = \sum_{k=0}^{\left[\frac{2n}{3}\right]}(-1)^k \binom{2n-2k}{k}, \; n = 0, 1, 2, 3, \ldots \tag{49}$$

Entry 7.

$$\pi = 4\tan^{-1}\left(\frac{3}{5}\right) + 4\tan^{-1}\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \frac{1}{3}\left(\frac{1}{3} + \ldots\right)^3\right)^3\right) + 4\tan^{-1}\left(\sqrt{3 - \frac{1}{\sqrt{3 - \frac{1}{\sqrt{3 - \ldots}}}}}\right) - 4\tan^{-1}\left(\sqrt[3]{1 + 3\sqrt[3]{1 + 3\sqrt[3]{1 + \ldots}}}\right) \tag{50}$$

Remark: $\pi = 4\sum_{n=0}^{\infty}(-1)^n (2n+1)^{-1}$.

4. References

[1] D.M.Burton, Abstract Algebra,Wm.C.Brown Company Publishers,Dubuque,IA,1988.

[2] L.Gilbert, J.Gilbert, Elementary Modern Algebra,8th.Ed,Cengage Learning,2015.

[3] R.W.D.Nickalls, Viete,Descartes and the cubic equation,Math. Gazette,90 (518), doi:10.1017/S0025557200179598, 2006.