“Graphical Representation of Quaternions and Their Concomitant Functions.”

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Abstract:-

In this particular paper we will demonstrate that, by invoking the concept of a ‘quaternionic quasi-complex component’, it is possible to graphically represent all quaternions and their concomitant functions with the aid of specific quationic analogues of the Argand diagram from complex variable analysis, bearing in mind that the algebraic and analytic properties of the aforesaid numbers and functions have been comprehensively elucidated in the author’s antecedent papers [2]; [3]; [4] & [5].

I. PRELIMINARY REMARK.

It should therefore be noted that the relevant details pertaining to each of the above mentioned papers are accordingly provided in the ensuing **Section V** (**i.e.** **BIBLIOGRAPHY**) of this paper.

II. COPY OF AUTHOR’S PAPER.

[**N.B**. Total number of A4 pages to follow after this page = 31.]

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PREFACE.

In this particular paper we will demonstrate that, by invoking the concept of a 'quaternionic quasi-complex component', it is possible to graphically represent all quaternions and their concomitant functions with the aid of specific quaternionic analogues of the Argand diagram from complex variable analysis, bearing in mind that the algebraic and analytic properties of the aforesaid numbers and functions have been comprehensively elucidated in the author's antecedent papers [2]; [3]; [4] & [5].

Stephen C. Pearson.

1st January 2023.

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I. Introduction.

From the author's paper [2] we recall the following fundamental definition pertaining to all quaternion (hypercomplex) numbers, namely –

Definition DI–1.

A quaternion number,

$q = x + iy + j\hat{x} + k\hat{y}$, where $x, y, \hat{x}, \hat{y} \in \mathbb{R}$,

is defined by the following rules with respect to its constituent basis elements, $1, i, j, k$:–

$1 \cdot i = i \cdot 1 = i$, $1 \cdot j = j \cdot 1 = j$, $1 \cdot k = k \cdot 1 = k$;

$1^2 = 1$, $i^2 = j^2 = k^2 = -1$;

$ij = k$, $jk = i$, $ki = j$, $ji = -k$; $kj = -i$, $ik = -j$,

where '1' denotes the identity element of multiplication. Furthermore, we define

$0 \cdot 1 = 1 \cdot 0 = 0 \cdot i = i \cdot 0 = 0 \cdot j = j \cdot 0 = 0 \cdot k = k \cdot 0 = 0$,

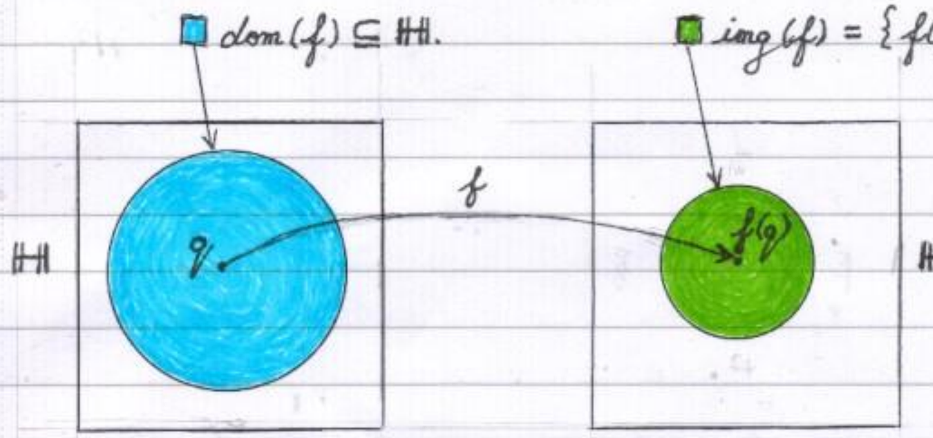
the identity element of addition.

Similarly, from this same paper we also recall the following generic formula pertaining to every quaternion (hypercomplex) function, in other words –

$f(q) = u_1(x, y, \hat{x}, \hat{y}) + i v_1(x, y, \hat{x}, \hat{y}) + j u_2(x, y, \hat{x}, \hat{y}) + k v_2(x, y, \hat{x}, \hat{y})$,

$\forall q \in dom(f) \subseteq \mathbb{H} \Longrightarrow$ the ordered quadruplet, $(x, y, \hat{x}, \hat{y}) \in R_{(x_0, y_0, \hat{x}_0, \hat{y}_0)} \subseteq \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^4$, insofar as $(x_0, y_0, \hat{x}_0, \hat{y}_0) \in R_{(x_0, y_0, \hat{x}_0, \hat{y}_0)}$,

whereupon these functions are customarily depicted with the aid of two <u>Venn diagrams</u>, as indicated in <u>Fig. 1</u> below:-



<u>N.B.</u>
By definition, the set of all quaternions, $\mathbb{H} = U$, the <u>universal set</u>, for the purposes of our ensuing discussion.

<u>Fig. 1.</u>

Whilst such <u>'coordinate free' graphical representations</u> are technically correct, their visual appeal to the reader may be somewhat limited, since it is simply impossible to visualise their constituent elements by utilising any <u>arbitrary four dimensional (4D) Euclidean or non-Euclidean coordinate system</u>. However, in view of the contents of <u>Definition DI-1</u>, we can now devise an alternative approach, bearing in mind our next definition, which we shall accordingly enunciate as follows -

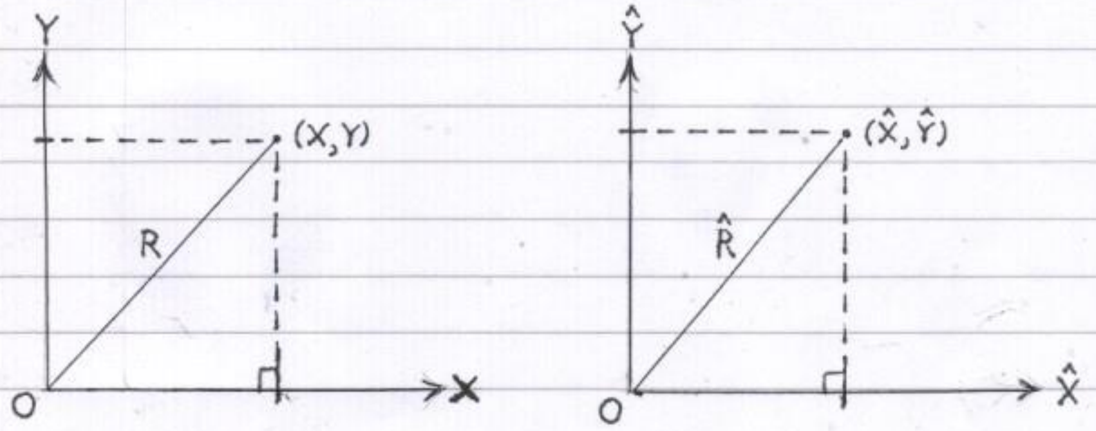
<u>Definition DI-2.</u>

Let there exist an ordered quadruplet, $(X, Y, \hat{X}, \hat{Y}) \in \mathbb{R}^4$. Subsequently, its concomitant quaternion (hypercomplex) number,

$$\begin{aligned} Q &= X + iY + j\hat{X} + k\hat{Y} \\ &= X + iY + j\hat{X} + ij\hat{Y} \\ &= X + iY + \hat{X}j + i\hat{Y}j \end{aligned}$$

$$= X + iY + (\hat{X} + i\hat{Y})j,$$

insofar as the terms, $X + iY$ and $\hat{X} + i\hat{Y}$, shall henceforth be referred to as being the respective quaternionic quasi-complex components of the quaternion, Q. Furthermore, we can graphically represent both of these components by means of the following quaternionic analogues of the Argand diagram from complex variable analysis [*], as is evident from Fig. 2 below:–



N.B. (a) The radius,

$$R = \|(X, Y)\| = \sqrt{X^2 + Y^2} = |X + iY|.$$

(b) The radius,

$$\hat{R} = \|(\hat{X}, \hat{Y})\| = \sqrt{\hat{X}^2 + \hat{Y}^2} = |\hat{X} + i\hat{Y}|.$$

Fig. 2.

Subsequently, having regard to what we have discussed thus far, we shall now proceed to elucidate the graphical representation of

(a) quaternionic smooth arcs and contours AND
(b) regions pertaining to quaternions and their concomitant functions,

throughout the remainder of this paper.

[*] N.B.

Churchill et al. [1], having rigorously defined this particular concept, have moreover provided specific examples thereof for the purposes of elucidating the fundamentals of complex variable analysis.

II. Graphical Representation of Quaternionic Smooth Arcs and Contours.

From the author's papers [2]; [3]; [4] & [5] the reader will recall that

(a) a smooth arc, K, is defined by the equation,

$$q(t) = x(t) + iy(t) + j\hat{x}(t) + k\hat{y}(t), \ \forall t \in [a, b] \qquad (2-1),$$

insofar as its resultant first derivative with respect to 't',

$$q'(t) = \frac{d}{dt}[q(t)] = \frac{d}{dt}(x(t)) + i\frac{d}{dt}(y(t)) + j\frac{d}{dt}(\hat{x}(t)) + k\frac{d}{dt}(\hat{y}(t))$$

$$= x'(t) + iy'(t) + j\hat{x}'(t) + k\hat{y}'(t), \ \forall t \in (a, b);$$

(b) a contour, $C = \bigcup_{n=1}^{N} K_n$, is defined by the sequence of equations,

$$q_n(t) = x_n(t) + iy_n(t) + j\hat{x}_n(t) + k\hat{y}_n(t), \ \forall t \in [a_n, b_n] \ \&$$

$$n \in \{1, 2, \ldots\ldots\ldots, N\} \qquad (2-2),$$

insofar as

(i) each resultant first derivative with respect to 't',

$$q_n'(t) = x_n'(t) + iy_n'(t) + j\hat{x}_n'(t) + k\hat{y}_n'(t), \ \forall t \in (a_n, b_n), \quad \&$$

(ii) each constituent endpoint,

$$q_{m+1}(a_{m+1}) = q_m(b_m), \ m \in \{1, 2, \ldots\ldots\ldots, N-1\},$$

whereupon in the particular case of a closed contour we also require that the endpoint,

$q_N(b_N) = q_1(a_1)$.

Subsequently, we will enunciate the graphical representation of such entities by means of the next two theorems –

Theorem TII–1.

Let there exist a quaternion parametric function, $q(t)$, whose corresponding real and imaginary parts are specified in Eq.(2–1). Hence, in view of its prerequisite analytic properties, it may subsequently be proven that any such parametric function can be graphically represented by means of two (2) distinct quaternionic analogues of the Argand diagram.

* * * * *

PROOF:–

From Definition DI–2 we recall that any quaternion (hypercomplex) number,

$$Q = X + iY + j\hat{X} + k\hat{Y} = X + iY + (\hat{X} + i\hat{Y})j.$$

Now, by setting

$$Q = q = q(t);\ X = x = x(t);\ Y = y = y(t);\ \hat{X} = \hat{x} = \hat{x}(t);\ \hat{Y} = \hat{y} = \hat{y}(t)$$

$\Longrightarrow$ the parametric function, $q(t) = x(t) + iy(t) + (\hat{x}(t) + i\hat{y}(t))j$, $\forall t \in [a,b]$,

we immediately perceive that the concomitant Fig. 2 may likewise be modified in such a way so as to generate the following two (2) quaternionic analogues of the Argand diagram, namely –

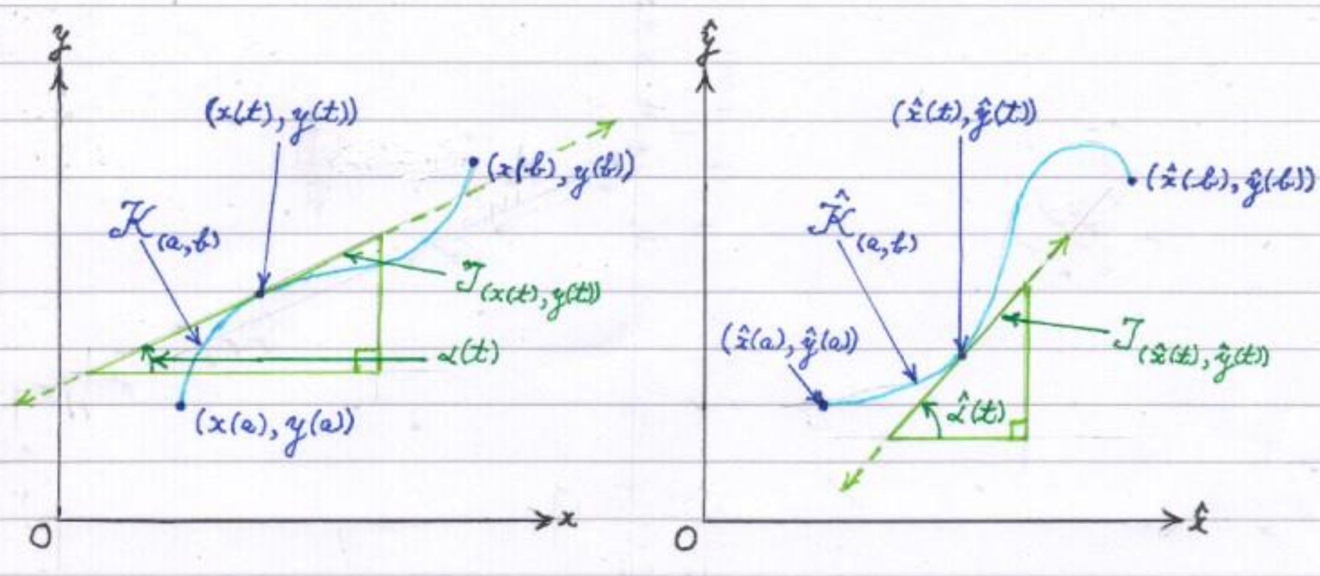


Fig. 3.

whereupon we subsequently deduce from these particular diagrams that

(a) the smooth arc, $\mathcal{K}_{(a,b)} = \{(x(t), y(t)) \mid t \in [a,b]\}$;

(b) the smooth arc, $\hat{\mathcal{K}}_{(a,b)} = \{(\hat{x}(t), \hat{y}(t)) \mid t \in [a,b]\}$;

(c) the tangent, $\mathcal{T}_{(x(t),y(t))} = \{(x,y) \mid x \in \mathbb{R} \;\&\; (y - y(t))/(x - x(t)) = \tan(\alpha(t))\}$;

(d) the tangent, $\mathcal{T}_{(\hat{x}(t),\hat{y}(t))} = \{(\hat{x},\hat{y}) \mid \hat{x} \in \mathbb{R} \;\&\; (\hat{y} - \hat{y}(t))/(\hat{x} - \hat{x}(t)) = \tan(\hat{\alpha}(t))\}$;

(e) the gradient of the tangent, $\mathcal{T}_{(x(t),y(t))}$,

$\tan(\alpha(t)) = y'(t)/x'(t)$;

(f) the gradient of the tangent, $\mathcal{T}_{(\hat{x}(t),\hat{y}(t))}$,

$\tan(\hat{\alpha}(t)) = \hat{y}'(t)/\hat{x}(t)$,

$\forall t \in (a,b)$, as required. Q.E.D.

Theorem TII-2.

Let there exist a sequence of quaternion parametric functions, $q_n(t)$, $\forall n \varepsilon \{1, 2, \ldots\ldots, N\}$, whose corresponding real and imaginary parts are specified in Eq (2-2). Hence, in view of their prerequisite analytic properties, it may subsequently be proven that any such sequence of parametric functions can be graphically represented by means of four (4) distinct quaternionic analogues of the Argand diagram.

* * * * *

PROOF:-

From Definition DI-2 we once again recall that any quaternion (hyper-complex) number,

$$Q = X + iY + j\hat{X} + k\hat{Y} = X + iY + (\hat{X} + i\hat{Y})j.$$

Now, by setting

$$Q = q = q_n(t);\ X = x = x_n(t);\ Y = y = y_n(t);\ \hat{X} = \hat{x} = \hat{x}_n(t);\ \hat{Y} = \hat{y} = \hat{y}_n(t),$$

it immediately follows that the sequence of quaternion parametric functions,

$$q_n(t) = x_n(t) + iy_n(t) + (\hat{x}_n(t) + i\hat{y}_n(t))j,\ \forall t \varepsilon [a_n, b_n]\ \&$$

$n \varepsilon \{1, 2, \ldots\ldots, N\}$, insofar as each constituent endpoint,

$$q_{m+1}(a_{m+1}) = q_m(b_m),\ \text{where } m \varepsilon \{1, 2, \ldots\ldots, N-1\},$$

likewise implies that

$$x_{m+1}(a_{m+1}) + iy_{m+1}(a_{m+1}) + (\hat{x}_{m+1}(a_{m+1}) + i\hat{y}_{m+1}(a_{m+1}))j =$$
$$x_m(b_m) + iy_m(b_m) + (\hat{x}_m(b_m) + i\hat{y}_m(b_m))j,$$

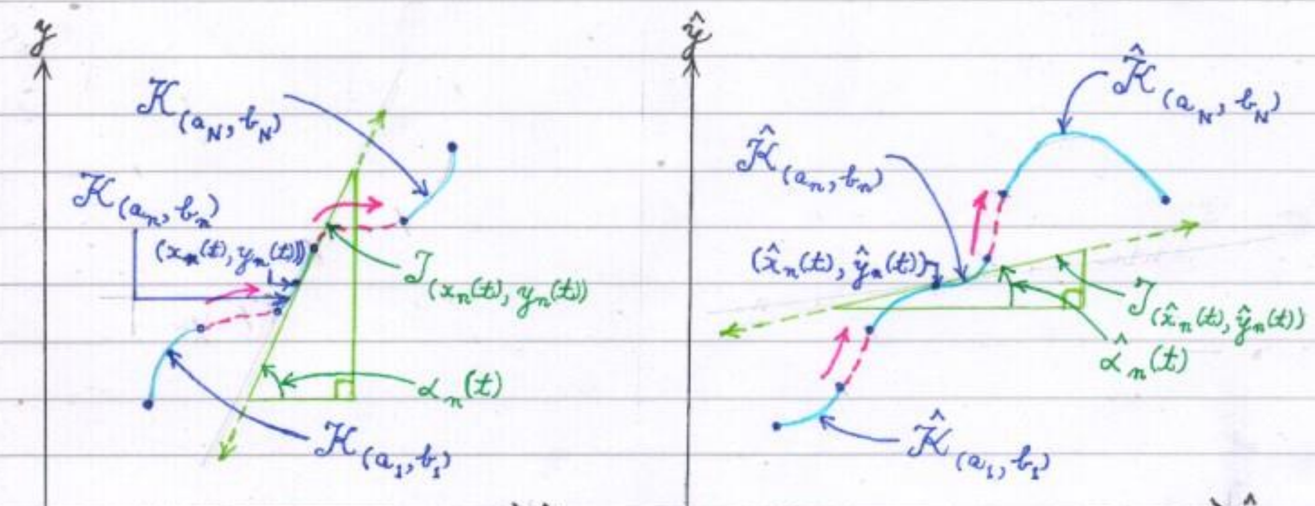
and hence the corresponding real and imaginary parts of this particular equation thus yields

$x_{m+1}(a_{m+1}) = x_m(b_m);$
$y_{m+1}(a_{m+1}) = y_m(b_m);$
$\hat{x}_{m+1}(a_{m+1}) = \hat{x}_m(b_m)$ &
$\hat{y}_{m+1}(a_{m+1}) = \hat{y}_m(b_m).$

Let us now consider the case of a <u>contour, $\mathsf{C} = \bigcup_{n=1}^{N} \underline{\mathsf{K}}_n$</u>, which is <u>open</u> thereby implying that the endpoint,

$q_N(b_N) \neq q_1(a_1).$

Subsequently, we perceive that the antecedent <u>Fig. 3</u> may likewise be modified in such a way so as to generate the following two (2) quaternionic analogues of the Argand diagram, namely –



<u>Fig. 4(a)</u>.

Similarly, let us consider the case of a <u>contour, $\mathsf{C} = \bigcup_{n=1}^{N} \underline{\mathsf{K}}_n$</u>, which is <u>closed</u> thereby implying that the endpoint,

$q_N(b_N) = q_1(a_1).$

Hence, we once again perceive that the antecedent Fig. 3 may likewise be modified in such a way so as to generate the following two (2) quaternionic analogues of the Argand diagram, namely –

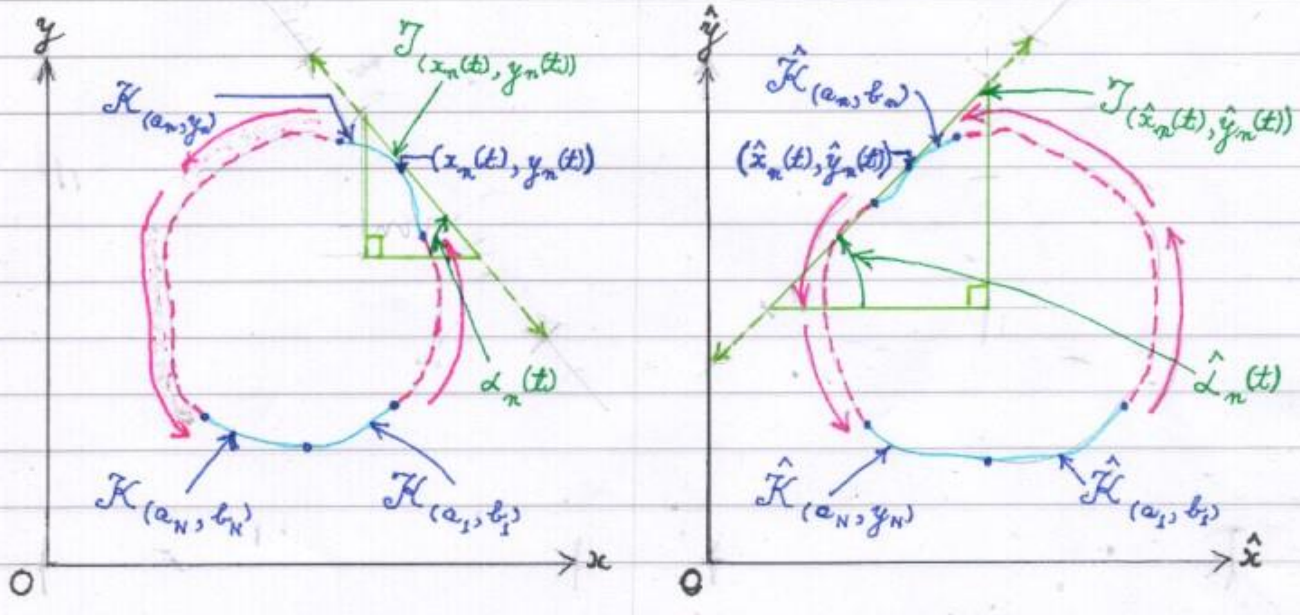


Fig. 4(b).

Finally, we deduce from both Fig. 4(a) and Fig. 4(b) that

(a) each smooth arc, $\left\{ \begin{array}{l} \mathcal{K}_{(a_n, b_n)} = \{(x_n(t), y_n(t)) \mid t \in [a_n, b_n]\} \ \& \\ \hat{\mathcal{K}}_{(a_n, b_n)} = \{(\hat{x}_n(t), \hat{y}_n(t)) \mid t \in [a_n, b_n]\} \end{array} \right\}$;

(b) each tangent, $\left\{ \begin{array}{l} \mathcal{T}_{(x_n(t), y_n(t))} = \{(x,y) \mid x \in \mathbb{R} \ \& \ (y - y_n(t))/(x - x_n(t)) = \tan(\alpha_n(t))\} \\ \qquad\qquad \& \\ \mathcal{T}_{(\hat{x}_n(t), \hat{y}_n(t))} = \{(\hat{x}, \hat{y}) \mid \hat{x} \in \mathbb{R} \ \& \ (\hat{y} - \hat{y}_n(t))/(\hat{x} - \hat{x}_n(t)) = \tan(\hat{\mathcal{L}}_n(t))\} \end{array} \right\}$;

(c) the gradient of $\mathcal{T}_{(x_n(t), y_n(t))}$, namely – $\tan(\alpha_n(t)) = y_n'(t)/x_n'(t)$;

(d) the gradient of $\mathcal{T}_{(\hat{x}_n(t), \hat{y}_n(t))}$, namely – $\tan(\hat{\mathcal{L}}_n(t)) = \hat{y}_n'(t)/\hat{x}_n'(t)$,

$\forall t \in (a_n, b_n) \ \& \ n \in \{1, 2, \ldots\ldots\ldots, N\}$, as required. Q.E.D.

III. Graphical Representation of Regions pertaining to Quaternions and Their Concomitant Functions.

From pages 9 and 10 of the previous section we recall that any quaternionic contour, $C = \bigcup_{n=1}^{N} K_n$, which is closed insofar as the endpoint,

$$q_N(b_N) = q_1(a_1),$$

subsequently generates two sets of smooth arcs, namely –

$$\bigcup_{n=1}^{N} \mathcal{K}_{(a_n, b_n)} \subset \mathbb{R} \times \mathbb{R} \;\&\; \bigcup_{n=1}^{N} \hat{\mathcal{K}}_{(a_n, b_n)} \subset \mathbb{R} \times \mathbb{R},$$

whose respective endpoints,

$$(x_N(b_N), y_N(b_N)) = (x_1(a_1), y_1(a_1));$$

$$(\hat{x}_N(b_N), \hat{y}_N(b_N)) = (\hat{x}_1(a_1), \hat{y}_1(a_1)),$$

as is evident from Fig. (4b), whereupon we shall henceforth enunciate the following definitions, namely –

Definition DIII-1.

Let there exist two regions, $R_{(x_0, y_0)} \subset \mathbb{R} \times \mathbb{R}$ & $R_{(\hat{x}_0, \hat{y}_0)} \subset \mathbb{R} \times \mathbb{R}$, which are accordingly characterised by the following properties:–

(a) the ordered pairs, $(x_0, y_0) \in R_{(x_0, y_0)}$ & $(\hat{x}_0, \hat{y}_0) \in R_{(\hat{x}_0, \hat{y}_0)}$;

(b) the ordered pairs, $(x, y) \in R_{(x_0, y_0)}$ & $(\hat{x}, \hat{y}) \in R_{(\hat{x}_0, \hat{y}_0)}$;

(c) the boundary of $R_{(x_0, y_0)}$,

$$\partial R_{(x_0, y_0)} = \bigcup_{n=1}^{N} \mathcal{K}_{(a_n, b_n)}, \text{ where}$$

(i) each set, $\mathcal{K}_{(a_n, b_n)} = \{(x_n(t), y_n(t)) \mid t \varepsilon [a_n, b_n]\}$ &

(ii) the endpoint, $(x_N(b_N), y_N(b_N)) = (x_1(a_1), y_1(a_1))$;

(d) the <u>boundary</u> of $R_{(\hat{x}_0, \hat{y}_0)}$,

$$\partial R_{(\hat{x}_0, \hat{y}_0)} = \bigcup_{n=1}^{N} \hat{\mathcal{K}}_{(a_n, b_n)}, \text{ where}$$

(i) each set, $\hat{\mathcal{K}}_{(a_n, b_n)} = \{(\hat{x}_n(t), \hat{y}_n(t)) \mid t \varepsilon [a_n, b_n]\}$ &

(ii) the endpoint, $(\hat{x}_N(b_N), \hat{y}_N(b_N)) = (\hat{x}_1(a_1), \hat{y}_1(a_1))$.

<u>Definition DIII-2.</u>

Let there exist two <u>regions</u>, $R_{(u_0, v_0)} \subset \mathbb{R} \times \mathbb{R}$ & $R_{(\hat{u}_0, \hat{v}_0)} \subset \mathbb{R} \times \mathbb{R}$, which are accordingly characterised by the following properties:-

(a) the ordered pairs, $(u_0, v_0) \varepsilon R_{(u_0, v_0)}$ & $(\hat{u}_0, \hat{v}_0) \varepsilon R_{(\hat{u}_0, \hat{v}_0)}$;

(b) the ordered pairs, $(u, v) \varepsilon R_{(u_0, v_0)}$ & $(\hat{u}, \hat{v}) \varepsilon R_{(\hat{u}_0, \hat{v}_0)}$;

(c) the <u>boundary</u> of $R_{(u_0, v_0)}$,

$$\partial R_{(u_0, v_0)} = \bigcup_{n=1}^{N} \mathcal{L}_{(a_n, b_n)}, \text{ where}$$

(i) each set, $\mathcal{L}_{(a_n, b_n)} = \{(u_n(t), v_n(t)) \mid t \varepsilon [a_n, b_n]\}$ &

(ii) the endpoint, $(u_N(b_N), v_N(b_N)) = (u_1(a_1), v_1(a_1))$;

(d) the <u>boundary</u> of $R_{(\hat{u}_0, \hat{v}_0)}$,

$$\partial R_{(\hat{u}_0, v_0)} = \bigcup_{n=1}^{N} \hat{\mathcal{L}}_{(a_n, b_n)}, \text{ where}$$

(i) each set, $\hat{\mathcal{L}}_{(a_n, b_n)} = \{(\hat{u}_n(t), \hat{v}_n(t)) \mid t \in [a_n, b_n]\}$ &

(ii) the endpoint, $(\hat{u}_N(b_N), \hat{v}_N(b_N)) = (\hat{u}_1(a_1), \hat{v}_1(a_1))$.

As a direct consequence of these particular definitions, we shall likewise enunciate three additional theorems as follows:-

Theorem TIII-1.

Let there exist a quaternion (hypercomplex) function,

$$f(q) = f(x + iy + j\hat{x} + k\hat{y}) = u_1(x, y, \hat{x}, \hat{y}) + iv_1(x, y, \hat{x}, \hat{y}) + ju_2(x, y, \hat{x}, \hat{y}) + kv_2(x, y, \hat{x}, \hat{y}),$$

such that the set,

$$\{q, q_0\} \subset \mathrm{dom}(f) \subseteq \mathbb{H} \Longrightarrow \{f(q), f(q_0)\} \subset \mathrm{img}(f) = \{f(s) \mid s \in \mathrm{dom}(f)\} \subseteq \mathbb{H}.$$

Subsequently, it may be proven that if the sets, $\{q, q_0\}$ & $\{f(q), f(q_0)\}$, also satisfy the criteria specified in Definitions DIII-1 & DIII-2, then the quaternion, q, and its concomitant function, $f(q)$, can likewise be graphically represented by means of four (4) distinct quaternionic analogues of the Argand diagram.

* * * * *

PROOF:-

Let the set,

$$\begin{aligned}\{q, q_0\} &= \{x + iy + j\hat{x} + k\hat{y}, x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0\} \\ &= \{x + iy + (\hat{x} + i\hat{y})j, x_0 + iy_0 + (\hat{x}_0 + i\hat{y}_0)j\} \subset \mathrm{dom}(f) \subseteq \mathbb{H},\end{aligned}$$

imply that there exist two regions, $R_{(x_0,y_0)} \subset \mathbb{R}\times\mathbb{R}$ & $R_{(\hat{x}_0,\hat{y}_0)} \subset \mathbb{R}\times\mathbb{R}$, whose respective properties are specified in <u>Definition DIII-1</u>.

Similarly, let the set,

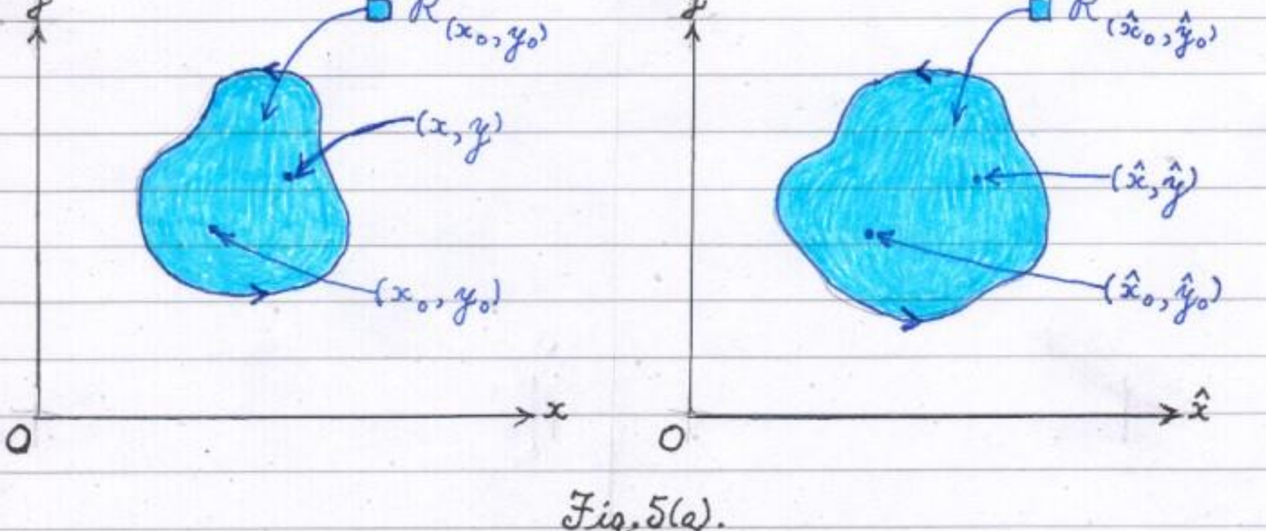
$$\{f(q), f(q_0)\} = \left\{\begin{array}{l} u_1(x,y,\hat{x},\hat{y}) + i v_1(x,\hat{y},\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k\, v_2(x,y,\hat{x},\hat{y}), \\ u_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + i v_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + j u_2(x_0,y_0,\hat{x}_0,\hat{y}_0) + \\ k\, v_2(x_0,y_0,\hat{x}_0,\hat{y}_0) \end{array}\right\}$$

$$= \left\{\begin{array}{l} u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + (u_2(x,y,\hat{x},\hat{y}) + i v_2(x,y,\hat{x},\hat{y}))\, j, \\ u_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + i v_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + (u_2(x_0,y_0,\hat{x}_0,y_0) + i v_2(x_0,y_0,x_0,y_0))\, j \end{array}\right\}$$

$$\subset img(f) = \{f(s) \mid s \in dom(f)\} \subseteq \mathbb{H},$$

likewise imply that there exist two regions, $R_{(u_0,v_0)} \subset \mathbb{R}\times\mathbb{R}$ & $R_{(\hat{u}_0,\hat{v}_0)} \subset \mathbb{R}\times\mathbb{R}$, whose respective respective properties are specified in <u>Definition DIII-2</u>.

In view of the preceding statements, we deduce that the quaternion, q, and its concomitant function, $f(q)$, can be graphically represented by modifying the antecedent <u>Fig. 4(b)</u> in such a way so as to generate the following quaternionic analogues of the Argand diagram, namely-



<u>Fig. 5(a).</u>

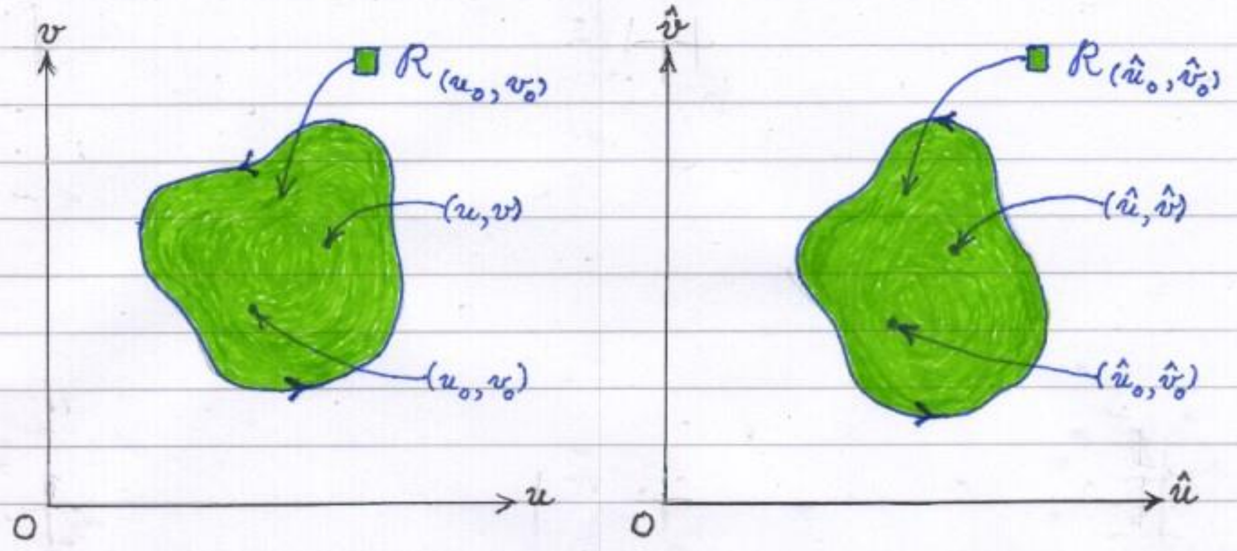


Fig. 5(b).

Subsequently, in the case of Fig. 5(b) it should therefore be noted that

(i) the ordered pair, $(u, v) = (u_1(x, y, \hat{x}, \hat{y}), v_1(x, y, \hat{x}, \hat{y}))$;

(ii) the ordered pair, $(u_0, v_0) = (u_1(x_0, y_0, \hat{x}_0, \hat{y}_0), v_1(x_0, y_0, \hat{x}_0, y_0))$;

(iii) the ordered pair, $(\hat{u}, \hat{v}) = (u_2(x, y, \hat{x}, \hat{y}), v_2(x, y, \hat{x}, \hat{y}))$;

(iv) the ordered pair, $(\hat{u}_0, \hat{v}_0) = (u_2(x_0, y_0, \hat{x}_0, \hat{y}_0), v_2(x_0, y_0, \hat{x}_0, \hat{y}_0))$,

as required. Q.E.D.

Theorem TIII-2.

Let there exist a quaternion (hypercomplex) function,

$$f(q) = u_1(x, y, \hat{x}, \hat{y}) + i v_1(x, y, \hat{x}, \hat{y}) + j u_2(x, y, \hat{x}, \hat{y}) + k v_2(x, y, \hat{x}, \hat{y}),$$

$\forall q \in dom(f) \subseteq \mathbb{H}$, whose concomitant limit,

$\lim_{q \to q_0}[f(q)] = l_0 = l_{10} + i\,l_{20} + j\,l_{30} + k\,l_{40} \,\varepsilon\, img(f) =$

$\{f(s) \mid s \,\varepsilon\, dom(f)\} \subseteq \mathbb{H}$.

Subsequently, it may be proven that this particular limit can likewise be graphically represented by means of four (4) distinct quaternionic analogues of the Argand diagram.

* * * * *

PROOF:-

From the author's paper [2] the reader will recall that the existence of the aforesaid limit accordingly implies the existence of two real numbers, $\delta > 0$ & $\epsilon > 0$, insofar as the inequality,

$|f(q) - l_0| < \epsilon$,

occurs simultaneously with the inequality,

$0 < |q - q_0| < \delta$,

such that we obtain, after making the appropriate algebraic substitutions, the pair of inequalities,

$$\left| \begin{array}{l} u_1(x,y,\hat{x},\hat{y}) + i\,v_1(x,y,\hat{x},\hat{y}) + j\,u_2(x,y,\hat{x},\hat{y}) + k\,v_2(x,y,\hat{x},\hat{y}) - \\ (l_{10} + i\,l_{20} + j\,l_{30} + k\,l_{40}) \end{array} \right| < \epsilon \quad \&$$

$$0 < |x + iy + j\hat{x} + k\hat{y} - (x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0)| < \delta$$

$$\therefore \left| \begin{array}{l} u_1(x,y,\hat{x},\hat{y}) - l_{10} + i(v_1(x,y,\hat{x},\hat{y}) - l_{20}) + j(u_2(x,y,\hat{x},\hat{y}) - l_{30}) \\ + k(v_2(x,y,\hat{x},\hat{y}) - l_{40}) \end{array} \right| < \epsilon \quad \&$$

$$0 < |x - x_0 + i(y - y_0) + j(\hat{x} - \hat{x}_0) + k(\hat{y} - \hat{y}_0)| < \delta$$

$$\therefore \sqrt{\begin{array}{l}(u_1(x,y,\hat{x},\hat{y})-l_{10})^2+(v_1(x,y,\hat{x},\hat{y})-l_{20})^2+(u_2(x,y,\hat{x},\hat{y})-l_{30})^2\\ +(v_2(x,y,\hat{x},\hat{y})-l_{40})^2\end{array}} < \epsilon$$

&

$$0 < \sqrt{(x-x_0)^2+(y-y_0)^2+(\hat{x}-\hat{x}_0)^2+(\hat{y}-\hat{y}_0)^2} < \delta$$

$$\therefore \left\{\begin{array}{l}(u_1(x,y,\hat{x},\hat{y})-l_{10})^2+(v_1(x,y,\hat{x},\hat{y})-l_{30})^2+\\ (u_2(x,y,\hat{x},\hat{y})-l_{30})^2+(v_2(x,y,\hat{x},\hat{y})-l_{40})^2\end{array}\right\} < \epsilon^2 \quad (1) \quad \&$$

$$0 < (x-x_0)^2+(y-y_0)^2+(\hat{x}-\hat{x}_0)^2+(\hat{y}-\hat{y}_0)^2 < \delta^2 \quad (2).$$

Now, let there exist four real numbers, $\delta_1 > 0$; $\delta_2 > 0$; $\delta_3 > 0$ & $\delta_4 > 0$, such that we respectively obtain the inequalities,

$$\delta_1^2 \leq (x-x_0)^2+(y-y_0)^2 \leq \delta_2^2 \quad (3) \quad \&$$

$$\delta_3^2 \leq (\hat{x}-\hat{x}_0)^2+(\hat{y}-\hat{y}_0)^2 \leq \delta_4^2 \quad (4).$$

Clearly, the addition of these two inequalities thus yields

$$\delta_1^2+\delta_3^2 \leq (x-x_0)^2+(y-y_0)^2+(\hat{x}-\hat{x}_0)^2+(\hat{y}-\hat{y}_0)^2 \leq \delta_2^2+\delta_4^2,$$

whereupon it immediately follows that by generating the inequalities,

$$\delta_1^2+\delta_3^2 > 0 \quad \& \quad \delta_2^2+\delta_4^2 < \delta^2,$$

we futher obtain the inequality,

$$0 < \delta_1^2+\delta_3^2 \leq (x-x_0)^2+(y-y_0)^2+(\hat{x}-\hat{x}_0)^2+(\hat{y}-\hat{y}_0)^2 \leq \delta_2^2+\delta_4^2 < \delta^2,$$

which is identical to inequality (2) and hence inequalities (3) & (4) can be respectively written as

$$\delta_1 \leq \sqrt{(x-x_0)^2 + (y-y_0)^2} \leq \delta_2 \quad (5) \;\&$$

$$\delta_3 \leq \sqrt{(\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2} \leq \delta_4 \quad (6).$$

Similarly, let there exist two real numbers, $\epsilon_1 > 0$ & $\epsilon_2 > 0$, such that we respectively obtain the inequalities,

$$(u_1(x,y,\hat{x},\hat{y}) - l_{10})^2 + (v_1(x,y,\hat{x},\hat{y}) - l_{20})^2 \leq \epsilon_1^2 \quad (7) \;\&$$

$$(u_2(x,y,\hat{x},\hat{y}) - l_{30})^2 + (v_2(x,y,\hat{x},\hat{y}) - l_{40})^2 \leq \epsilon_2^2 \quad (8).$$

Clearly, the addition of these two inequalities thus yields

$$\left\{\begin{matrix}(u_1(x,y,\hat{x},\hat{y}) - l_{10})^2 + (v_1(x,y,\hat{x},\hat{y}) - l_{20})^2 + \\ (u_2(x,y,\hat{x},\hat{y}) - l_{30})^2 + (v_2(x,y,\hat{x},\hat{y}) - l_{40})^2\end{matrix}\right\} \leq \epsilon_1^2 + \epsilon_2^2 ,$$

whereupon it immediately follows that by generating the inequality,

$$\epsilon_1^2 + \epsilon_2^2 < \epsilon^2,$$

we further obtain the inequality,

$$\left\{\begin{matrix}(u_1(x,y,\hat{x},\hat{y}) - l_{10})^2 + (v_1(x,y,\hat{x},\hat{y}) - l_{20})^2 + \\ (u_2(x,y,\hat{x},\hat{y}) - l_{30})^2 + (v_2(x,y,\hat{x},\hat{y}) - l_{40})^2\end{matrix}\right\} \leq \epsilon_1^2 + \epsilon_2^2 < \epsilon^2,$$

which is identical to inequality (1) and hence inequalities (7) & (8) can be respectively written as

$$\sqrt{(u_1(x,y,\hat{x},\hat{y}) - l_{10})^2 + (v_1(x,y,\hat{x},\hat{y}) - l_{20})^2} \leq \epsilon_1 \quad (9) \;\&$$

$$\sqrt{(u_2(x,y,\hat{x},\hat{y}) - l_{30})^2 + (v_2(x,y,\hat{x},\hat{y}) - l_{40})^2} \leq \epsilon_2 \quad (10).$$

In view of inequalities (5); (6); (9) & (10), we deduce that the limit,

$$\lim_{q \to q_0}\left[f(q)\right] = \ell_0 = \ell_{10} + i\ell_{20} + j\ell_{30} + k\ell_{40},$$

can be graphically represented by modifying the antecedent Fig. 5(a) & Fig. 5(b) in such a way as to generate the following quaternionic analogues of the Argand diagram, namely –

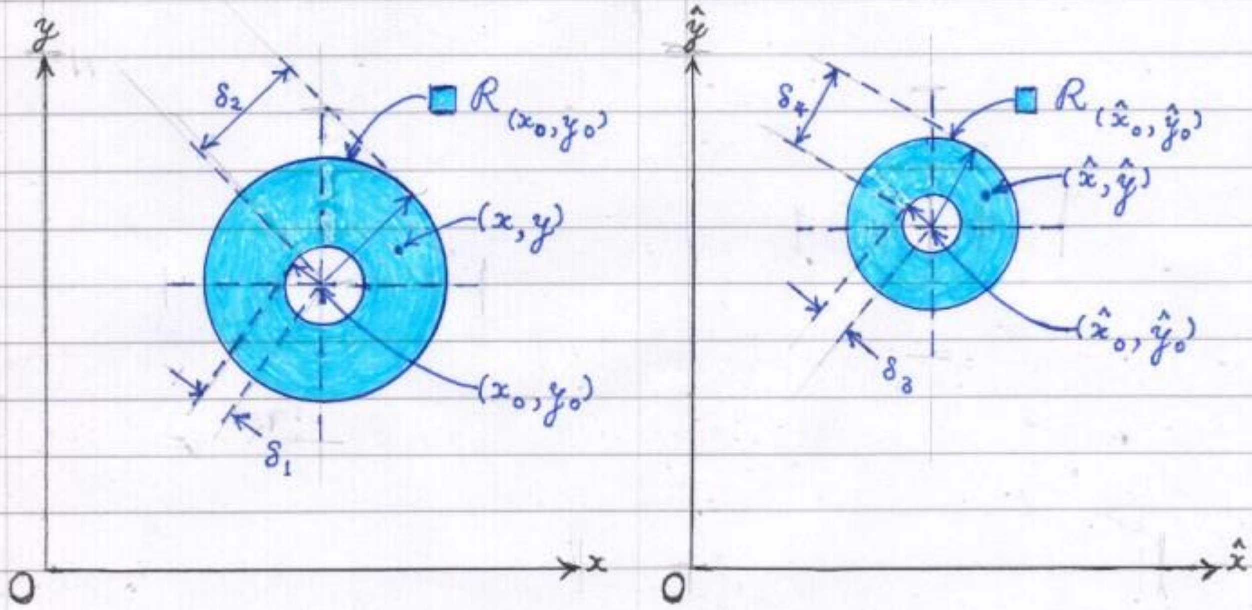


Fig. 6(a).

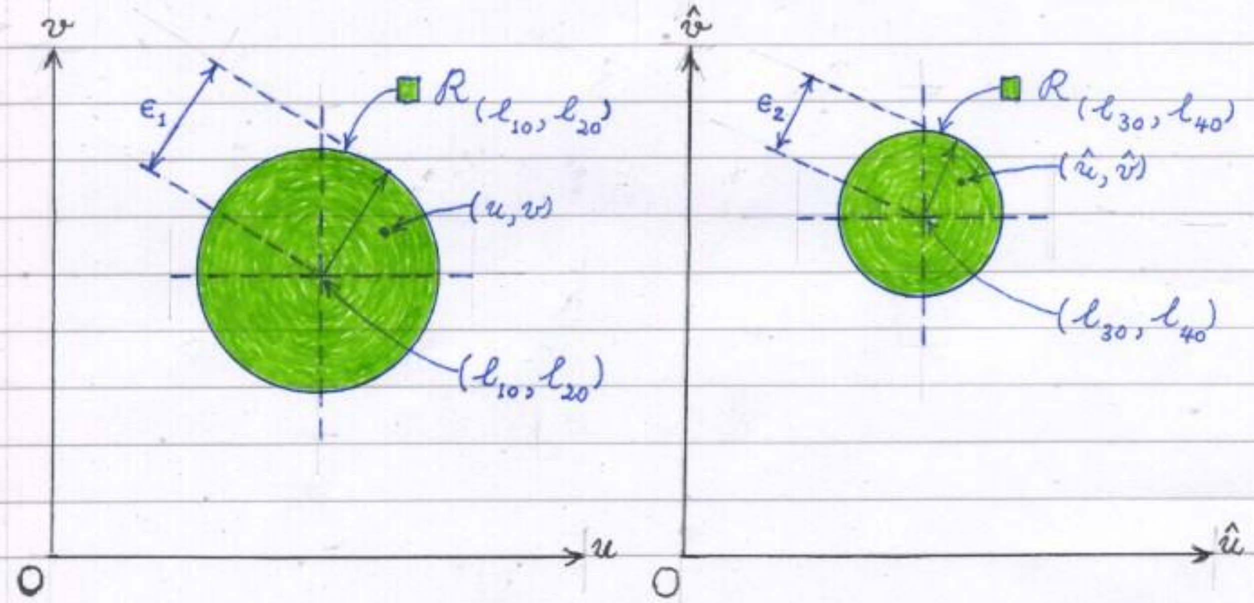


Fig. 6(b).

Subsequently, in the case of Fig. 6(a) it should therefore be noted that

(i) the region,

$$R_{(x_0,y_0)} = \{(s_1,s_2) \mid \delta_1 \leq \sqrt{(s_1-x_0)^2+(s_2-y_0)^2} \leq \delta_2\};$$

(ii) the region,

$$R_{(\hat{x}_0,\hat{y}_0)} = \{(s_1,s_2) \mid \delta_3 \leq \sqrt{(s_1-\hat{x}_0)^2+(s_2-\hat{y}_0)^2} \leq \delta_4\};$$

(iii) the inequalities, $\delta_1^2+\delta_3^2 > 0$ & $0 < \delta_2^2+\delta_4^2 < \delta^2$,

and, similarly, in the case of Fig. 6(b) it should likewise be noted that

(i) the ordered pairs,

$$(u,v) = (u_1(x,y,\hat{x},\hat{y}), v_1(x,y,\hat{x},\hat{y})) \quad \&$$
$$(\hat{u},\hat{v}) = (u_2(x,y,\hat{x},\hat{y}), v_2(x,y,\hat{x},\hat{y})) \quad ;$$

(ii) the region,

$$R_{(\ell_{10},\ell_{20})} = \{(s_1,s_2) \mid \sqrt{(s_1-\ell_{10})^2+(s_2-\ell_{20})^2} \leq \epsilon_1\};$$

(iii) the region,

$$R_{(\ell_{30},\ell_{40})} = \{(s_1,s_2) \mid \sqrt{(s_1-\ell_{30})^2+(s_2-\ell_{40})^2} \leq \epsilon_2\};$$

(iv) the inequality, $0 < \epsilon_1^2+\epsilon_2^2 < \epsilon^2$,

as required. Q.E.D.

Theorem TIII-3.

Let, there exist a quaternion (hypercomplex) function,

$$f(q) = u_1(x,y,\hat{x},\hat{y}) + i v_1(x,y,\hat{x},\hat{y}) + j u_2(x,y,\hat{x},\hat{y}) + k v_2(x,y,\hat{x},\hat{y}),$$

which is continuous, $\forall q \in dom(f) \subseteq \mathbb{H}$, insofar as its concomitant limit,

$$\lim_{q \to q_0}[f(q)] = f(q_0) = f(x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0)$$

$$= u_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + i v_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + j u_2(x_0,y_0,\hat{x}_0,\hat{y}_0) + k v_2(x_0,y_0,\hat{x}_0,\hat{y}_0),$$

$$\in img(f) = \{f(s) \mid s \in dom(f)\} \subseteq \mathbb{H}.$$

Subsequently, it may be proven that this particular limit can likewise be graphically represented by means of four (4) distinct quaternionic analogues of the Argand diagram.

* * * * *

PROOF:-

From the author's paper [2] the reader will recall that the existence of the aforesaid limit accordingly implies the existence of two real numbers, $\delta > 0$ & $\epsilon > 0$, insofar as the inequality,

$$|f(q) - f(q_0)| < \epsilon,$$

occurs simultaneously with the inequality,

$$|q - q_0| < \delta,$$

such that we obtain, after making the appropriate algebraic substitutions, the pair of inequalities,

$$\left| \begin{matrix} u_1(x,y,\hat{x},\hat{y}) + i\,v_1(x,y,\hat{x},\hat{y}) + j\,u_2(x,y,\hat{x},\hat{y}) + k\,v_2(x,y,\hat{x},\hat{y}) - \\ \left[\begin{matrix} u_1(x_0,y_0,\hat{x}_0,y_0) + i v_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + j u_2(x_0,y_0,\hat{x}_0,\hat{y}_0) + \\ k v_2(x_0,y_0,\hat{x}_0,\hat{y}_0) \end{matrix} \right] \end{matrix} \right| < \epsilon \ \&$$

$$|x + iy + j\hat{x} + k\hat{y} - (x_0 + iy_0 + j\hat{x}_0 + k\hat{y}_0)| < \delta$$

$$\therefore \left| \begin{matrix} u_1(x,y,\hat{x},\hat{y}) - u_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + i(v_1(x,y,\hat{x},\hat{y}) - v_1(x_0,y_0,\hat{x}_0,\hat{y}_0)) \\ j(u_2(x,y,\hat{x},\hat{y}) - u_2(x_0,y_0,\hat{x}_0,\hat{y}_0)) + k(v_2(x,y,\hat{x},\hat{y}) - v_2(x_0,y_0,\hat{x}_0,\hat{y}_0)) \end{matrix} \right| < \epsilon \ \&$$

$$|x - x_0 + i(y - y_0) + j(\hat{x} - \hat{x}_0) + k(\hat{y} - \hat{y}_0)| < \delta$$

$$\therefore \sqrt{\begin{matrix} (u_1(x,y,\hat{x},\hat{y}) - u_1(x_0,y_0,\hat{x}_0,y_0))^2 + (v_1(x,y,\hat{x},\hat{y}) - v_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + \\ (u_2(x,y,\hat{x},\hat{y}) - u_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_2(x,y,\hat{x},\hat{y}) - v_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 \end{matrix}} < \epsilon \ \&$$

$$\sqrt{(x-x_0)^2 + (y-y_0)^2 + (\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2} < \delta$$

$$\therefore \left[\begin{matrix} (u_1(x,y,\hat{x},\hat{y}) - u_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_1(x,y,\hat{x},\hat{y}) - v_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + \\ (u_2(x,y,\hat{x},\hat{y}) - u_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_2(x,y,\hat{x},\hat{y}) - v_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 \end{matrix} \right]$$

$$< \epsilon^2 \quad (1) \quad \&$$

$$(x-x_0)^2 + (y-y_0)^2 + (\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2 < \delta^2 \qquad (2).$$

Now, let there exist two real numbers, $\delta_1 > 0$ & $\delta_2 > 0$, such that we respectively obtain the inequalities,

$$(x-x_0)^2 + (y-y_0)^2 \leq \delta_1^2 \quad (3) \quad \&$$

$$(\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2 \leq \delta_2^2 \quad (4).$$

Clearly, the addition of these two inequalities thus yields

$$(x-x_0)^2 - (y-y_0)^2 + (\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2 \leq \delta_1^2 + \delta_2^2,$$

whereupon it immediately follows that by generating the inequality,

$$\delta_1^2 + \delta_2^2 < \delta^2,$$

we further obtain the inequality,

$$(x-x_0)^2 + (y-y_0)^2 + (\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2 \leq \delta_1^2 + \delta_2^2 < \delta^2,$$

which is identical to inequality (2) and hence inequalities (3) & (4) can be respectively written as

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} \leq \delta_1 \quad (5) \quad \&$$

$$\sqrt{(\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2} \leq \delta_2 \quad (6).$$

Similarly, let there exist two real numbers, $\epsilon_1 > 0$ & $\epsilon_2 > 0$, such that we respectively obtain the inequalities,

$$(u_1(x,y,\hat{x},\hat{y}) - u_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_1(x,y,\hat{x},\hat{y}) - v_1(x_0,y_0,\hat{x}_0,y_0))^2 \leq \epsilon_1^2 \quad (7)$$

&

$$(u_2(x,y,\hat{x},\hat{y}) - u_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_2(x,y,\hat{x},\hat{y}) - v_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 \leq \epsilon_2^2 \quad (8).$$

Clearly, the addition of these two inequalities thus yields

$$\begin{bmatrix} (u_1(x,y,\hat{x},\hat{y}) - u_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_1(x,y,\hat{x},\hat{y}) - v_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + \\ (u_2(x,y,\hat{x},\hat{y}) - u_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_2(x,y,\hat{x},\hat{y}) - v_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 \end{bmatrix}$$

$$\leq \epsilon_1^2 + \epsilon_2^2,$$

whereupon it immediately follows that by generating the inequality,

$\epsilon_1^2 + \epsilon_2^2 < \epsilon^2$,

we further obtain the inequality,

$$\begin{bmatrix}(u_1(x,y,\hat{x},\hat{y}) - u_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_1(x,y,\hat{x},\hat{y}) - v_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + \\ (u_2(x,y,\hat{x},\hat{y}) - u_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_2(x,y,\hat{x},\hat{y}) - v_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2\end{bmatrix}$$

$\leq \epsilon_1^2 + \epsilon_2^2 < \epsilon^2$,

which is identical to inequality (1) and hence inequalities (7) & (8) can be respectively written as

$$\sqrt{(u_1(x,y,\hat{x},\hat{y}) - u_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_1(x,y,\hat{x},\hat{y}) - v_1(x_0,y_0,\hat{x}_0,\hat{y}_0))^2} \leq \epsilon_1 \quad (9)$$

&

$$\sqrt{(u_2(x,y,\hat{x},\hat{y}) - u_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2 + (v_2(x,y,\hat{x},\hat{y}) - v_2(x_0,y_0,\hat{x}_0,\hat{y}_0))^2} \leq \epsilon_2 \quad (10).$$

In view of inequalities (5); (6); (9) & (10), we deduce that the limit,

$$\lim_{q \to q_0} [f(q)] = f(q_0) = \begin{matrix} u_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + i v_1(x_0,y_0,\hat{x}_0,\hat{y}_0) + \\ j u_2(x_0,y_0,\hat{x}_0,\hat{y}_0) + k v_2(x_0,y_0,\hat{x}_0,\hat{y}_0) \end{matrix},$$

can be graphically represented by modifying the antecedent Fig. 6(a) & Fig. 6(b) in such a way so as to generate the following quaternionic analogues of the Argand diagram, namely –

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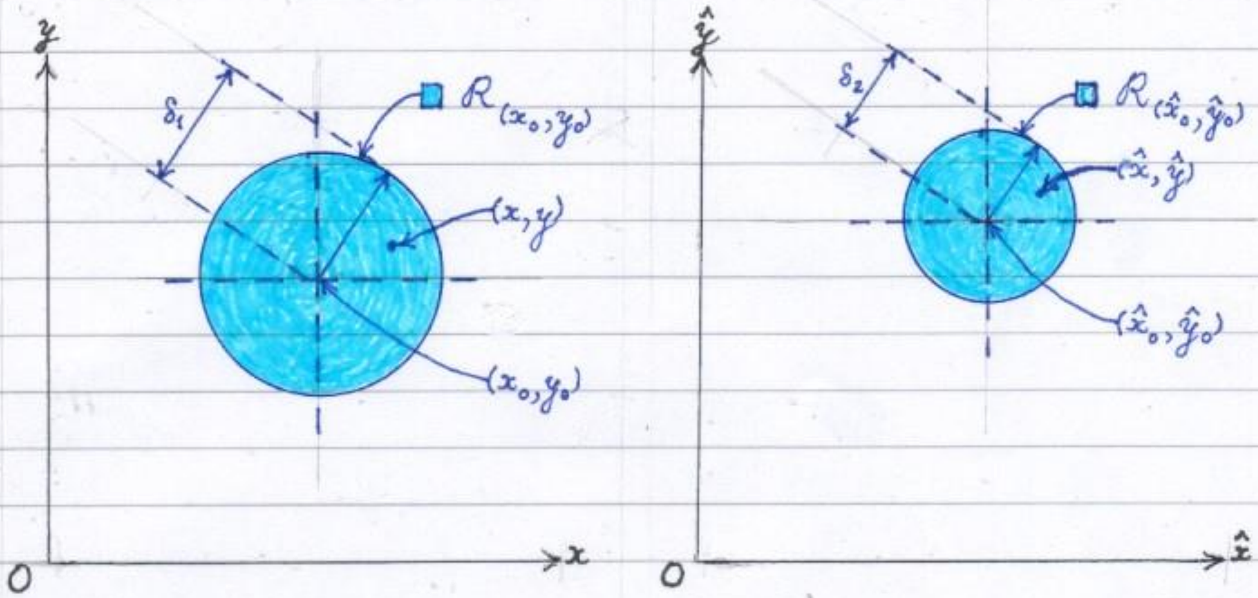

Fig. 7(a).

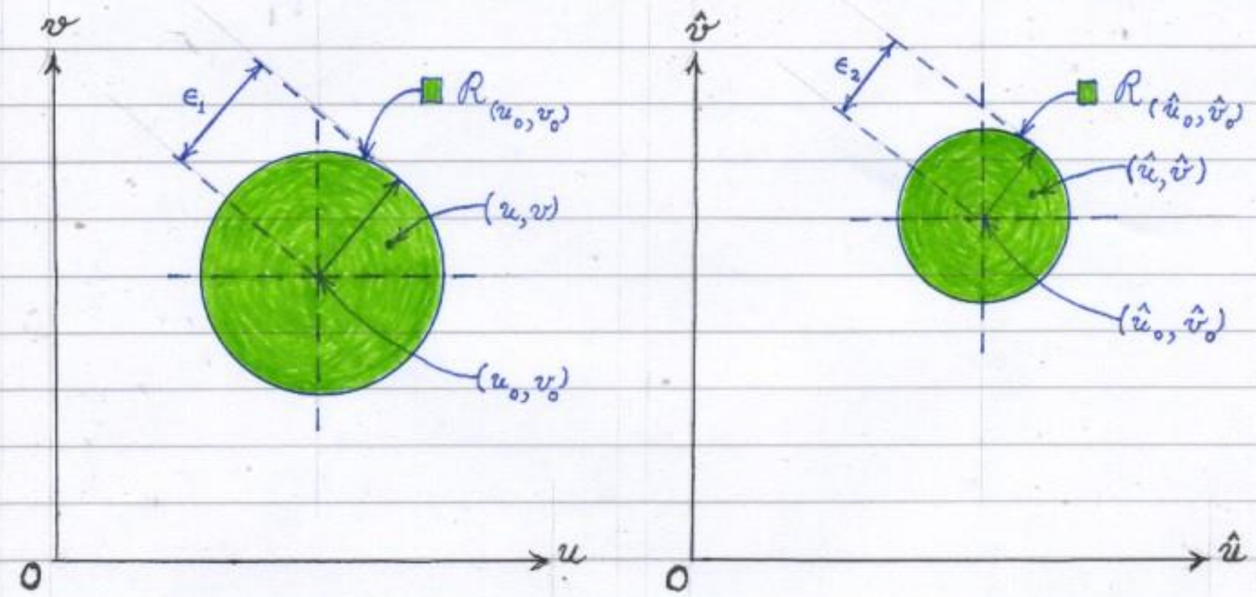

Fig. 7(b).

Subsequently, in the case of Fig. 7(a) it should therefore be noted that

(i) the region,

$$R_{(x_0,y_0)} = \{(s_1,s_2) \mid \sqrt{(s_1-x_0)^2+(s_2-y_0)^2} \leqslant \delta_1\};$$

(ii) the region,

$$R_{(\hat{x}_0,\hat{y}_0)} = \{(s_1,s_2) \mid \sqrt{(s_1-\hat{x}_0)^2+(s_2-\hat{y}_0)^2} \leq \delta_2\};$$

(iii) the inequality, $0 < \delta_1^2 + \delta_2^2 < \delta^2$,

and, similarly, in the case of Fig. 7(b) it should likewise be noted that

(i) the ordered pairs,

$$(u,v) = (u_1(x,y,\hat{x},\hat{y}),\ v_1(x,y,\hat{x},\hat{y}));$$
$$(u_0,v_0) = (u_1(x_0,y_0,\hat{x}_0,\hat{y}_0),\ v_1(x_0,y_0,\hat{x}_0,\hat{y}_0));$$
$$(\hat{u},\hat{v}) = (u_2(x,y,\hat{x},\hat{y}),\ v_2(x,y,\hat{x},\hat{y}));$$
$$(\hat{u}_0,\hat{v}_0) = (u_2(x_0,y_0,\hat{x}_0,\hat{y}_0),\ v_2(x_0,y_0,\hat{x}_0,\hat{y}_0));$$

(ii) the region,

$$R_{(u_0,v_0)} = \{(s_1,s_2) \mid \sqrt{(s_1-u_0)^2+(s_2-v_0)^2} \leq \epsilon_1\};$$

(iii) the region,

$$R_{(\hat{u}_0,\hat{v}_0)} = \{(s_1,s_2) \mid \sqrt{(s_1-\hat{u}_0)^2+(s_2-\hat{v}_0)^2} \leq \epsilon_2\};$$

(iv) the inequality, $0 < \epsilon_1^2 + \epsilon_2^2 < \epsilon^2$,

as required. Q.E.D.

IV. Appraisal of Results with a View To Further Theoretical Development.

As is evident from the contents of the preceding sections, we have enunciated the graphical representation of quaternions and their concomitant functions merely as a generalisation of the same representations having been utilised in complex variable anlysis without providing any specific examples to that effect. By direct contrast, however, Churchill et al. [1] have provided many such examples in relation to

(a) the mapping of elementary functions;
(b) the conformal mapping of analytic functions;
(c) specific applications of the conformal mapping of analytic functions &
(d) the Schwarz-Cristoffel transformation.

Subsequently, the author envisages that quaternionic analogues of these particular mappings and transformations should therefore be formulated by means of any additional papers to be published in the forseeable future.

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[*] Website address:- https://vixra.org/author/stephen_c_pearson.

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