THE RIEMANN HYPOTHESIS IS TRUE: THE END OF THE MYSTERY

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This paper is dedicated to the memory of my Father who taught me arithmetic,
To my wife Wahida, my daughter Sinda and my son Mohamed Mazen
'I feel that these aren't the right techniques to solve the Riemann hypothesis itself, it's going to need some big idea from somewhere else.'

James Maynard (07/15/2024)

Abstract. In 1859, Georg Friedrich Bernhard Riemann had announced the following conjecture, called Riemann Hypothesis : *The nontrivial roots (zeros)* $s = \sigma + it$ *of the zeta function, defined by:*

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \quad \Re(s) > 1$$

have real part $\sigma = \frac{1}{2}$. In this note, I give the proof that $\sigma = \frac{1}{2}$ using an equivalent statement of the Riemann Hypothesis concerning the Dirichlet η function.

1. Introduction

In 1859, G.F.B. Riemann had announced the following conjecture [2] known Riemann Hypothesis:

Conjecture 1.1. Let $\zeta(s)$ be the complex function of the complex variable $s = \sigma + it$ defined by the analytic continuation of the function:

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

over the whole complex plane, with the exception of $s = 1$. Then the nontrivial zeros of $\zeta(s) = 0$ are written as :

$$s = \frac{1}{2} + it$$

In this paper, our idea is to start from an equivalent statement of the Riemann Hypothesis, namely the one concerning the Dirichlet η function. The latter is

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related to Riemann's ζ function where we do not need to manipulate any expression of $\zeta(s)$ in the critical band $0 < \Re(s) < 1$. In our calculations, we will use the definition of the limit of real sequences. We arrive to give the proof that $\sigma = \dfrac{1}{2}$.

1.1. **The function zeta(s).** We denote $s = \sigma + it$ the complex variable of $\mathbb{C}$. For $\Re(s) = \sigma > 1$, let ζ_1 be the function defined by :

$$\zeta_1(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}, \text{ for } \Re(s) = \sigma > 1$$

We know that with the previous definition, the function ζ_1 is an analytical function of s. Denote by $\zeta(s)$ the function obtained by the analytic continuation of $\zeta_1(s)$ to the whole complex plane, minus the point $s = 1$, then we recall the following theorem [3]:

Theorem 1.2. *The function $\zeta(s)$ satisfies the following :*

1. $\zeta(s)$ has no zero for $\Re(s) > 1$;
2. the only pole of $\zeta(s)$ is at $s = 1$; it has residue 1 and is simple;
3. $\zeta(s)$ has trivial zeros at $s = -2, -4, \dots$;
4. the nontrivial zeros lie inside the region $0 \leq \Re(s) \leq 1$ (called the critical strip) and are symmetric about both the vertical line $\Re(s) = \dfrac{1}{2}$ and the real axis $\Im(s) = 0$.

The vertical line $\Re(s) = \dfrac{1}{2}$ is called the critical line.

For our proof, we will use the function presented by G.H. Hardy [4] namely Dirichlet eta function [3]:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)$$

The function eta is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$ [3].

We have also the theorem (see page 16, [4]):

Theorem 1.3. *For all $t \in \mathbb{R}$, $\zeta(1 + it) \neq 0$.*

So, we take the critical strip as the region defined as $0 < \Re(s) < 1$.

1.2. **A Equivalent statement to the Riemann Hypothesis.** Among the equivalent statements to the Riemann Hypothesis is that of the Dirichlet eta function which is stated as follows [3]:

Equivalence 1.4. The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function :

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s), \quad \sigma > 1 \tag{1.1}$$

that fall in the critical strip $0 < \Re(s) < 1$ lie on the critical line $\Re(s) = \dfrac{1}{2}$.

The series (1.1) is convergent, and represents $(1-2^{1-s})\zeta(s)$ for $\Re(s) = \sigma > 0$ ([4], pages 20-21). We can rewrite:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = (1-2^{1-s})\zeta(s), \quad \Re(s) = \sigma > 0 \tag{1.2}$$

$\eta(s)$ is a complex number, it can be written as :

$$\eta(s) = \rho.e^{i\alpha} \Longrightarrow \rho^2 = \eta(s).\overline{\eta(s)} \tag{1.3}$$

and $\eta(s) = 0 \Longleftrightarrow \rho = 0$.

2. Preliminaries of the proof that the zeros of the function eta(s) are on the critical line $\Re(s) = 1/2$

Proof. We denote $s = \sigma + it$ with $0 < \sigma < 1$. We consider one zero of $\eta(s)$ that falls in critical strip and we write it as $s = \sigma + it$, then we obtain $0 < \sigma < 1$ and $\eta(s) = 0 \Longleftrightarrow (1-2^{1-s})\zeta(s) = 0$. We verify easily the two propositions:

$$\boxed{s, \textit{ is one zero of } \eta(s) \textit{ that falls in the critical strip, is also one zero of}}$$
$$\boxed{\zeta(s) \text{ in the critical strip}} \tag{2.1}$$

Conversely, if s is a zero of $\zeta(s)$ in the critical strip, let $\zeta(s) = 0 \Longrightarrow \eta(s) = (1-2^{1-s})\zeta(s) = 0$, then s is also one zero of $\eta(s)$ in the critical strip. We can write:

$$\boxed{s, \textit{ is one zero of } \zeta(s) \textit{ that falls in the critical strip, is also one zero of}}$$
$$\boxed{\eta(s) \text{ in the critical strip}} \tag{2.2}$$

Let us write the function η:

$$\eta(s) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-sLogn} = \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-(\sigma+it)Logn} =$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}.e^{-itLogn}$$
$$= \sum_{n=1}^{+\infty} (-1)^{n-1} e^{-\sigma Logn}(cos(tLogn) - isin(tLogn))$$

The function η is convergent for all $s \in \mathbb{C}$ with $\Re(s) > 0$, but not absolutely convergent. We definite the sequence of functions $((\eta_n)_{n \in \mathbb{N}^*}(s))$ as:

$$\eta_n(s) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^s} = \sum_{k=1}^{n} (-1)^{k-1} \frac{cos(tLogk)}{k^\sigma} - i \sum_{k=1}^{n} (-1)^{k-1} \frac{sin(tLogk)}{k^\sigma}$$

with $s = \sigma + it$ and $t \neq 0$.

Let $s=\sigma+it$ with $0<\sigma<1$ be one zero of the function eta, then :

$$\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^s}=0$$

or:

$$\forall\epsilon'>0\quad\exists n_0,\forall N>n_0,\left|\sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^s}\right|<\epsilon'$$

It follows that we can write $lim_{n\longrightarrow+\infty}\eta_n(s)=0=\eta(s)$. We obtain:

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{cos(tLogk)}{k^\sigma}=0$$

$$lim_{n\longrightarrow+\infty}\sum_{k=1}^{n}(-1)^{k-1}\frac{sin(tLogk)}{k^\sigma}=0$$

Using the definition of the limit of a sequence, we can write:

$$\forall\epsilon_1>0\,\exists n_r,\forall N>n_r,\mid\Re(\eta(s)_N)\mid<\epsilon_1\Longrightarrow\Re^2(\eta(s)_N)<{\epsilon_1}^2\tag{2.3}$$

$$\forall\epsilon_2>0\,\exists n_i,\forall N>n_i,\mid\Im(\eta(s)_N)\mid<\epsilon_2\Longrightarrow\Im^2(\eta(s)_N)<{\epsilon_2}^2\tag{2.4}$$

Then:

$$0<\sum_{k=1}^{N}\frac{cos^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}cos(tLogk).cos(tLogk')}{k^\sigma k'^\sigma}<\epsilon_1^2$$

$$0<\sum_{k=1}^{N}\frac{sin^2(tLogk)}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}\frac{(-1)^{k+k'}sin(tLogk).sin(tLogk')}{k^\sigma k'^\sigma}<\epsilon_2^2$$

Taking $\epsilon=\epsilon_1=\epsilon_2$ and $N>max(n_r,n_i)$, we get by making the sum member to member of the last two inequalities:

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}<2\epsilon^2\tag{2.5}$$

We can write the above equation as :

$$0<\ \rho_N^2\ <2\epsilon^2\tag{2.6}$$

or $\rho(s)=0$.

3. Case $0<\Re(s)<1/2$

Suppose there exists $s=\sigma+it$ one zero of $\eta(s)$ or $\eta(s)=0\Longrightarrow\rho^2(s)=0$ with $0<\sigma<\frac{1}{2}\Longrightarrow s$ lies inside the critical band. We write the equation (2.5):

$$0<\sum_{k=1}^{N}\frac{1}{k^{2\sigma}}+2\sum_{k,k'=1;k<k'}^{N}(-1)^{k+k'}\frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}<2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k^{2\sigma}} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma}$$

But $2\sigma < 1$, it follows that $lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} \frac{1}{k^{2\sigma}} \longrightarrow +\infty$ and then, we obtain :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(tLog(k/k'))}{k^\sigma k'^\sigma} = -\infty} \tag{3.1}$$

4. Case $\Re(s) = 1/2$

We suppose that $\sigma = \frac{1}{2}$. Let's start by recalling Hardy's theorem (1914) ([3], page 24):

Theorem 4.1. *There are infinitely many zeros of $\zeta(s)$ on the critical line.*

From the propositions (2.1-2.2), it follows the proposition :

Proposition 4.2. *There are infinitely many zeros of $\eta(s)$ on the critical line.*

Let $s_j = \frac{1}{2} + it_j$ one of the zeros of the function $\eta(s)$ on the critical line, so $\eta(s_j) = 0$. The equation (2.5) is written for s_j:

$$0 < \sum_{k=1}^{N} \frac{1}{k} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} < 2\epsilon^2$$

or:

$$\sum_{k=1}^{N} \frac{1}{k} < 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

If $N \longrightarrow +\infty$, the series $\sum_{k=1}^{N} \frac{1}{k}$ is divergent and becomes infinite. then:

$$\sum_{k=1}^{+\infty} \frac{1}{k} \leq 2\epsilon^2 - 2 \sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}}$$

Hence, we obtain the following result:

$$\boxed{lim_{N \longrightarrow +\infty} \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t_j Log(k/k'))}{\sqrt{k}\sqrt{k'}} = -\infty} \tag{4.1}$$

if not, we will have a contradiction with the fact that :

$$lim_{N \longrightarrow +\infty} \sum_{k=1}^{N} (-1)^{k-1} \frac{1}{k^{s_j}} = 0 \Longleftrightarrow \eta(s) \text{ is convergent for } s_j = \frac{1}{2} + it_j$$

5. Case $1/2 < \Re(s) < 1$

Let $s = \sigma + it$ be the zero of $\eta(s)$ in $0 < \Re(s) < \frac{1}{2}$, object of the section 3. From the proposition (2.1), $\zeta(s) = 0$. According to point 4 of theorem 1.2, the complex number $s' = 1 - \sigma + it = \sigma' + it'$ with $\sigma' = 1 - \sigma$, $t' = t$ and $\frac{1}{2} < \sigma' < 1$ verifies $\zeta(s') = 0$, so s' is also a zero of the function $\zeta(s)$ in the band $\frac{1}{2} < \Re(s) < 1$, it follows from the proposition (2.2) that $\eta(s') = 0 \Longrightarrow \rho(s') = 0$. By applying (2.5), we get:

$$0 < \sum_{k=1}^{N} \frac{1}{k^{2\sigma'}} + 2 \sum_{k,k'=1;k<k'}^{N} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} < 2\epsilon^2 \tag{5.1}$$

As $0 < \sigma < \frac{1}{2} \Longrightarrow 2 > 2\sigma' = 2(1-\sigma) > 1$, then the series $\sum_{k=1}^{N} \dfrac{1}{k^{2\sigma'}}$ is convergent to a positive constant not null $C(\sigma')$. As $1/k^2 < 1/k^{2\sigma'}$ for all $k > 0$, then :

$$0 < \zeta(2) = \frac{\pi^2}{6} = \sum_{k=1}^{+\infty} \frac{1}{k^2} < \sum_{k=1}^{+\infty} \frac{1}{k^{2\sigma'}} = C(\sigma') = \zeta_1(2\sigma') = \zeta(2\sigma')$$

From the equation (5.1), it follows that :

$$\boxed{\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty} \tag{5.2}$$

5.0.1. *Case $t = 0$.* We suppose that $t = 0 \Longrightarrow t' = 0$. We known the following proposition:

Proposition 5.1. *For all $s = \sigma$ real with $0 < \sigma < 1$, $\eta(s) > 0$ and $\zeta(s) < 0$.*

We deduce the contradiction with the hypothesis $s' = \sigma'$ is a zero of $\eta(s)$ and:

$$\boxed{\text{The equation (5.2) is false for the case } t' = t = 0.} \tag{5.3}$$

5.0.2. *Case $t' = t \neq 0$.* We suppose that $t' \neq 0$. Let $s' = \sigma' + it' = 1 - \sigma + it$ a zero of $\eta(s)$, we have:

$$\sum_{k,k'=1;k<k'}^{+\infty} (-1)^{k+k'} \frac{cos(t'Log(k/k'))}{k^{\sigma'} k'^{\sigma'}} = -\frac{C(\sigma')}{2} = -\frac{\zeta(2\sigma')}{2} > -\infty \tag{5.4}$$

the left member of the equation (5.4) above is finite and depends of σ' and t', but the right member is a function only of σ' equal to $-\zeta(2\sigma')/2$.

We recall the following theorem (see page 140, [4]):

Theorem 5.2.

$$lim_{T \longrightarrow +\infty} \frac{1}{T} \int_1^T \mid \zeta(\sigma" + i\tau) \mid^2 d\tau = \zeta(2\sigma") \quad (\sigma" > \frac{1}{2}) \tag{5.5}$$

Let t_0 so that $t_0 \geq 1$. As the integral of the left member of the above equation is convergent, the equation (5.5) can be written as:

$$lim_{T \longrightarrow +\infty} \frac{1}{T} \int_{t_0}^{T} |\zeta(\sigma" + i\tau)|^2 d\tau = \zeta(2\sigma")$$

and $\zeta(2\sigma")$ is independent of any t_0 then in particular for $t_0 = t'$. As $\sigma"$ is any $\sigma" > 1/2$, I choose $\sigma" = \sigma'$ and $t_0 = t'$, it follows that $\zeta(2\sigma')$ does not depend of t' so that $s' = \sigma' + it'$ is a root of η. Hence, the contradiction with equation (5.2). Then the equation (5.4) is false.

$$\boxed{\text{It follows that the equation (5.4) is false for the case } t' \neq 0.} \tag{5.6}$$

It follows that the equation (5.2) is false and $\eta(s')$ does not vanish for $\sigma' \in]1/2, 1[$.

From (5.3-5.6), we conclude that the function $\eta(s)$ has no zeros for all $s' = \sigma' + it'$ with $\sigma' \in]1/2, 1[$, it follows that the case of the section (3) above concerning the case $0 < \Re(s) < \dfrac{1}{2}$ is false too. Then, the function $\eta(s)$ has all its zeros on the critical line $\sigma = \dfrac{1}{2}$. From the equivalent statement (1.4), it follows that **the Riemann hypothesis is verified**. $\square$

We therefore announce the important theorem as follows:

Theorem 5.3. *The Riemann Hypothesis is true:*
All nontrivial zeros of the function $\zeta(s)$ with $s = \sigma + it$ lie on the vertical line $\Re(s) = \dfrac{1}{2}$.

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