EASY AERODYNAMICS
KUTTA-ZHUKOWSKY (JOUKOWSKI) LIFT

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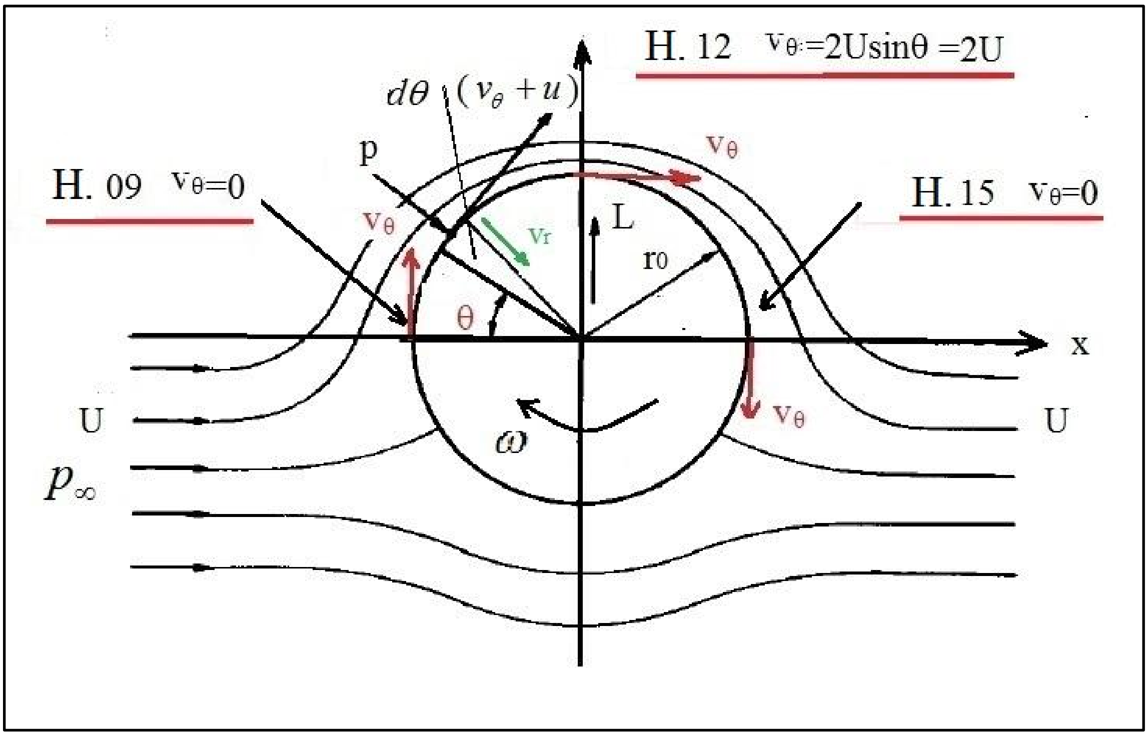
Abstract: calculation of the lift in flight – Kutta-Zhukowsky's Equation – a personal summary.

Contents:

KUTTA-ZHUKOWSKY'S EQUATION



We have a long cylinder in a uniform flow U and spinning with a constant angular speed ω. This makes the fluid over the cylinder surface have a speed component $u = r_0\omega$, because of the viscosity. Still on the cylinder surface there is the v_θ component from the U flow. In Appendix 2 (eq. (2.5)) we prove the component v_θ at h. 12.00 is 2U, so around the cylinder at different θ angles, we will have: $v_\theta = 2U\sin\theta$. Therefore, around the cylinder we will have tangentially a $v_T = v_\theta + u = 2U\sin\theta + r_0\omega$.

After naming p_∞ the pressure far away from the cylinder and p that over the cylinder, according to the Bernoulli Equation (see Appendix 3): $p + \frac{1}{2}\rho v^2 + \rho g h = const$, that is:

$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2$ and being all at the same height h, we have, after a comparison between the flow on the left side of the cylinder and that over the cylinder:

$p_\infty + \frac{\rho}{2}U^2 = p + \frac{\rho}{2}(2U\sin\theta + r_0\omega)^2$. Therefore:

$$\frac{p - p_\infty}{\frac{\rho}{2}U^2} = 1 - \left(\frac{2U\sin\theta + r_0\omega}{U}\right)^2 \qquad (1)$$

If we now consider the infinitesimal element of arc $r_0 d\theta$, it makes along an l (el) stretch of cylinder (which ideally extends into the sheet), a small surface $l r_0 d\theta$. Its projection over the orizontal plane is obviously: $l r_0 \sin\theta d\theta$. Over it the pressure $(p - p_\infty)$ acts, and as it is drawn downwards in the drawing, the relevant force over the small surface is upwards and its sign will be opposite: $dL_{[N]} = -(p - p_\infty) l r_0 \sin\theta d\theta$. Let's evaluate now the force per unit of length of the cylinder: $dL_{[N/m]} = dL = -(p - p_\infty) r_0 \sin\theta d\theta$ (with no more the small l - el).

About the total lift L, we will integrate from bottom to top of the cylinder, that is between $-\pi/2$ and $+\pi/2$; not only; we will multiply by two, as the cylinder has got two "sides", back and front:

$$L = L_{[N/m]} = 2\int_{-\pi/2}^{+\pi/2} -(p - p_\infty) r_0 \sin\theta d\theta = -r_0 \rho U^2 \int_{-\pi/2}^{+\pi/2} [1 - (\frac{2U\sin\theta + r_0\omega}{U})^2] \sin\theta d\theta =$$

$$= -r_0 \rho U^2 \int_{-\pi/2}^{+\pi/2} [1 - (\frac{r_0\omega}{U})^2 - \frac{4r_0\omega}{U}\sin\theta - 4\sin^2\theta] \sin\theta d\theta = 2\pi \cdot r_0^2 \omega\rho U = 2\pi \cdot r_0 u \rho U$$

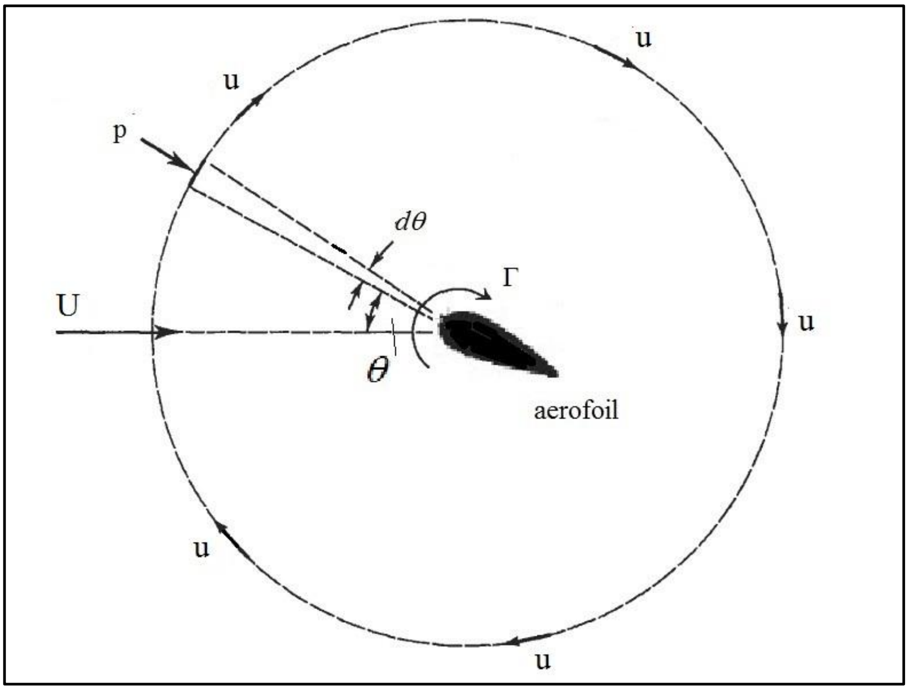
or: $L = 2\pi \cdot r_0 u \rho U$ (2)

Now, as the peripheral speed u of the cylinder is all over the circumference, the circulation Γ (speed by space of action) is: $\Gamma = 2\pi \cdot r_0 u$, from which:

$L = \rho U \Gamma$ (Kutta-Joukowski Equation for the spinning cylinder) (3)

Therefore, we can state that when a circulation Γ is generated and not only by a spinning cylinder, but also by an aerofoil (wing), a lift L is generated (see here below).

KUTTA-JOUKOWSKI FOR A GENERAL AEROFOIL
(Kutta-Joukowski Theorem)



Here we have an aerofoil which generates a circulation Γ (a speed u taken on a crown at a certain distance r from the foil, in order to be under the approximation limits).

If we consider here, as well, a small surface created by $d\theta$ and if we use the Bernoulli equation far from the calculation point (where the pressure is p_∞ and the speed of the flow is U) and on the calculation point (where we have p and $(\vec{U} + \vec{u})^2 = U^2 + u^2 + 2Uu\cos(90° - \theta) = U^2 + u^2 + 2Uu\sin\theta$):

$$p_\infty + \frac{\rho}{2}U^2 = p + \frac{\rho}{2}(\vec{U} + \vec{u})^2 = p + \frac{\rho}{2}(U^2 + u^2 + 2Uu\sin\theta)$$

As we are at a certain distance from the aerofoil, r is relatively large and u^2 can be neglected with respect to U^2, from which: $p = p_\infty - \rho U u \sin\theta$. (4)

Now, as well as in the previous case of the cylinder, the force due to such a pressure p vertically on the small surface (per metre of wing length):

$lr\sin\theta d\theta / l = r\sin\theta d\theta$ (5)

is: $dL_{[N/m]} = dL_C = -pr\sin\theta d\theta$, (L on the crown) and according to (4):

$dL_{[N/m]} = dL_C = -(p_\infty - \rho U u \sin\theta) r \sin\theta d\theta$, from which, after integrating between 0 and 2π (so avoiding also to multiply by two), we have:

$$L_{[N/m]} = L_C = -\int_0^{2\pi} (p_\infty - \rho U u \sin\theta) r \sin\theta d\theta = \rho u U r \pi \qquad (6)$$

This is just one of the vertical contributions of force, as in opposition to the case of the cylinder, where we made calculations on the edge of the cylinder, there was no access of fluid into the cylinder itself, as it was impenetrable.

But here we are considering a crown which is far away from the foil, so through the infinitesimal surface given by (5) in its horizontal component, some fluid has access, and it's an access of mass. As we have flow from left to right, the access of mass is through the vertical component of the surface, so through:

$lr\cos\theta d\theta / l = r\cos\theta d\theta$ and such a flow dQ, in [(kg/s)/m], due to U, will be:

$$dQ = \rho U r \cos\theta d\theta \ . \qquad (7)$$

And we also have to say that such a mass, entered into the crown, undergoes an increase of vertical speed (due to the existence of u) which is:

$$\Delta v = u\cos\theta \ , \qquad (8)$$

from which, as the change of the linear momentum is a force, we have:

$$dQ \cdot \Delta v = \rho U u r \cos^2\theta d\theta = dL_I \ , \qquad (9)$$

where L_I is the inertial contribution to the lift. By integrating now the (9) between 0 and 2π , we get L_I:

$$L_I = \int_0^{2\pi} dL_I = \int_0^{2\pi} \rho U u r \cos^2\theta d\theta = \rho u U r \pi \qquad (10)$$

and for the total lift $L = L_C + L_I$ we sum up the contributions of the (6) and (10), so having:

$L = 2\rho u U r \pi$ and as above we defined the circulation $\Gamma = 2\pi \cdot ru$, we get, once again, the Kutta Joukowski Equation:

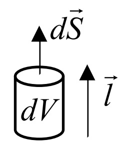
$$L = \rho U \Gamma \quad \text{(Kutta Joukowski Equation for an aerofoil).} \qquad (11)$$

APPENDIX 1 – The Continuity Equation

Continuity Equation $\dfrac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$; proof:

$\rho\vec{v} = \vec{J}$ is the mass current density $[\dfrac{kg/s}{m^2}]$ (dimensionally obvious)

$M = \int_V \rho \cdot dV$ (held obvious)



We have: $\dfrac{\partial}{\partial t} M = \dfrac{\partial}{\partial t}\int_V \rho \cdot dV = \int_V \dfrac{\partial \rho}{\partial t} \cdot dV = -\int_S \rho\vec{v} \cdot d\vec{S}$,in fact, <u>in terms of dimensions</u>:

$dV = \vec{l} \cdot d\vec{S}$ and so $\dfrac{\partial}{\partial t} dV = d\vec{S}\dfrac{\partial \vec{l}}{\partial t} = d\vec{S} \cdot \vec{v}$ and sign – is in case of "escaping" mass.

So: $\int_V \dfrac{\partial \rho}{\partial t} dV = -\int_S (\rho\vec{v}) \cdot d\vec{S} = -\int_V \vec{\nabla} \cdot (\rho\vec{v}) \cdot dV$, after having used the Divergence Theorem in the last equality.

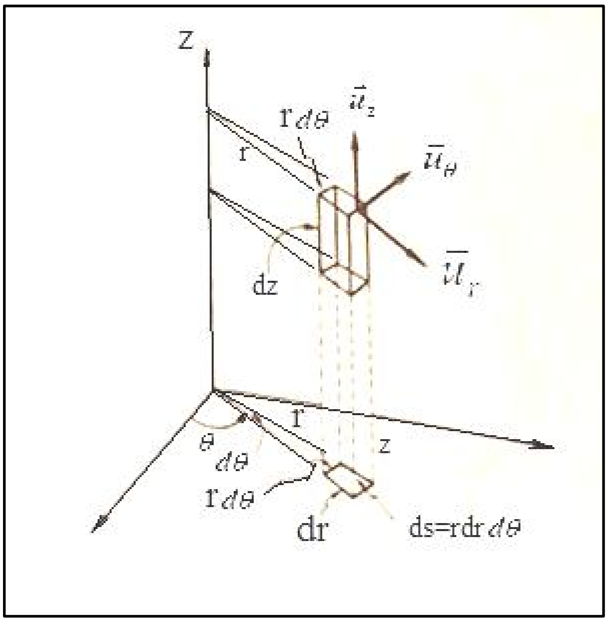
Therefore: $\int_V [\dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho\vec{v})] dV = 0$, from which we get the Continuity Equation.

APPENDIX 2 – The Potential of the flow

About the speed v on the circumference of the spinning cylinder, and in particular about the component v_θ due to the external flow, whose horizontal speed far away is U, let's find a potential function ϕ so that $v = \nabla\phi$; from the Continuity Equation (see App. 1) $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v})$ we have, if ρ is constant, $\vec{\nabla}\cdot(\rho\vec{v}) = \rho\vec{\nabla}\cdot\vec{v} = 0$, so $\vec{\nabla}\cdot\vec{v} = 0$, and $\vec{\nabla}\cdot\vec{v} = \vec{\nabla}\cdot(\nabla\phi) = \Delta\phi = 0$ and $\Delta\phi = 0$ is the Laplace's Equation:

Laplacian: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$, $\Delta\phi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})\phi = 0$.

Let's remind now the expression for the Laplacian in cylindrical coordinates (very suitable for us, as our spinning cylinder has got a cylindrical symmetry indeed):

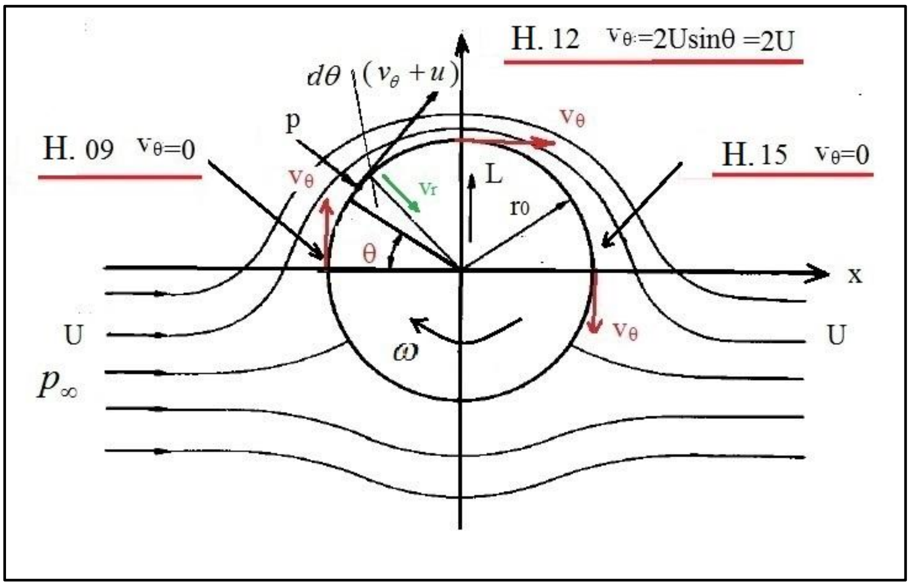


$\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$, and with reference to our spinning cylinder, let's consider the z axis as ideally penetrating into the sheet. Moreover, out of symmetry reasons of our long cylinder, the component with z is zero, so: $\Delta\phi = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$ and for our new Laplace's Equation, we have: $\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$, that is:

$$r\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r}) + \frac{\partial^2\phi}{\partial\theta^2} = 0. \quad (2.1)$$

Therefore, in cylindrical coordinates we can say that the v speed components around the cylinder and due to the speed of the external flow (U, from far away), will be formally three: v_θ, v_r and v_z, but out of geometric reasons and out of reasons of uniformity of the cylinder, we have that $v_z = 0$. We realize that the function $\phi = A_n r^n \cos(n\theta)$ is a solution for the (2.1) both with either n=+1 or n=-1 (and A_n / A_{-n} = constants to be found), so we have at least two solutions: $\phi_1 = A_1 r\cos\theta$ and $\phi_2 = A_{-1}r^{-1}\cos(-\theta) = A_{-1}r^{-1}\cos\theta$. So, the sum of them will be a more general solution:

$$\phi = \phi_1 + \phi_2 = A_1 r\cos\theta + A_{-1}r^{-1}\cos\theta\ . \quad (2.2)$$



With reference to the above figure and after suitably reconciling all the algebraic signs, in order to make the equations fit the figure, we see that a bit on the left of the point at h. 09 (as if it was about a clock…) the speed of the flow is U and is horizontal, from left to right, while over the cylinder, right at h. 09, the speed gets obviously zero, and it will eventually increase upwards (v_θ) as long as θ increases. The same is about the point at h. 15, on the right side of which the flow will then get back a U horizontal speed, towards right, that is -U.

Let's use those two points (h. 09 and h. 15) to evaluate ϕ and then, from the equation we will find, we will calculate the value of v_θ at h. 12:

$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}(A_1 r\cos\theta + A_{-1}r^{-1}\cos\theta) = -A_1\sin\theta - A_{-1}r^{-2}\sin\theta$, from which:

$v_{\theta=0} = -A_1\sin 0 - A_{-1}r^{-2}\sin 0 = 0$. Same value for θ=180°.

Moreover, obviously: $v_r = \frac{\partial\phi}{\partial r} = \frac{\partial}{\partial r}(A_1 r\cos\theta + A_{-1}r^{-1}\cos\theta) = A_1\cos\theta - A_{-1}r^{-2}\cos\theta$, from which, as we must also have $v_{r_0} = 0$, we have: $0 = A_1\cos\theta - A_{-1}r_0^{-2}\cos\theta$, that is: $A_{-1} = r_0^2 A_1$, and so:

$\phi = A_1 r\cos\theta + r_0^2 A_1 r^{-1}\cos\theta = A_1\cos\theta(r + \frac{r_0^2}{r})$. Therefore:

$$\phi = A_1\cos\theta(r + \frac{r_0^2}{r}) \qquad (2.3)$$

Now, in order to figure out A1, let's get on the horizontal axis (θ=0 or θ=180°) and let's evaluate v_r, which must be, for $r \to +\infty$, respectively U and –U:

$v_r = \frac{\partial\phi}{\partial r} = A_1\cos\theta(1 - \frac{r_0^2}{r^2})$, $(v_r)_{\theta=0} = A_1(1 - \frac{r_0^2}{r^2})$, $(v_r)_{\theta=180°} = -A_1(1 - \frac{r_0^2}{r^2})$, that is:

$(v_r)_{\theta=0\_r\to+\infty} = A_1 = U$, $(v_r)_{\theta=180°\_r\to+\infty} = -A_1 = -U$ and so, finally:

$$\phi = U\cos\theta(r + \frac{r_0^2}{r}) \qquad (2.4)$$

At last, about v_θ evaluated in a generic θ and in θ=90°, we have:

$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}[U\cos\theta(r + \frac{r_0^2}{r})] = -U(1 + \frac{r_0^2}{r^2})\sin\theta$ and over the circumference of the cylinder (r=r0): $v_\theta = -2U\sin\theta$, and also:

$$v_{\theta=90°} = -2U . \qquad (2.5)$$

Moreover, still vertically (θ=90°), or at a certain θ= θ0, but far away from the cylinder, ($r \to \infty$) the flow must get back to the imperturbated flow when there's no cylinder, that is with v=U:

$v_\theta = -U(1+\frac{r_0^2}{r^2})\sin\theta$, $v_r = U\cos\theta(1-\frac{r_0^2}{r^2})$,

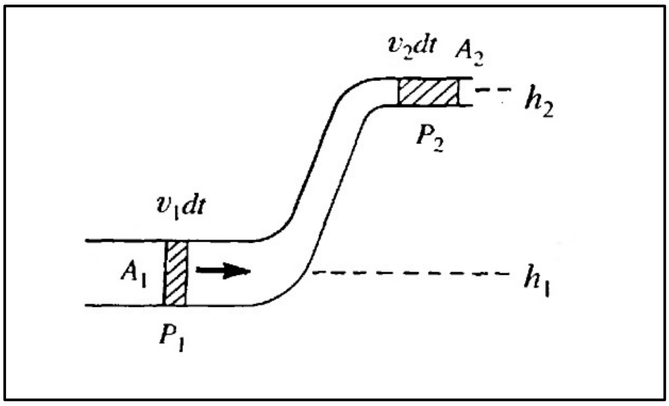
so, when θ=90° and $r \to \infty$, $v_\theta = -U$ and $v_r = 0$, that is, according to Pithagora: v=U.

On the contrary, for θ= θ0 and $r \to \infty$, $v_\theta = -U\sin\theta_0$ and $v_r = U\cos\theta_0$, that is, according to Pithagora: $v = \sqrt{[(U\cos\theta_0)^2 + (-U\sin\theta_0)^2]} = U$, so the (2.4) represents very well our flow, everywhere around the cylinder.

APPENDIX 3 – Bernoulli's Equation

According to the energy conservation:

$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 = const$



Or, with reference to the Navier Stokes Equation (8.1) in Appendix 8, if we are in a stationary situation, whereas $\vec{v} \neq f(t)$ >> $\frac{\partial \vec{v}}{\partial t} = 0$, and then $\rho = const$, and where there's no viscous forces, the Navier-Stokes Equation for sure reduce sto the Euler's one (but added with the gravitational component):

$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{g}$, and, better, as we said that $\frac{\partial \vec{v}}{\partial t} = 0$, we have:

$$(\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{g} \ . \qquad (3.1)$$

If now we consider the divergence and the gradient in terms of directional derivative, on direction $d\vec{l}$, specifically, then we have in (3.1): $\frac{dv}{dl}$ instead of $\vec{\nabla}\cdot\vec{v}$, and $\frac{dp}{dl}$ instead of $\vec{\nabla}p$ and then, still in (3.1), the gravitational acceleration $\vec{g}$ (which exerts along z, downwards) must be projected along $d\vec{l}$ ($\frac{dz}{dl}$ is the relevant direction cosine), and so (3.1) becomes:

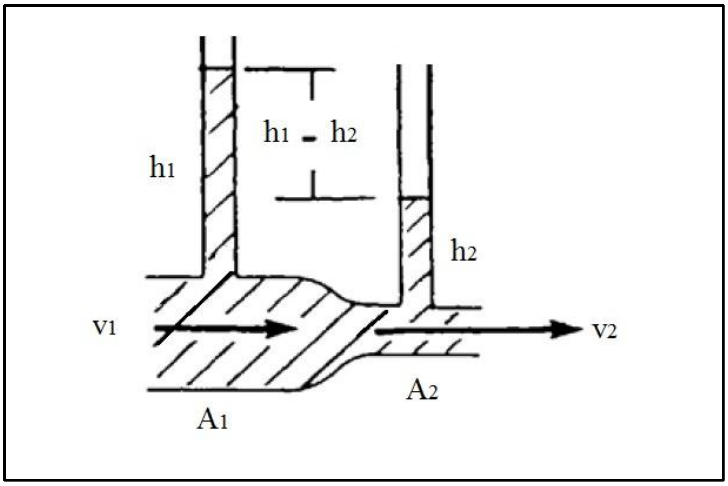
$v\frac{dv}{d\vec{l}} = -\frac{1}{\rho}\frac{dp}{dl} - g\frac{dz}{dl}$, from which: $vdv + \frac{1}{\rho}dp + gdz = 0$ and by integrating it:

$\frac{1}{2}v^2 + \frac{p}{\rho} + gz = 0$, and by multiplying by the density ρ , we get: $\frac{1}{2}\rho v^2 + p + \rho gz = 0$

that is, really the statement!

APPENDIX 4 – Venturi and Pitot

Venturi pipe and evaluation of the flow.



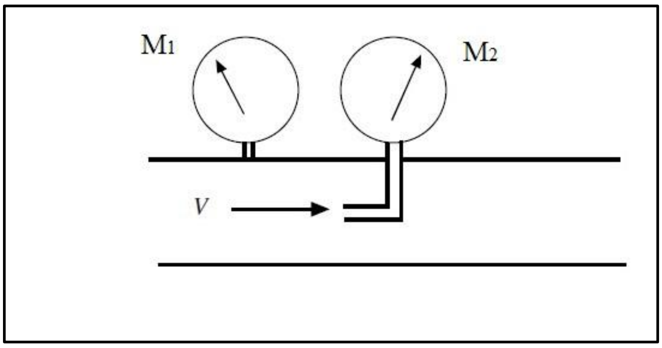
Where the pipe gets narrower, the fluid speed is higher, but the flow is uniform anyway along all the pipe ($Q = vS = v_1 S_1 = v_2 S_2$); by using the Bernoulli's Equation in these two points at different sections and same height (horizontally):

$p_1 + \frac{1}{2}\rho v_1^2 + \rho g h = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h$, so: $p_1 + \frac{1}{2}\rho v_1^2 = p_2 + \frac{1}{2}\rho v_2^2$, from which:

$$p_1 - p_2 = \frac{1}{2}\rho(v_2^2 - v_1^2) = \frac{1}{2}\rho v_2^2 (1 - \frac{v_1^2}{v_2^2}) = \frac{1}{2}\rho v_2^2 (1 - \frac{Q^2 / S_1^2}{Q^2 / S_2^2}) = \frac{1}{2}\rho v_2^2 (1 - \frac{S_2^2}{S_1^2})$$, from which:

$$Q = \sqrt{v_2^2 S_2^2} = \sqrt{\frac{2(p_1 - p_2)}{\rho(\frac{1}{S_2^2} - \frac{1}{S_1^2})}}$$

Pitot pipe and evaluation of the external flow speed.

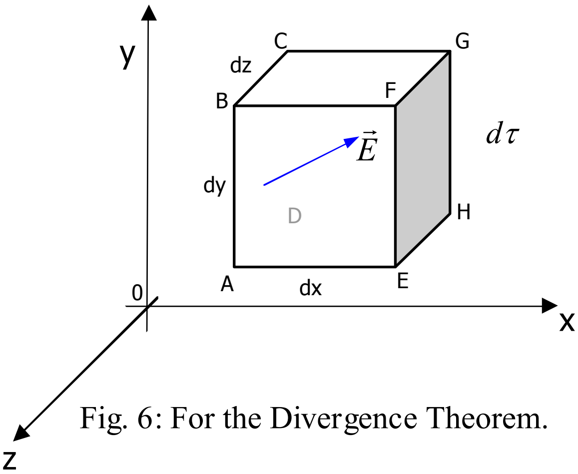


Starting from the Bernoulli's Equation, let p_1 be the external ambient pressure measured by the pressure gauge M_1 and p_2 that measured by pressure gauge M_2, which receives the fluid with speed:

$p_1 + \frac{1}{2}\rho 0 + \rho g h = p_2 + \frac{1}{2}\rho v_2^2 + \rho g h$, from which: $p_1 = p_2 + \frac{1}{2}\rho v_2^2$, so the speed of the fluid outside is: $v_2 = \sqrt{\frac{2}{\rho}(p_1 - p_2)}$.

APPENDIX 5 – Divergence and Rotor Theorems

Divergence Theorem (practical proof):



$$\int_S \vec{E}\cdot d\vec{S} = \int_V div\vec{E}\cdot dV$$

Fig. 6: For the Divergence Theorem.

Name ϕ the flux of the vector $\vec{E}$; we have:

$d\phi_{ABCD} = \vec{E}\cdot d\vec{S} = -E_x(x,\bar{y},\bar{z})dydz$ ($\bar{y}$ means y "mean")

$d\phi_{EFGH} = E_x(x+dx,\bar{y},\bar{z})dydz$, but we obviously know that also: (as a development):

$E_x(x+dx,\bar{y},\bar{z}) = E_x(x,\bar{y},\bar{z}) + \dfrac{\partial E_x(x,\bar{y},\bar{z})}{\partial x}dx$ so:

$d\phi_{EFGH} = E_x(x,\bar{y},\bar{z})dydz + \dfrac{\partial E_x(x,\bar{y},\bar{z})}{\partial x}dxdydz$ and so:

$d\phi_{ABCD} + d\phi_{EFGH} = \dfrac{\partial E_x}{\partial x}dV$. We similarly act on axes y and z:

$$d\phi_{AEHD} + d\phi_{BCGF} = \frac{\partial E_y}{\partial y}dV$$

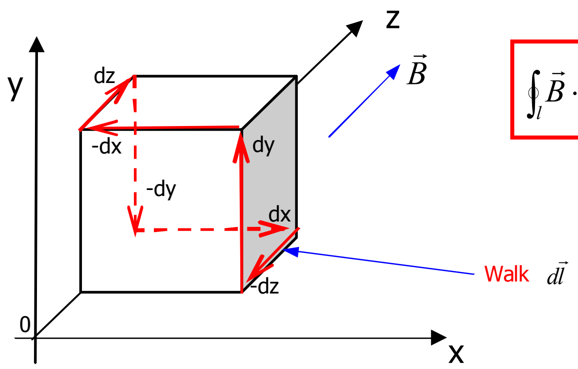
$$d\phi_{ABFE} + d\phi_{CGHD} = \frac{\partial E_z}{\partial z}dV$$

And then we sum up the fluxes so found, having totally:

$d\phi = (\dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z})dV = (div\cdot\vec{E})dV = (\vec{\nabla}\cdot\vec{E})dV$ therefore:

$\phi_S(\vec{E}) = \int_\phi d\phi = \int_S \vec{E}\cdot d\vec{S} = \int_V div\vec{E}\cdot dV = \int_V (\vec{\nabla}\cdot\vec{E})\cdot dV$ that is the statement.

Rotor or di Stokes' Theorem (practical proof-**by Rubino!**):



$$\oint_l \vec{B}\cdot d\vec{l} = \int_S rot\vec{B}\cdot d\vec{S} = \int_S \vec{\nabla}\times\vec{B}\cdot d\vec{S}$$

Fig. 7: For the Rotor Theorem (proof by Rubino).

Let's figure out $\vec{B}\cdot d\vec{l}$:
On dz B is B_z; on dx B is B_x; on dy B is B_y;
on -dz B is $B_z + \frac{\partial B_z}{\partial x}dx - \frac{\partial B_z}{\partial y}dy$, for 3-D Taylor's development and also because to go from the center of dz to that of –dz we go up along x, then we go down along y and nothing along z itself.
Similarly, on -dx B is $B_x - \frac{\partial B_x}{\partial z}dz + \frac{\partial B_x}{\partial y}dy$ and on -dy B is $B_y - \frac{\partial B_y}{\partial x}dx + \frac{\partial B_y}{\partial z}dz$.
By summing up all contributions:

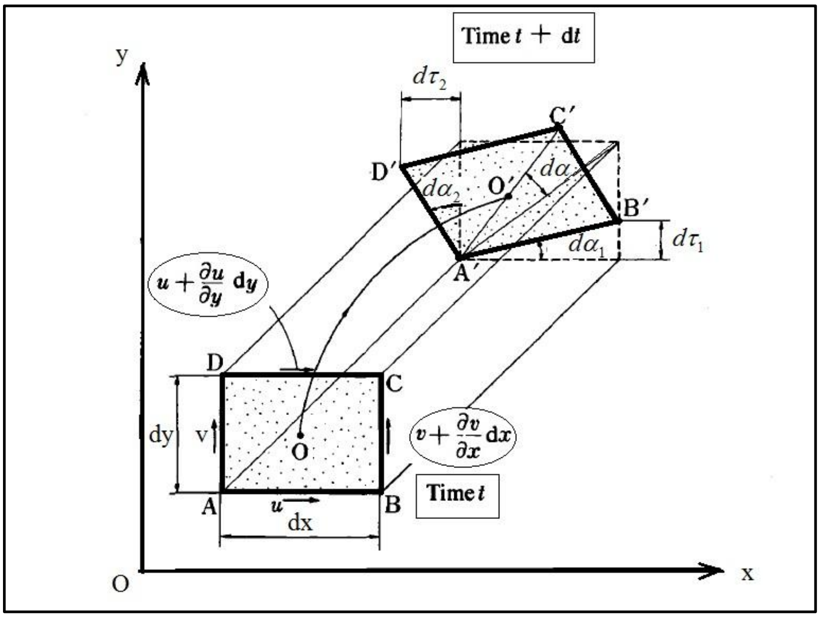
$$\vec{B}\cdot d\vec{l} = B_z dz - (B_z + \frac{\partial B_z}{\partial x}dx - \frac{\partial B_z}{\partial y}dy)dz + B_x dx - (B_x - \frac{\partial B_x}{\partial z}dz + \frac{\partial B_x}{\partial y}dy)dx + B_y dy -$$

$$+ (B_y - \frac{\partial B_y}{\partial x}dx + \frac{\partial B_y}{\partial z}dz)dy = (\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z})dydz + (\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x})dxdz + (\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y})dxdy =$$

$= rot\vec{B}\cdot d\vec{S} = \vec{\nabla}\times\vec{B}\cdot d\vec{S}$, where $d\vec{S}$ has got components $[\hat{x}(dydz), \hat{y}(dxdz), \hat{z}(dxdy)]$
that is, the statement: $\oint_l \vec{B}\cdot d\vec{l} = \int_S rot\vec{B}\cdot d\vec{S} = \int_S \vec{\nabla}\times\vec{B}\cdot d\vec{S}$, after having reminded of:

$$rot\vec{B} = \vec{\nabla}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix} .$$

APPENDIX 6 – Rotor and rotation



Why does the rotor is so called? Let's consider the elementary rectangle of flow ABCD in the above figure; at time t, it moves from O to O' and then, after a time dt, it wraps in A'B'C'D'.
AB on the x axis moves towards A'B' and also rotates about $d\tau_1$. AD, as well, in the y direction, rotates about $d\tau_2$.
So: $d\tau_1 = \frac{\partial v}{\partial x}dxdt$, $d\tau_2 = -\frac{\partial u}{\partial y}dydt$, from which: $d\alpha_1 = \frac{d\tau_1}{dx} = \frac{\partial v}{\partial x}dt$ and $d\alpha_2 = \frac{d\tau_2}{dy} = -\frac{\partial u}{\partial y}dt$.
The angular speeds of AB and AD are: $\omega_1 = \frac{d\alpha_1}{dt} = \frac{\partial v}{\partial x}$ and $\omega_2 = \frac{d\alpha_2}{dt} = -\frac{\partial u}{\partial y}$.

About the centre O, the mean angular speed is: $\omega = \frac{1}{2}(\omega_1 + \omega_2) = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) = \frac{1}{2}\Omega_z$, where $\Omega_z = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ which is the vorticity for the z axis. If the vorticity is zero, the flow is said irrotational. In three dimensions, we have: $\Omega = rotV = (\Omega_x, \Omega_y, \Omega_z) =$
$= [(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}),(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}),(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})]$, that is, if the rotor is zero, the fluid didn't rotate.

APPENDIX 7 – TheEuler's Equation

Euler's Equation: $(\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho})$.

(p is the pressure; moreover, this equation is a sketch of the Navier-Stokes Equation, whereas we're not yet taking into account the gravitational field and the viscous forces)
The force acting on a small fluid volume dV is $d\vec{f} = -p \cdot d\vec{S}$, with sign -, as we are dealing with a force towards the small volume. Moreover:
$\vec{f} = -\int_S p \cdot d\vec{S} = -\int_V \vec{\nabla} p \cdot dV$, after having used a dual of the Divergence theorem (a Green's formula), to go from the surface integral to the volume one.

We also have: $\frac{\partial \vec{f}}{\partial V} = \frac{\partial}{\partial V}[-\int_V \vec{\nabla} p \cdot dV] = -\vec{\nabla} p$, but, in terms of dimensions, it's simultaneously true that:
$\frac{\partial \vec{f}}{\partial V} = \frac{d}{dV}[M \frac{d\vec{v}}{dt}] = \frac{dM}{dV}\frac{d\vec{v}}{dt} = \rho \frac{d\vec{v}}{dt} =$ and from these two equations, we have:

$$\rho \frac{d\vec{v}}{dt} = -\vec{\nabla} p. \tag{7.1}$$

Now we remind that: $d\vec{l} = (dx, dy, dz)$, $\vec{\nabla} = (\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz})$ and $\vec{v} = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})$, so we can easily write that:

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{\partial \vec{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \vec{v}}{\partial y}\frac{dy}{dt} + \frac{\partial \vec{v}}{\partial z}\frac{dz}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = \frac{d\vec{v}}{dt}$$

and for (7.1) we finally have:

$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v} = -\frac{\vec{\nabla} p}{\rho}$ that is the Euler's Equation, indeed.

APPENDIX 8 – The Navier Stokes Equation

The Navier-Stokes Equation in the case o fan incompressibile fluid, that is $\rho = const$ and $\vec{\nabla}\cdot\vec{v} = 0$: (this situations is about most of practical cases)

$$\rho[\frac{\partial\vec{v}}{\partial t} + \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2] = -\vec{\nabla}p - \rho\vec{\nabla}\phi + \eta\nabla^2\vec{v} \qquad (8.1)$$

where $\vec{\Omega} = \vec{\nabla}\times\vec{v}$ (vorticity), η (viscosity), ϕ (gravitational potential), ρ (density), $\vec{v}$ (velocity), t (time).

Now, the terms of this Euler's Equation have the dimension of an acceleration $\vec{a}$; so, if we want to take into account the gravitational field, too, on the right side we can algebraically add the gravitational acceleration $\vec{g}$, with a negative sign, as it's downwards.

But we know that the gradient of the potential ϕ is really $\vec{g}$ ($\vec{\nabla}\phi = \vec{g}$), so:

$\frac{\partial\vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi$. As the following vectorial identity is in force:

$(\vec{v}\cdot\vec{\nabla})\vec{v} = (\vec{\nabla}\times\vec{v})\times\vec{v} + \frac{1}{2}\vec{\nabla}(\vec{v}\cdot\vec{v})$, and if we take the expression for the vorticity ($\vec{\Omega} = \vec{\nabla}\times\vec{v}$), we have: $\frac{\partial\vec{v}}{\partial t} + \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2 = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi$ and, so far, we have also taken into account the gravitational field. In the most general case where we have to do with a <u>viscous</u> fluid , we'll also add a viscous force component:

$$\frac{\partial\vec{v}}{\partial t} + \vec{\Omega}\times\vec{v} + \frac{1}{2}\vec{\nabla}v^2 = -\frac{\vec{\nabla}p}{\rho} - \vec{\nabla}\phi + \frac{\vec{f}_{visc}}{\rho} \qquad (8.2)$$

whereas $\vec{f}_{visc}$ is divided by the density because of the dimension compatibility with other terms in that equation.

(8.2) is already the Navier-Stokes Equation, whereas the viscous force $\vec{f}_{visc}$ is still to be evaluated.

We will evaluate $\vec{f}_{visc}$ in the case of incompressible fluids, that is fluids with $\rho = const$, >> $\frac{\partial\rho}{\partial t} = 0$ so, for the Continuity Equations, $\vec{\nabla}(\rho\vec{v}) = 0$, >> $\vec{\nabla}\cdot\vec{v} = 0$.

<u>Calculation of</u> $\vec{f}_{visc}$:

<u>VISCOSITY:</u>

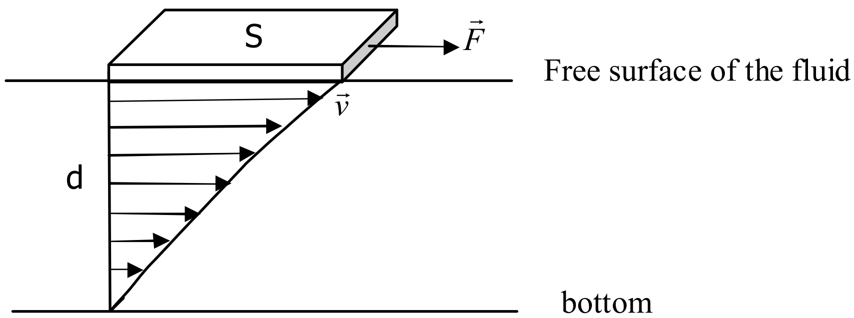


Fig. 1.

We know from general physics that: $\frac{\vec{F}}{S} = \eta\frac{\vec{v}}{d}$, (8.3) that is, in order to drag the slab whose base surface is S, over the fluid, at a d distance from the bottom, and drag it at a $\vec{v}$ speed, we need a force $\vec{F}$

Now, let's write down (8.3) in a differential form, for stresses $\vec{\tau}$ and for components: (x)

$\tau_x = \frac{F_x}{S} = \eta \frac{\partial u}{\partial y}$, having set $\vec{v} = (u, v, w)$, and so:

$$F_x = \eta \frac{\partial u}{\partial y} \cdot S \tag{8.4}$$

We now use (8.4) on a small fluid volume dV in Fig. 2:

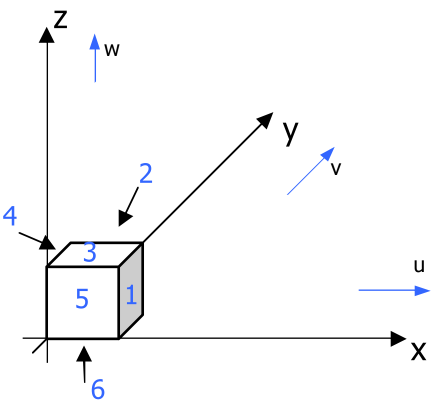


Fig. 2: Small volume of fluid dV.

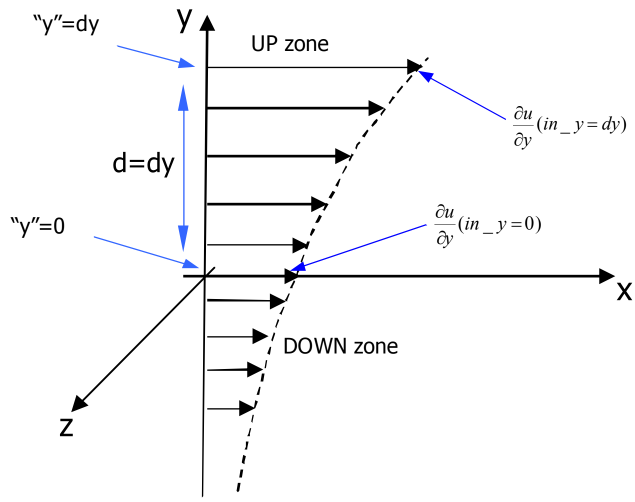


Fig. 3: Axis y, faces 2 and 5.

In Fig. 3 we have reproduced what shown in Fig. 1, but in a three-dimension context.

<u>Faces 2 and 5:</u>

so, with reference to Fig. 3, let's figure out the viscous forces (due to variations of u) on faces 2 and 5 of the small volume, that is those we meet when moving along the y axis, by using (8.4):

<u>Viscous shear stress on face 2</u> $= +\eta[\frac{\partial u}{\partial y}(in\_y = dy)]dxdz$

$\frac{\vec{v}}{d} in\_(8.3)$ $\quad S\_in\_(8.3)$

This force acting on face 2 is positive (+) because the fluid over the point where it's figured out (UP zone) has got a higher speed (longer horizontal arrows) which drags S along the positive x.

On face 5, on the contrary, we'll have a (-) negative sign, because the fluid under such S surface has got a lower speed (down) and want to be dragged, so making a resistance, that is a negative force:

<u>Viscous shear stress on face 5</u> $= -\eta[\frac{\partial u}{\partial y}(in\_y = 0)]dxdz$

The resultant on x is the difference between the two equations, or better, the algebraic sum:

$F_{x(y)} = \eta[\frac{\partial u}{\partial y}(y = dy) - \frac{\partial u}{\partial y}(y = 0)]dxdz = \eta \frac{[\frac{\partial u}{\partial y}(y = dy) - \frac{\partial u}{\partial y}(y = 0)]}{dy} dxdydz = \eta \frac{\partial^2 u}{\partial y^2} dV$, after having multiplied numerator and denominator by dy. Therefore:

$$F_{x(y)} = \eta \frac{\partial^2 u}{\partial y^2} dV \quad \text{(viscous force on x due to variations of u along y)} \tag{8.5}$$

<u>Faces 3 and 6:</u>

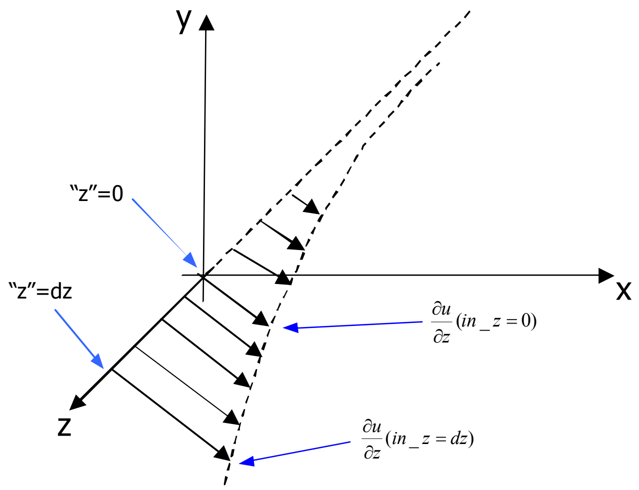


Fig. 4: Axis z, faces 3 and 6.

Similarly to the previous case, we have, as a result:

$$F_{x(z)} = \eta \frac{\partial^2 u}{\partial z^2} dV \quad \text{(viscous force on x due to variation of u along z)} \tag{8.6}$$

<u>Faces 1 and 4:</u>

For what case $F_{x(x)}$ is concerned, that is the viscous force on x due to variations of u (which is a component on x) along x itself, we will not talk about shear stresses, as, in such a case, the relevant force is still about x, but acts on S=dydz, which is orthogonal to x; so, it's about a NORMAL force, a tensile/compression one, and we refer to Fig. 5 below:

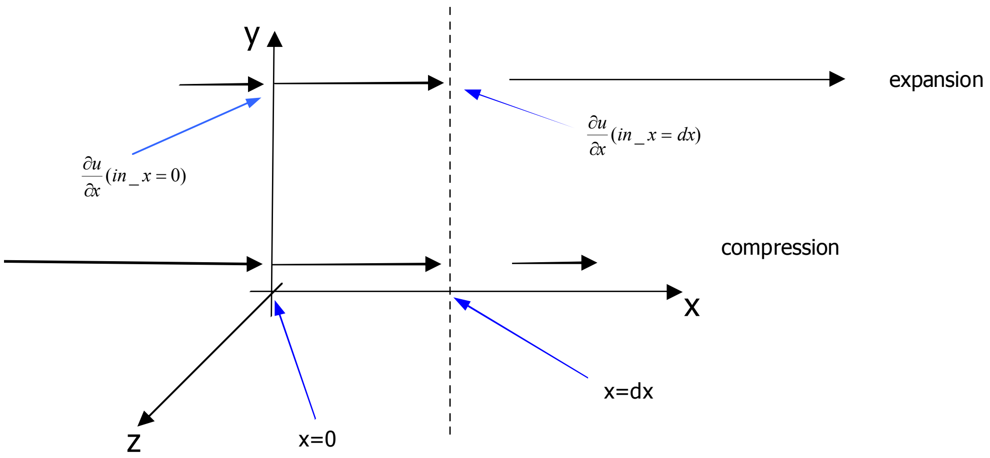


Fig. 5: Axis x, faces 1 and 4.

Anyway, nothing changes with numbers, with respect to previous cases, and we have:

$$F_{x(x)} = \eta \frac{\partial^2 u}{\partial x^2} dV \quad \text{(viscous force on x due to variations of u along x itself)} \tag{8.7}$$

Now that we have three components of the viscous forces acting along x (that is those due to variations of the u component (comp. x) of speed $\vec{v}$, with respect to y, z and x itself), let's sum them up and get F_{x-visc} :

$F_{x-visc} = \eta \frac{\partial^2 u}{\partial y^2} dV + \eta \frac{\partial^2 u}{\partial z^2} dV + \eta \frac{\partial^2 u}{\partial x^2} dV = \eta \cdot dV (\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial x^2}) = \eta \cdot dV \cdot \nabla^2 u$, and we rewrite it below:

$$F_{x-visc} = \eta \cdot dV \cdot \nabla^2 u \tag{8.8}$$

Now we carry out the same reasonings for an evaluation of F_{y-visc} and of F_{z-visc}, and obviously get ($\vec{v} = (u, v, w)$):

$$F_{y-visc} = \eta \cdot dV \cdot \nabla^2 v \tag{8.9}$$

$$F_{z-visc} = \eta \cdot dV \cdot \nabla^2 w \tag{8.10}$$

from which, finally, by adding (8.8), (8.9), and (8.10), we have:

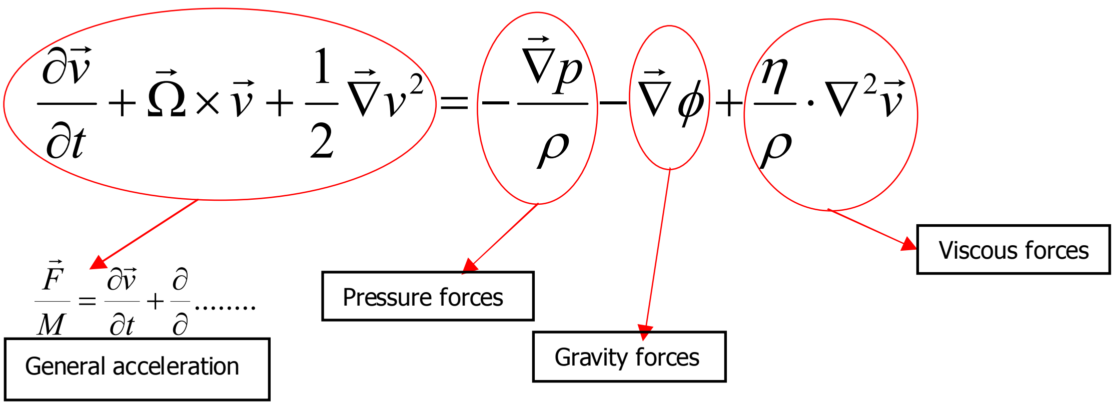
$\vec{F}_{visc} = F_{x-visc}\hat{x} + F_{y-visc}\hat{y} + F_{z-visc}\hat{z} = \eta \cdot dV[\hat{x}\nabla^2 u + \hat{y}\nabla^2 v + \hat{z}\nabla^2 w] = \eta \cdot dV \cdot \nabla^2 \vec{v}$ and we report here:

$$\vec{F}_{visc} = \eta \cdot dV \cdot \nabla^2 \vec{v} \tag{8.11}$$

Now, such a $\vec{F}_{visc}$ must be used in (8.2), after having divided it by ρ and by dV (that is, by $M = \rho \cdot dV$), as both sides of (8.2) have got the dimension of a force divided by a mass, indeed, so:

$$\frac{\partial \vec{v}}{\partial t} + \vec{\Omega} \times \vec{v} + \frac{1}{2}\vec{\nabla} v^2 = -\frac{\vec{\nabla} p}{\rho} - \vec{\nabla}\phi + \frac{\eta}{\rho} \cdot \nabla^2 \vec{v} \tag{8.12}$$

And therefore, finally, the **<u>Navier-Stokes Equation</u>**, and we write it better again:



<u>Compressible fluids – very rare cases</u>:

for those cases, $\rho \neq const$, >>$\frac{\partial \rho}{\partial t} \neq 0$, >>$\vec{\nabla}(\rho\vec{v}) \neq 0$, and to (8.12) we have to add the following term: $+\frac{(\eta + \eta')}{\rho}\vec{\nabla}(\vec{\nabla} \cdot \vec{v})$, but (8.12) already enclose a big series of practical cases…