Equation of motion for the Throat of Wormhole in three dimensional gravity

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Abstract

We investigate the equation of motion for the throat of wormhole in three dimensions , both classically and quantum mechanically. Minisuperspace model is applied to the latter case. Our main purpose is to treat the motion of the throat in the same way as the wave function of the universe by Hartle-Hawking. The resulting wave function may have the Yukawa potential like solution. In this article some part is preliminary.

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Contents

1 Introduction

Quantum gravity is not yet established in spite of lots of efforts of many researchers. As we know well, black holes have been deeply studied as suitable objects of quantum gravity ; black hole entropy, black hole evaporation associated with the information paradox, inside of the horizon etc. Especially those have been investigated vigorously through AdS/CFT correspondence in string theory which is one of the strong candidates of quantum gravity. However we cannot catch an even glimpse of quantum gravity , though it has contributed many things to theoretical physics.
In this report we will focus on the wormhole in three dimensions which may suggest the different aspects from black hole .Wormholes are the solutions to the Einstein field equations that have topological structures with a throat connecting two asymptotically regions of spacetime (Mouth).[6], [7], [8]. It is known as the Einstein-Rosen bridge[6]. On the other hand microscopically wormhole is one of the topologies which are the fluctuations of the Planck scale spacetime (spacetime foam [9][10]). A microscopic wormhole might be extracted from the spacetime foam to produce a traversable macroscopic wormhole [12]. The traversable wormhole was first studied by Morris and Thorn in [1]. We treat three dimensional case. Because it looks simple and gives many suggestions to the case of our real four dimensions . As is well known, wormholes are collapsed if the exotic matter which violates the energy conditions, does not exist. The classical equation of motion for the wormhole throat is obtained from both the Einstein field equations and a suitable equation of state for the matter at the throat. The equation of motion for the throat of wormhole is treated classically and quantum mechanically . The properties of an hyperbolically expanding throat are studied. In this report the throat expansion is thought to correspond to the expansion of the universe in quantum cosmology[2]. So we deal with the throat $R(t)$

like the scale factor $a(t)$ in the quantum cosmology. In addition the degree of freedom is restricted to one, $R(t)$, namely treated as a minisuperspace model. We study the equation of motion of the throat of wormhole classically in Section 2. In section 3 we try to treat it as the wave function quantum mechanically. In the last section we will have the discussion and conclusion.

2 Classical equation of motion of the throat

We will study the equation of motion for the throat of wormhole classically. In the story, microscopically , namely on the Planck scale, spacetime may fluctuate quantum mechanically . On the macroscopic scale , the classical spacetime appears smooth connected developing all kinds of topological structures including wormholes. A microscopic wormhole may be extracted from the spacetime foam to induce the birth of a macroscopic traversable wormhole. Redmount and Suen studied microscopic wormholes in Lorentzian spacetime and found those wormholes are quantum mechanically unstable in four dimensions[13][14].

2.1 classical dynamics

A spherically symmetric Minkowski wormhole provides a very simple model of a mode of topological fluctuation in Lorentzian spacetime foam which was proposed by J. Wheeler[9]. The classical geometry of such a wormhole in 3 dimensions is constructed by removing a circle of time-dependent radius $r = R(t)$ from two Minkowski space regions and identifying the two boundary surfaces $r = R(t)$, and incorporating an appropriate surface-layer stress-energy on the boundary to satisfy the Einstein field equations. Off the boundary of throat exterior spacetime regions are empty , so the Einstein field equations are satisfied trivially. On the boundary of the throat of the wormhole the Einstein field equations are equivallent with the Lanczos equations.

$$-8\pi S^i{}_j = [K^i{}_j - \delta^i{}_j K^m{}_m] \tag{1}$$

where $S^i{}_j$ is the surface stress-energy tensor and the right-hand side is the discontinuity in the extrinsic curvature $K^i{}_j$ on the boundary(Σ) , minus its trace $K^m{}_m$ across the boundary (throat).Here $(i, j) = (\tau, \theta)$ Now we will find the extrinsic curvature of the surface; $r - R(t) = 0$. The spacetime metric on the boundary Σ can be written as

$$ds^2_\Sigma = -d\tau^2 + R^2 d\theta^2 = g_{ij} dx^i dx^j \tag{2}$$

where $d\tau = \sqrt{1 - \dot{R}^2}dt$, τ is the proper time on Σ (surface, boundary)and $\dot{R} = \frac{dR}{dt}$.

$$g_{ij} = \begin{pmatrix} -1 & 0 \\ 0 & R^2 \end{pmatrix} \tag{3}$$

we use the same meaning in different words.

Hyperurface=Boundary=Throat

The velocity 3-vector is

$$u^b = \frac{dx^b}{d\tau} = \left(\frac{dt}{d\tau}, \frac{dR}{d\tau}, 0\right) \tag{4}$$

$$u_b = \left(-\frac{dt}{d\tau}, \frac{dR}{d\tau}, 0\right) \tag{5}$$

$$u^b u_b = -1 \tag{6}$$

$$-\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{dR}{d\tau}\right)^2 = -1 \tag{7}$$

$$\dot{R} \equiv \frac{dR}{dt} \tag{8}$$

$$\begin{aligned} \left(\frac{dt}{d\tau}\right)^2 &= 1 + \left(\frac{dR}{d\tau}\right)^2 = 1 + \left(\frac{dR}{dt}\frac{dt}{d\tau}\right)^2 = 1 + \left(\frac{dR}{dt}\right)^2 \left(\frac{dt}{d\tau}\right)^2 \\ &= 1 + \dot{R}^2 \left(\frac{dt}{d\tau}\right)^2 \end{aligned} \tag{9}$$

From (9) we obtain

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-\dot{R}^2}}, \quad \frac{dR}{d\tau} = \frac{\dot{R}}{\sqrt{1-\dot{R}^2}} \tag{10}$$

So we obtain,

$$u^b = \left(\frac{1}{\sqrt{1-\dot{R}^2}}, \frac{\dot{R}}{\sqrt{1-\dot{R}^2}}, 0\right), u_b = \left(-\frac{1}{\sqrt{1-\dot{R}^2}}, \frac{\dot{R}}{\sqrt{1-\dot{R}^2}}, 0\right) \tag{11}$$

The unit normal to $\Sigma :\ n_b = (n_0, n_1, n_2)$ may found from

$$n^b n_b = 1, \tag{12}$$

$$n_b u^b = 0 \tag{13}$$

$$n_b u^b = n_0 u^0 + n_1 u^1 + n_2 u^2 = n_0 \frac{1}{\sqrt{1-\dot{R}^2}} + n_1 \frac{\dot{R}}{\sqrt{1-\dot{R}^2}} = 0 \tag{14}$$

$$n^0 n_0 + n^1 n_1 + n^2 n_2 = 1 \tag{15}$$

We obtain the unit normal vectors to Σ after some simple calculations.

$$n_b = \left(-\frac{\dot{R}}{\sqrt{1-\dot{R}^2}}, \frac{1}{\sqrt{1-\dot{R}^2}}, 0\right) \tag{16}$$

$$n^b = \left(\frac{\dot{R}}{\sqrt{1-\dot{R}^2}}, \frac{1}{\sqrt{1-\dot{R}^2}}, 0\right) \tag{17}$$

Next, we calculate the second fundamental form to Σ, which is defined as

$$K_{ij} = \frac{\partial x^\alpha}{\partial \xi^i}\frac{\partial x^\beta}{\partial \xi^j}\Delta_\alpha n_\beta \tag{18}$$

$$= -n_\gamma\left(\frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma^\gamma_{\alpha\beta}\frac{\partial x^\alpha}{\partial \xi^i}\frac{\partial x^\beta}{\partial \xi^j}\right) \tag{19}$$

Here $(\alpha, \beta) = (x_{0,} x_1, x_2) = (t, r = R(t), \theta)$ and $(i, j) = (\tau, \theta)$

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -dt^2 + dr^2 + r^2 d\theta^2 \tag{20}$$

So,

$$g_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^2 \end{pmatrix} \tag{21}$$

$$g^{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r^{-2} \end{pmatrix} \tag{22}$$

Christoffel symbol is calculated as

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2}g^{\gamma\lambda}(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta}) \tag{23}$$

Non vanishing terms are

$$\Gamma^1_{22} = \frac{1}{2}(\partial_2 g_{21} + \partial_2 g_{12} - \partial_1 g_{22} = \frac{1}{2}(-\partial_r(r^2)) = -r \tag{24}$$

$$\Gamma^2_{12} = \Gamma_{21} = \frac{1}{2}g^{22}(\partial_1 g_{22} + \partial g_2 g_{12} - \partial_2 g_{12}) = \frac{1}{2}r^{-2}\partial_r(r^2) = \frac{1}{r} \tag{25}$$

So we can calculate the second fundamental form explicitly;

$$
\begin{aligned}
K_{\tau\tau} &= \frac{\partial x^\alpha}{\partial \tau}\frac{\partial x^\beta}{\partial \tau}\,\nabla_\alpha\, n_\beta \\
&= -n_\gamma \left(\frac{\partial^2 x^\gamma}{\partial\tau\partial\tau} + \Gamma^\gamma_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tau}\frac{x^\beta}{\partial \tau}\right) \\
&= -n_0 \left(\frac{\partial^2 x^0}{\partial\tau\partial\tau} + \Gamma^0_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tau}\frac{x^\beta}{\partial \tau}\right) - n_1 \left(\frac{\partial^2 x^1}{\partial\tau\partial\tau} + \Gamma^1_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tau}\frac{x^\beta}{\partial \tau}\right) - n_2 \left(\frac{\partial^2 x^2}{\partial\tau\partial\tau} + \Gamma^2_{\alpha\beta}\frac{\partial x^\alpha}{\partial \tau}\frac{x^\beta}{\partial \tau}\right) \\
&= -n_0 \frac{\partial^2 x^0}{\partial\tau\partial\tau} - n_1 \frac{\partial^2 x^1}{\partial\tau\partial\tau} \\
&= \frac{\dot{R}}{\sqrt{1-\dot{R}^2}}\frac{\partial^2 t}{\partial\tau\partial\tau} - \frac{1}{\sqrt{1-\dot{R}^2}}\frac{\partial^2 R}{\partial\tau\partial\tau} \\
&= \frac{\dot{R}}{\sqrt{1-\dot{R}^2}}\frac{\partial}{\partial\tau}\left(\frac{\partial t}{\partial\tau}\right) - \frac{1}{\sqrt{1-\dot{R}^2}}\frac{\partial}{\partial\tau}\left(\frac{\partial R}{\partial\tau}\right) \\
&= \frac{\dot{R}}{\sqrt{1-\dot{R}^2}}\frac{\partial t}{\partial\tau}\frac{\partial}{\partial t}\left(\frac{\partial t}{\partial\tau}\right) - \frac{1}{\sqrt{1-\dot{R}^2}}\frac{\partial t}{\partial\tau}\frac{\partial}{\partial t}\left(\frac{\partial R}{\partial\tau}\right) \\
&= \frac{\dot{R}}{\sqrt{1-\dot{R}^2}}\frac{1}{\sqrt{1-\dot{R}^2}}\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1-\dot{R}^2}}\right) - \frac{1}{\sqrt{1-\dot{R}^2}}\frac{1}{\sqrt{1-\dot{R}^2}}\frac{\partial}{\partial t}\left(\frac{\dot{R}}{\sqrt{1-\dot{R}^2}}\right) \\
&= \frac{R^2\ddot{R}}{(1-\dot{R}^2)^{5/2}} - \frac{1}{1-\dot{R}^2}\left(\frac{\ddot{R}}{\sqrt{1-\dot{R}^2}} + \frac{\dot{R}^2\ddot{R}}{(1-\dot{R}^2)^{3/2}}\right) \\
&= \frac{-\ddot{R}}{(1-\dot{R}^2)^{3/2}}
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
K_{\theta\theta} &= -n_\gamma \left(\frac{\partial^2 x^\gamma}{\partial\theta\partial\theta} + \Gamma^\gamma_{\alpha\beta}\frac{\partial x^\alpha}{\partial \theta}\frac{\partial x^\beta}{\partial \theta}\right) \\
&= -n_0 \left(\frac{\partial^2 x^0}{\partial\theta\partial\theta} + \Gamma^0_{\alpha\beta}\frac{\partial x^\alpha}{\partial \theta}\frac{\partial x^\beta}{\partial \theta}\right) - n_1 \left(\frac{\partial^2 x^1}{\partial\theta\partial\theta} + \Gamma^1_{\alpha\beta}\frac{\partial x^\alpha}{\partial \theta}\frac{\partial x^\beta}{\partial \theta}\right) - n_2 \left(\frac{\partial^2 x^2}{\partial\theta\partial\theta} + \Gamma^2_{\alpha\beta}\frac{\partial x^\alpha}{\partial \theta}\frac{\partial x^\beta}{\partial \theta}\right) \\
&= -n_1 \left(\frac{\partial^2 x^1}{\partial\theta\partial\theta} + \Gamma^1_{\alpha\beta}\frac{\partial x^\alpha}{\partial \theta}\frac{\partial x^\beta}{\partial \theta}\right) \\
&= -\frac{1}{\sqrt{1-\dot{R}^2}}\left(\Gamma^1_{22}\frac{\partial x^2}{\partial\theta}\frac{\partial x^2}{\partial\theta}\right) \\
&= -\frac{1}{\sqrt{1-\dot{R}^2}}\left((-R)\frac{\partial\theta}{\partial\theta}\frac{\partial\theta}{\partial\theta}\right) \\
&= \frac{R}{\sqrt{1-\dot{R}^2}}
\end{aligned}
\tag{27}
$$

So the race is

$$\begin{aligned} K &\equiv K^i{}_i = g^{ik}K_{ki} = g^{00}K_{00} + g^{11}K_{11} = (-1)K_{\tau\tau} + R^{-2}K_{\theta\theta} \\ &= \frac{\ddot{R}}{(1-\dot{R}^2)^{3/2}} + R^{-2}\frac{R}{\sqrt{1-\dot{R}^2}} \end{aligned} \tag{28}$$

Using the above results we can obtain the surface stress tensor $S^i{}_j$

$$\begin{aligned} -8\pi S^\tau{}_\tau &= K^\tau{}_\tau - \delta^\tau{}_\tau K \\ -8\pi g^{\tau\lambda}S_{\lambda\tau} &= g^{\tau\lambda}K_{\lambda\tau} - K \\ -8\pi g^{\tau\tau}S_{\tau\tau} &= g^{\tau\tau}K_{\tau\tau} - K \\ -8\pi(-1)S_{\tau\tau} &= (-1)K_{\tau\tau} - K \end{aligned} \tag{29}$$

We can obtain $S_{\tau\tau}$ by substituting (26) and (28) to (29)

$$S_{\tau\tau} = -\frac{1}{8\pi R\sqrt{1-\dot{R}^2}} \tag{30}$$

Similarly

$$\begin{aligned} -8\pi g^{\theta\theta}S_{\theta\theta} &= g^{\theta\theta}K_{\theta\theta} - K \\ -8\pi R^{-2}S_{\theta\theta} &= R^{-2}K_{\theta\theta} - K \end{aligned} \tag{31}$$

We obtain $S_{\theta\theta}$ by substituting

$$S_{\theta\theta} = \frac{R^2}{8\pi}\frac{\ddot{R}}{(1-\dot{R}^2)^{3/2}} \tag{32}$$

We suppose that S_{ij} on the throat (boundary, surface, Σ) corresponds to a perfect fluid :

$$S_{ij} = (p_s + \sigma)u_i u_j + p_s g_{ij} \tag{33}$$

Here σ is the surface energy density and p_s is the surface pressure on the boundary.

$$S^i{}_j = \begin{pmatrix} -\sigma & 0 \\ 0 & p_s \end{pmatrix} \tag{34}$$

Then,

$$S_{\tau\tau} = \sigma \tag{35}$$

$$S_{\theta\theta} = R^2 p_s \tag{36}$$

So we obtain a couple of equaitons,

$$-\frac{1}{8\pi R\sqrt{1-\dot{R}^2}} = \sigma \tag{37}$$

$$\frac{R^2}{8\pi}\frac{\ddot{R}}{(1-\dot{R}^2)^{3/2}} = R^2 p_s \tag{38}$$

In order to find the equation of motion for the throat, we need an equation of state relating σ and p_s. Generally it is represented as

$$p_s = \omega\sigma \tag{39}$$

After some calculations, we obtain the equation of motion of the throat which is simple but based on the Einstein field Equation.

$$R\ddot{R} - \omega\dot{R}^2 + \omega = 0 \tag{40}$$

If we choose $\omega = -1$ for simplicity, the equation of motion is

$$R\ddot{R} + \dot{R}^2 - 1 = 0 \tag{41}$$

This resulting equation (40) in three dimensional gravity is the same as the one in four dimensional gravity[4][5], surprisingly. The solution is

$$R(t) = \sqrt{t^2 + b^2} \tag{42}$$

So we can obtain the σ by substituting (42) into (37)

$$\sigma = -\frac{1}{8\pi b} < 0 \tag{43}$$

3 Quantum Aspect

Next we investigate quantum aspect of the equation of motion of wormhole throat using the classical result. An action corresponding to the above equation of motion Eq.(41) appears as

$$S = \int \frac{b^2}{R}\sqrt{1-\dot{R}^2}dt \tag{44}$$

The Lagrangean is given by

$$\mathcal{L} = \frac{b^2}{R}\sqrt{1-\dot{R}^2} \tag{45}$$

The canonical momentum will be

$$p = \frac{\partial \mathcal{L}}{\partial \dot{R}} = -\frac{b^2 \dot{R}}{R\sqrt{1-\dot{R}^2}} \tag{46}$$

Inserting $R(t)$ into eq(44),

$$S = b^2 \int \frac{b}{b^2+t^2} dt = b^2 \arctan\left(\frac{t}{b}\right) \tag{47}$$

Next the Hamiltonian is given

$$\mathcal{H} = p\dot{R} - \mathcal{L} = \frac{p}{\dot{R}} = -\sqrt{p^2 + \frac{b^4}{R^2}} = -\sqrt{p^2 + \left(-\frac{b^2}{R}\right)^2} \tag{48}$$

We see

$$M(t) = -\frac{b^2}{R(t)} = -\frac{\hbar}{cR(t)} \tag{49}$$

In quantum cosmology theory, the evolution of the universe is completely determined by its quantum state that should satisfy the Wheeler-DeWitt equation. Now we assume the same situation in the evolution of the throat. namely, the evolution of the throat is completely determined by its quantum state that should satisfy the Wheeler-DeWitt equation. Fundamentally the quantum wormhole is to be described by a wave function $\psi(R,t)$ in a minisuperspace model which has a few number of degree of freedom, for example scale factor a in quantum cosmology.

$$\textbf{Evolution of R(t)} \Longleftrightarrow \textbf{Evolution of a(t)} \tag{50}$$

As you know well about quantization , two ways exist, canonical method and path integral method. Here we use the canonical method.

3.1 Canonical Method

3.1.1 rough estimation

Now we quantize the classical motion canonically in minisuperspace model which is ristricted the degree of freedom to a finite number of it. But the hamiltonian is represented by square root.so we will approximate (this is a very rough treatment)

$$H = -\frac{b^2}{R}\sqrt{1+\frac{R^2}{b^4}p^2} \approx -\frac{b^2}{R}(1+\frac{1}{2}\frac{R^2}{b^4}p^2) \tag{51}$$

$$= -\frac{R}{2b^2}p^2 - \frac{b^2}{R} \tag{52}$$

We do with the ordinary procedure of canonical quantization:

$$p \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial R} \tag{53}$$

We can obtain the Wheeler-DeWitt equation:

$$\hat{H}\psi(R) = 0 \tag{54}$$

where

$$\hat{H} = \frac{\hbar^2 R}{2b^2}\frac{\partial^2}{\partial R^2} - \frac{b^2}{R}$$

$\psi(R)$ is the wavefunction of the motion of the throat. Our Wheeler-DeWitt equation is

$$\frac{d^2\psi}{dR^2} - \frac{2b^4}{\hbar^2 R^2}\psi = 0 \tag{55}$$

The potential is

$$V(R) = -\frac{2b^4}{\hbar^2 R^2} \tag{56}$$

The solutions of the above Wheeler-DeWitt equation (Cauchy-Euler equation) are

$$\psi(R) = AR^{\frac{1+\sqrt{\alpha}}{2}} + BR^{\frac{1-\sqrt{\alpha}}{2}} \tag{57}$$

where $\alpha \equiv 1 + \frac{8b^4}{\hbar^2}$, A and B are constant defined by the boundary conditions.

3.1.2 Klein-Gordon equation

Next we estimate the Hamiltonian, From the Hamiltonian we obtain the equation;

$$E^2 = p^2 + \frac{b^4}{R^2} \tag{58}$$

The above equation corresponds to the usual relativistic energy-momentum relation:

$$E^2 = p^2c^2 + M^2c^4 \tag{59}$$

We find

$$M^2c^4 = \frac{b^4}{R^2}, M = \pm\frac{b^2}{Rc^2} \tag{60}$$

Next we do the ordinary canonical replacement for quantization;

$$E \rightarrow i\hbar\frac{\partial}{\partial t},\ \ p \rightarrow \frac{\hbar}{i}\frac{\partial}{\partial R} \tag{61}$$

So we obtain the Klein-Gordon equation:

$$\left(-\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial R^2}-\frac{b^4}{\hbar^2 R^2}\right)\psi(R,t)=0 \tag{62}$$

We start at the Einstein field equation to arrive at the Klein-Gordon equation in minisuperspace model.

The above equation is resolved by using inverse Fourier transform. $\psi(R,t)$ is represented by Inverse Fourier Transform.

$$\psi(R,t)=\frac{1}{\sqrt{2\pi}}\int\hat{\psi}(k,t)e^{ikR}dk \tag{63}$$

So we obtain the equation;

$$\frac{\partial^2\psi(R,t)}{\partial R^2} = \frac{1}{\sqrt{2\pi}}\int\hat{\psi}(k,t)\left(\frac{\partial^2}{\partial R^2}e^{ikR}\right)dk \tag{64}$$

$$= \frac{1}{\sqrt{2\pi}}\int\hat{\psi}(k,t)(ik)^2e^{ikR}dk \tag{65}$$

$$\frac{\partial^2\psi(R,t)}{\partial t^2}=\frac{1}{\sqrt{2\pi}}\int\frac{\partial^2\hat{\psi}(k,t)}{\partial t^2}e^{ikR}dk \tag{66}$$

The above equations are substituted into the original Klein-Gordon equation. We obtain

$$\frac{\partial^2\hat{\psi}(k,t)}{\partial t^2}+k^2\hat{\psi}(k,t)+\frac{b^4}{\hbar^2R^2}\hat{\psi}(k,t)=0 \tag{67}$$

At last we obtain the equation:

$$\frac{\partial^2}{\partial t^2}\hat{\psi}(k,t)=-\omega^2\hat{\psi}(k,t) \tag{68}$$

where

$$\omega^2\equiv k^2+\frac{b^4}{\hbar^2R^2} \tag{69}$$

where $\frac{b^2}{R}=M$ corresponds to the rest mass of a particle. This equation describes the Harmonic Oscillation ; The solutions are

$$\hat{\psi}(k,t)=A(k)e^{-i\omega t}+B(k)e^{i\omega t} \tag{70}$$

$$\tag{71}$$

Next we consider the stationary case of the Klein-Gordon equation, namely time independent case. The equation is

$$\left(\frac{\partial^2}{\partial R^2} - \frac{b^4}{\hbar^2 R^2}\right)\psi(R) = 0 \tag{72}$$

The corresponding Green function is

$$\left(\frac{\partial^2}{\partial R^2} - \alpha^2\right) G(R) = -\delta(R) \tag{73}$$

where

$$\alpha^2 = \frac{b^4}{\hbar^2 R^2} \tag{74}$$

and $\delta(R)$ is the delta function. Here we set the inverse Fourier transform of $G(R)$:

$$G(R) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{G}(k) e^{ikR} dk \tag{75}$$

$$\delta(R) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikR} dk \tag{76}$$

The delta function is defined as the Fourier transform of "1". Substituting eq (72),(73) into (71), we obtain

$$\widetilde{G}(k)\left(k^2 + \alpha^2\right) = 1 \tag{77}$$

So

$$\widetilde{G}(k) = \frac{1}{k^2 + \alpha^2} \tag{78}$$

Substituting (76) into (73),

$$G(R) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{k^2 + \alpha^2} e^{ikR} dk \tag{79}$$

Considering the complex plane, we can perform the above integration by use of the resdue of the poles at $k = \pm i\alpha$. Concretely we consider the next function on the complex plane;

$$f(k) = \frac{e^{ikR}}{k^2 + \alpha^2} = \frac{e^{ikR}}{(k + i\alpha)(k - i\alpha)} \tag{80}$$

So by the suitable integral pathway , we obtain the resdue as

$$Res(i\alpha) = (k - i\alpha)f(k)|_{k=i\alpha} = \frac{e^{ikR}}{k + i\alpha}|_{k=i\alpha} = \frac{e^{-\alpha R}}{2i\alpha} \tag{81}$$

So we obtain

$$\oint_C f(k)dk = \oint_C \frac{e^{ikR}}{k^2 + \alpha^2}dk = \oint_C \frac{e^{ikR}}{(k + i\alpha)(k - i\alpha)}dk = 2\pi i Res(i\alpha) = \frac{\pi}{\alpha}e^{-\alpha R} \tag{82}$$

We obtain , using $\alpha^2 = \frac{b^4}{\hbar^2 R^2}$

$$\psi(R) = \frac{1}{\sqrt{2\pi}}\frac{\pi}{\alpha}e^{-\alpha R} = \sqrt{\frac{\pi}{2}}\frac{e^{-\alpha R}}{\alpha} = \frac{1}{\sqrt{2\pi}}R\frac{e^{-\frac{b^2}{\hbar}}}{\frac{b^2}{\hbar}} \tag{83}$$

This solution may look like the deformation of the Yukawa potential[11].

4 Conclusions

In this report we investigated the equation of motion of wormhole throat in three dimensions in both classical level and quantum level. In minisuperspace model , the quantized equation of motion is the Klein -Gordon like equation. The solutions describe the ordinary Harmonic Oscilattion. Especially in a stationary case, the solution has the Yukawa potential like solutions. However the object we dealt with here is the gravitational equation of motion for the throat of wormhole in general relativity. We intended to describe the evolution of the throat in the same way as Hawking described the wave function of the universe..

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