Lecture Notes on Symmetry Optics

Lecture 11:
Uncertainty in the Provolving Beam

To accompany https://youtu.be/3BIVmSGA3EE
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1 Introduction

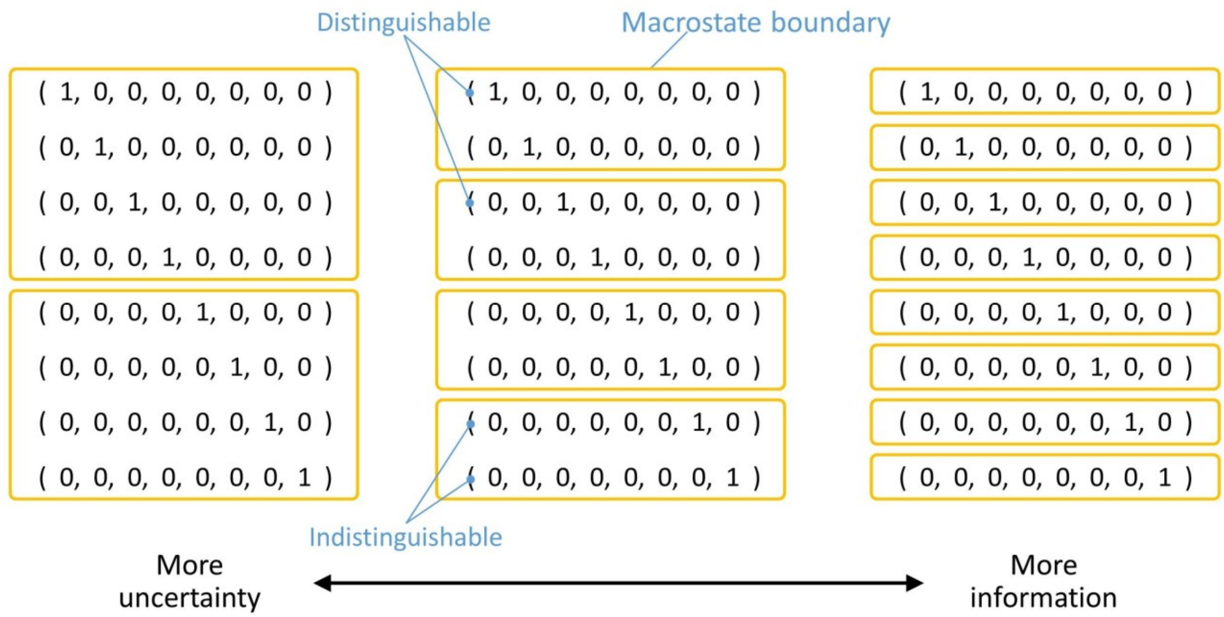
1.1 Intro

Hello, and welcome to Lectures on Symmetry Optics. I'm Paul Mirsky. This is lecture 11, and the topic is: *Uncertainty in the Provolving Beam*.

The previous lecture showed how information and uncertainty can be described by groups. Now, we'll show how these groups change as a beam of light propagates. The lecture starts by reviewing some topics within quantum mechanics, and it assumes that the audience already has a lot of background knowledge. Unfortunately, without that knowledge you may find this lecture pretty difficult to follow.

2 Review: Subgroups in Quantum Mechanics

2.1 State spaces and subspaces



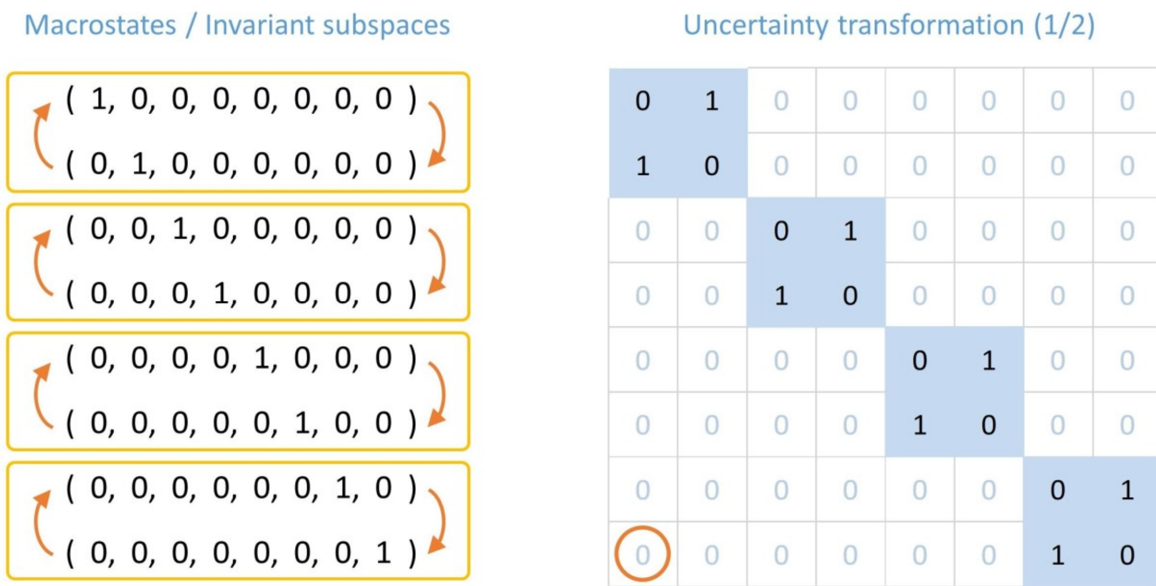
Let's consider some hypothetical quantum object than can be in eight different states. In quantum mechanics, the state space is represented by a vector space. In this example, these 8 different basis vectors represent the 8 states, and together they span the state space. Note that state vectors are usually written as columns, but in this lecture they are written as rows.

The state space can be divided into subspaces, where each subspace consists of many individual basis vectors. For instance, these 8 dimensions can be partitioned into 4 sets of 2 basis vectors each. Each subspace represents a different macrostate, so an observer can distinguish which subspace the object is in.

But the individual basis vectors are microstates, so the observer can not distinguish precisely which individual state vector an object is in. Also, the observer cannot distinguish between different overall phases of the state vector.

The state space can be partitioned with various degrees of granularity. For example, the 8 basis vectors could also be partitioned more coarsely into two macrostates, with 4 basis vectors in each. Alternatively, there might be 8 macrostates with only 1 basis vector in each; this the finest-possible division. When each macrostate is smaller, there is more information – meaning, more distinguishable macrostates. When each macrostate is larger, there is more uncertainty – meaning, more possible microstates for each macrostate.

2.2 Uncertainty group and invariant subspaces



A quantum object also has a set of uncertainty transformations. These are also often called *the symmetries of the Hamiltonian*. To illustrate, we choose a very simple uncertainty group with only two elements – one: the identity, and two: the matrix drawn here, which simply exchanges the two basis vectors in each subspace. Applying it twice gets back to the identity, showing the closure of the group.

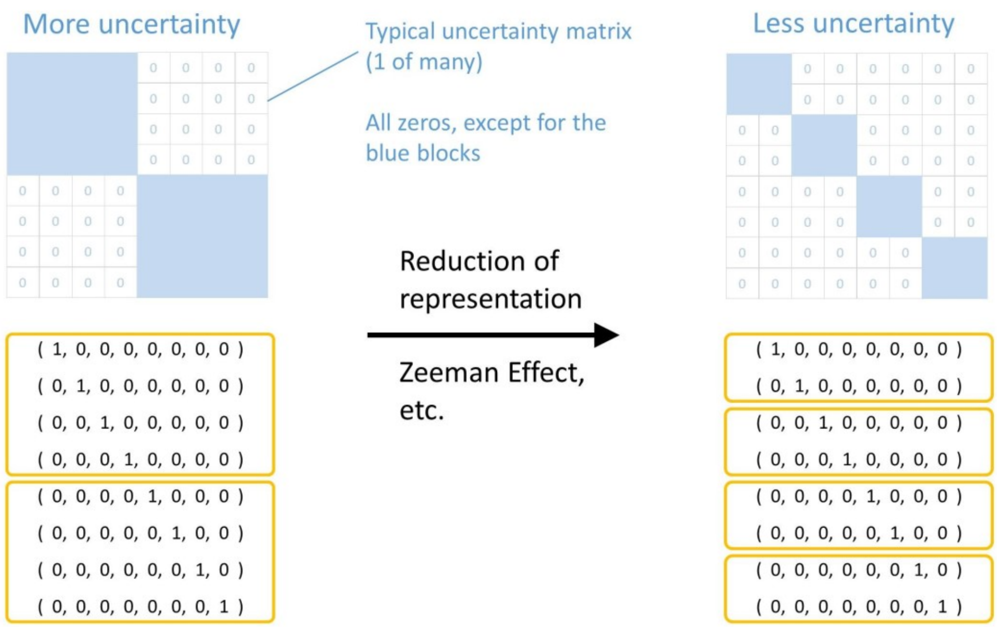
In reality, the uncertainty group typically has far more than 2 transformations, and they usually involve linear combinations of basis vectors. Also, because it's expressed using matrices, this is technically a representation of the uncertainty group. Groups and their representations are formally distinct, but they often seem to blur together into a single concept.

The key point of this slide is that all macrostates remain invariant under all uncertainty transformations. If the object begins in one microstate, for instance the topmost state vector, it may be transformed to various other microstates within the top subspace. If the object begins in a different microstate within in a different subspace, for instance the bottommost state vector, it will likewise remain in the bottom subspace. But the object will never move from one subspace into another. In other words, the subspaces are invariant under the uncertainty transformations.

In this example, it's very easy to spot the invariant subspaces because the matrix is block diagonal, meaning that all the matrix entries are zero outside of these blue blocks. For example, the lower-left matrix entry indicates how much the topmost vector is projected onto the lowermost vector. Those are two different subspaces, so the entry is equal to zero.

It's possible in principle to treat each individual block and its corresponding subspace as its own separate, independent object. This is called *reducing* the representation. The full matrices are a *reducible representation*, and each individual block is an *irreducible representation*, because it cannot be reduced to any smaller blocks.

2.3 Subgroups and ir/reducible representations



Now, we'll examine what happens when the uncertainty group changes. For example, suppose that a state space begins like the one on the left. It has two large subspaces. It also has a set of many uncertainty matrices. The slide doesn't list them all, but only shows the general pattern in which all of the entries are zero except for the two large blue diagonal blocks.

Now, suppose that the uncertainty group shrinks to become a subgroup of the old one. Each old subspace breaks up into smaller new subspaces, as we see on the right. The diagonal blocks also become smaller. In other words, each irreducible representation of the larger group is replaced by a reducible representation of the smaller group, which in turn can be reduced to multiple irreducible representations of the smaller group.

One example of such a change is the Zeeman effect. An electron in an atom experiences the same forces for any possible rotation in 3 dimensions. Each subspace corresponds to a certain amount of angular momentum. But if the atom is placed into a magnetic field, the angular momentum can be aligned either with or against the field, so each subspace breaks up into smaller subspaces. Now, the forces are the same for all 2-d rotations that are parallel to the magnetic field. Those are a subgroup of the original 3-d rotation group.

In general, the uncertainty might continue shrinking to ever-smaller-and-smaller subgroups, keeping finer-and-finer subspaces invariant.

It's also possible for the changes to occur in the other direction. In that case, the uncertainty group grows and multiple separate subspaces merge together to become larger subspaces.

2.4 Different bases / coordinate systems

Default basis	Complementary basis ($u = e^{-2\pi i/8}$)
(1, 0, 0, 0, 0, 0, 0, 0)	$(u^{0}, u^{-4}, u^{0}, u^{-4}, u^{0}, u^{-4}, u^{0}, u^{-4})$
(0, 1, 0, 0, 0, 0, 0, 0)	$(u^{-4}, u^{+1}, u^{-2}, u^{+3}, u^{0}, u^{-3}, u^{+2}, u^{-1})$
(0, 0, 1, 0, 0, 0, 0, 0)	$(u^{0}, u^{-2}, u^{-4}, u^{+2}, u^{0}, u^{-2}, u^{-4}, u^{+2})$
(0, 0, 0, 1, 0, 0, 0, 0)	$(u^{-4}, u^{+3}, u^{+2}, u^{+1}, u^{0}, u^{-1}, u^{-2}, u^{-3})$
(0, 0, 0, 0, 1, 0, 0, 0)	$(u^{0}, u^{0}, u^{0}, u^{0}, u^{0}, u^{0}, u^{0}, u^{0})$
(0, 0, 0, 0, 0, 1, 0, 0)	$(u^{-4}, u^{-3}, u^{-2}, u^{-1}, u^{0}, u^{+1}, u^{+2}, u^{+3})$
(0, 0, 0, 0, 0, 0, 1, 0)	$(u^{0}, u^{+2}, u^{-4}, u^{-2}, u^{0}, u^{+2}, u^{-4}, u^{-2})$
(0, 0, 0, 0, 0, 0, 0, 1)	$(u^{-4}, u^{-1}, u^{+2}, u^{-3}, u^{0}, u^{+3}, u^{-2}, u^{+1})$

$$(u^{0}, u^{-4}, u^{0}, u^{-4}, u^{0}, u^{-4}, u^{0}, u^{-4}) \xrightarrow{\text{Expressed in comp. basis}} (1, 0, 0, 0, 0, 0, 0, 0)$$

Now let's address a related topic: Any time we write a vector, we express it using some basis for the vector space, which is essentially the same thing as a coordinate system. The most familiar example is a set of X, Y and Z axes, which can be set at various orientations in space. When the vector space represents different quantum states, the meaning is more abstract but it works the same mathematically.

Up to now we have used the basis on the left implicitly, by default. But the same space can also have other coordinate systems, such as the complementary basis on the right.

The same physical state can be expressed in multiple alternative ways, depending on the coordinate system. For example, consider the first vector from the complementary basis. Its amplitude is spread over all the many vector components, when it's expressed in the default basis. But if we express it in the complimentary basis itself, then it appears concentrated at a single component, because it projects completely onto one of the basis vectors.

In quantum mechanics, a single state space can be observed multiple ways, and each observable is a different coordinate system.

2.5 Different coordinates hide diagonal blocks

Invariant subspaces

Uncertainty transformation (1/2)

-1	0	0	0	0	0	0	0
0	-0.7	0	0	0	0.7i	0	0
0	0	0	0	0	0	1i	0
0	0	0	0.7	0	0	0	0.7i
0	0	0	0	1	0	0	0
0	-0.7i	0	0	0	0.7	0	0
0	0	-1i	0	0	0	0	0
0	0	0	-0.7i	0	0	0	-0.7

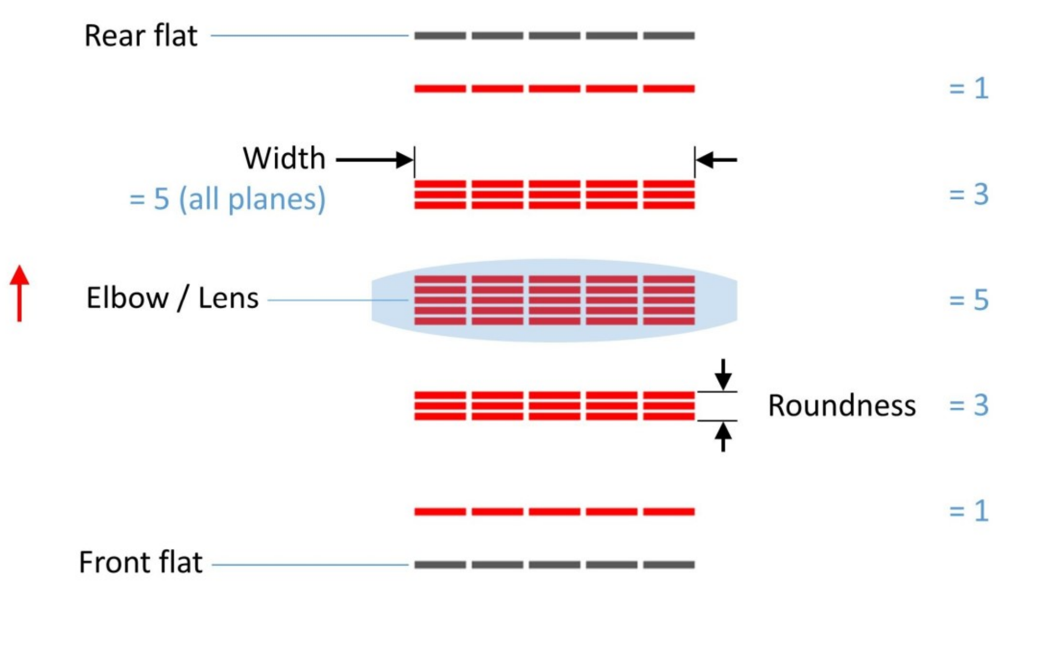
This implies that our discussion of invariant subspaces has been somewhat imprecise, because we have only considered a single coordinate system. Now we must add that also for all other coordinate systems, there exist uncertainty representations which may not be block-diagonal, but behave the same way. These occur when the invariant subspaces are not aligned with the coordinate system.

For example, consider the subspaces and the matrix shown here. These are actually the very same subspaces and uncertainty generator that we showed three slides ago, but now it is expressed in a basis of complementary vectors so that it is not block-diagonal. Alternatively, you can say that it has undergone a similarity transformation. While it’s harder to see at a glance, this matrix still keeps each of these subspaces invariant.

This concludes the first section of the lecture.

3 Uncertainties in the Lens-balanced Beam

3.1 Lens-balanced beam



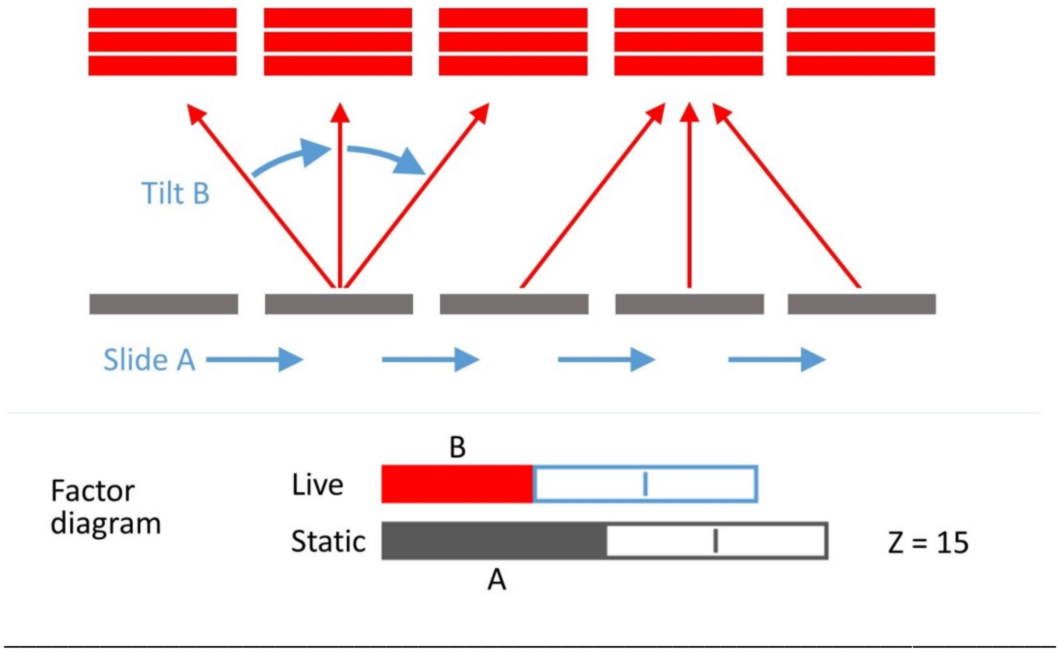
Up to this point, we have been reviewing very general principles, which apply to all kinds of quantum objects and systems. Now we'll focus on just one kind of object: the lens-balanced beam. This is the simplest object in symmetry optics, and it occurs when the elbow coincides with the lens. In this example, the focal length is 25 wavelengths.

The width of the stripe is 5, which is the square root of the focal length. The width remains constant all the way from the front flat, through the elbow, to the rear flat.

The roundness is hard to define precisely at the flat, but it is equal to 1 very near the flat, then rises linearly to reach a maximum of 5 at the elbow.

After the lens, the same changes occur in reverse. The roundness decreases linearly and falls to 1 near the rear flat.

3.2 Visualizing slide and tilt transformations

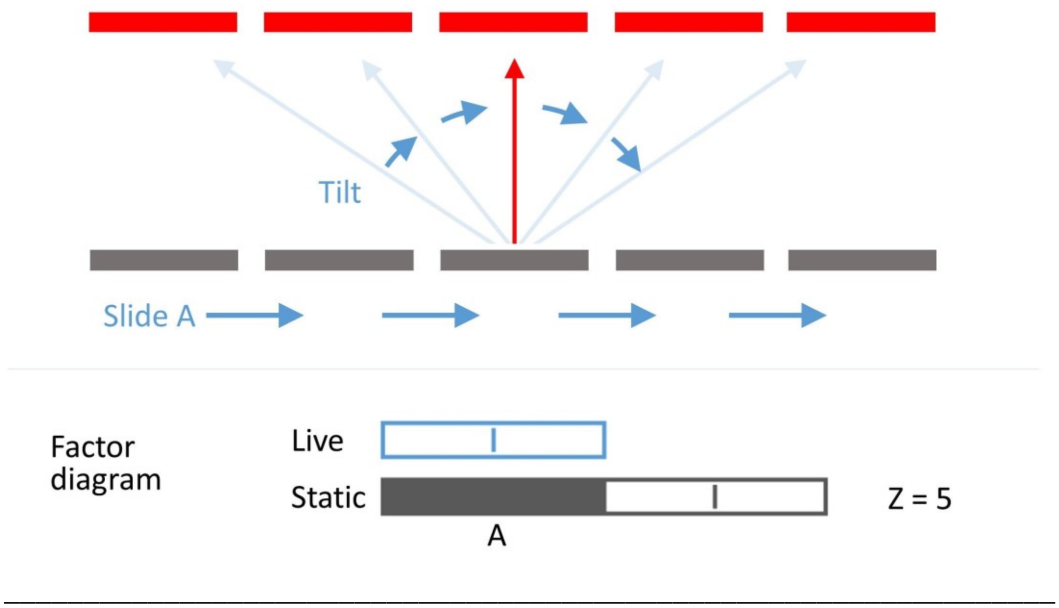


Here's a closer look at the plane where roundness is 3. Each source patch emits to 3 target patches, and each target patch receives light from 3 different source patches, so the light is propagating at 3 different angles.

A tilt applied at the flat transforms each angle into the next angle. Also, the last angle circles around and becomes the first. That tilt changes each microstate into another indistinguishable microstate, so it's an uncertainty generator. The second uncertainty generator is to Slide each patch to the next, which also circles around. The third uncertainty generator, which is not suggested by this visualization, is an overall phase change.

Slide and tilt also correspond to two different factors, as we have discussed in earlier lectures. Factor A is size 5, and in this plane it corresponds to the width of the beam, or the slide uncertainty. Factor B is size 3, and in this plane it corresponds to the roundness of the beam, or the tilt uncertainty.

3.3 Roundness at elbow = information at/near flat

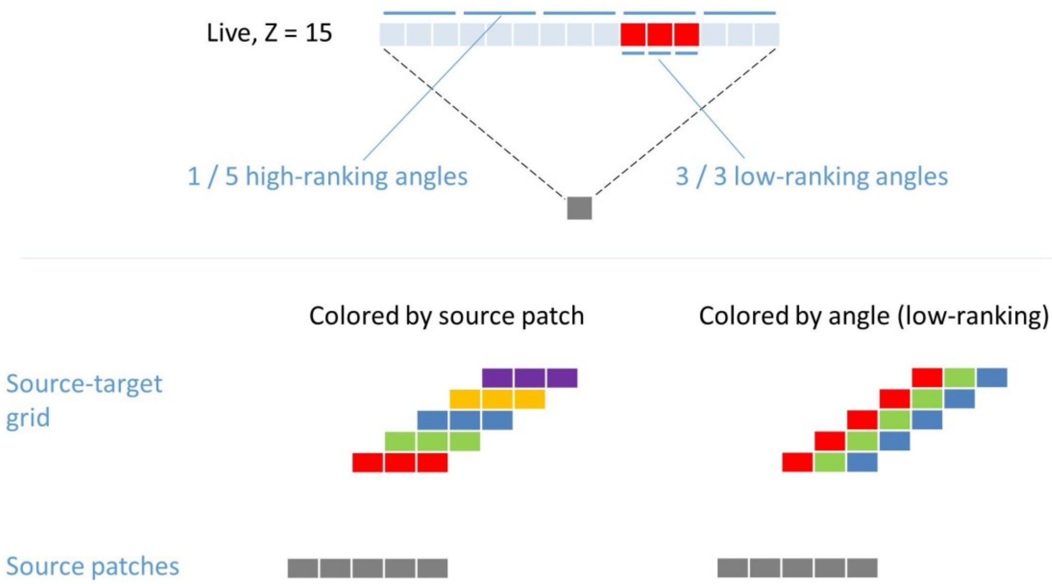


For this beam, the maximum possible roundness is 5. The minimum roundness is 1; this occurs close to the flat at Z = 5, where factor B is size 1 and therefore isn't visible on the factor diagram.

In the last slide, Tilt was an uncertainty transformation. Now, Tilt acts as an information transformation which would shift the light from one angle to another. Equivalently, tilt describes the potential roundness which will exist closer to the elbow, but does not yet exist.

One mystery which still remains in symmetry optics is that there is no way to calculate the distribution of light at the flat itself. The method which works fine in other planes simply doesn't give an answer when Z = 0. For this reason we permit a certain amount of imprecision, and say that the symmetry in this plane very near the flat is also the symmetry exactly at the flat. Tilt is the information; slide and phase are the uncertainty.

3.4 Higher- and lower-ranking angles

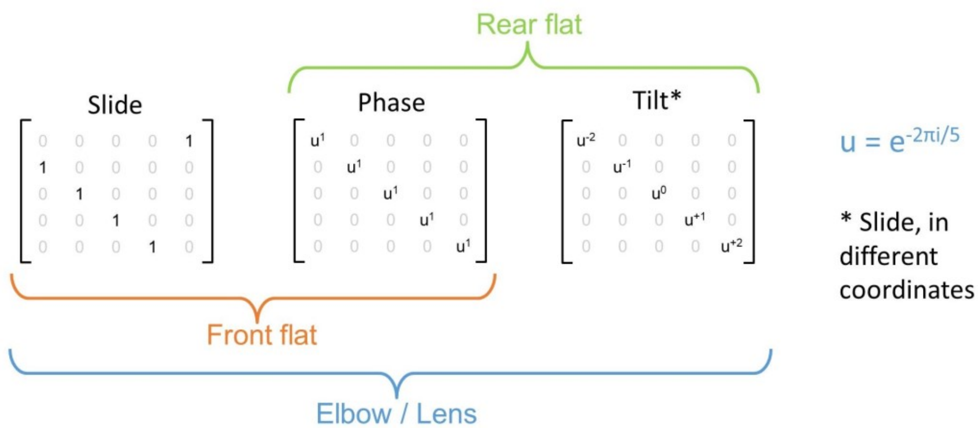


As we discuss angles, we need to clarify the difference between two different types of angle: high-ranking, and low-ranking. The live consists of a bright stripe within a wide limit which is otherwise dark. As the light propagates, the wide limit expands with a divergence of 1 radian. In this example, the stripe expands with a divergence of 1/5 radians and the stripe can be at one of 5 possible high-ranking angles. In the last 2 lectures, when we've discussed the angle of the beam, we've meant this high-ranking angle. It's the overall direction of the beam.

But within the stripe are multiple patches and each one is at a slightly different low-ranking angle relative to the source patch. These are the different angles that make up the roundness, and these are the focus of this present lecture.

When we create the source target grid, we tabulate a single instance of the live for each source patch. Usually we think of it like the picture on the left, where each color corresponds to a different source patch. But we can also interpret the exact same source-target grid according to angle, like the picture on the right. Each color corresponds to a different low-ranking angle, and each angle is spread over the width of the stripe. For the purpose of this lecture, this is the most useful way to visualize the different angles.

3.5 Uncertainty generators in various planes



$$\langle T, S, P \rangle = T^J \cdot S^K \cdot P^L$$

$$J, K, L \in \{-2, -1, 0, +1, +2\}$$

Now we'll summarize the uncertainties in the major planes. This is the most important slide in the whole lecture.

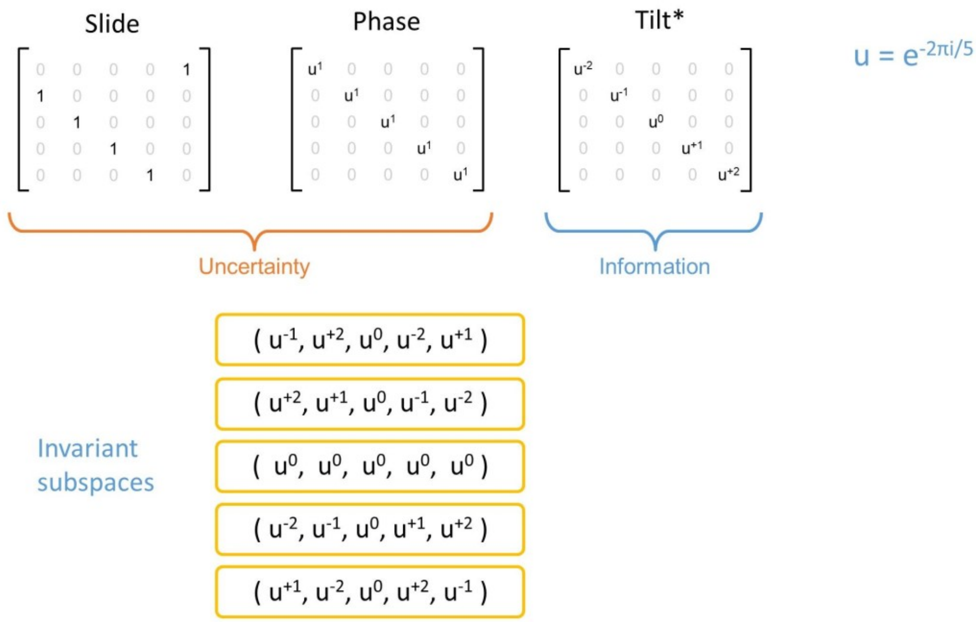
At the elbow, the uncertainty includes all three of the major transformations – Slide, Phase, and Tilt. These generate the group <Tilt, Slide, Phase>, consisting of all elements of the form $\text{Tilt}^J \cdot \text{Slide}^K \cdot \text{Phase}^L$, where the exponents J, K, and L can each be any integer from -2 to +2. The number of different elements in this group is 125, which is 5·5·5. The elbow is is the plane of maximum uncertainty in the beam. In terms of an analogy with the Zeeman effect, it's like an atom with no magnetic field.

At the flats, the uncertainties are smaller: <Slide, Phase> at the front flat, and <Tilt, Phase> at the rear flat. The flat planes are analogous to atoms inside a magnetic field. Pushing the analogy even further, one flat is like an atom in a vertical field, and the other flat is like an atom in a horizontal field, because each flat's uncertainty is a different subgroup of <Tilt, Slide, Phase>, just like each atom's uncertainty is a 2-d rotation around a different axis.

Also, note that there is an asterisk on the label Tilt, which indicates that 'Tilt' is only one interpretation. Later on we will re-interpret the same transformation as Slide for a different coordinate system.

Now, we'll examine the invariant subspaces in the major planes.

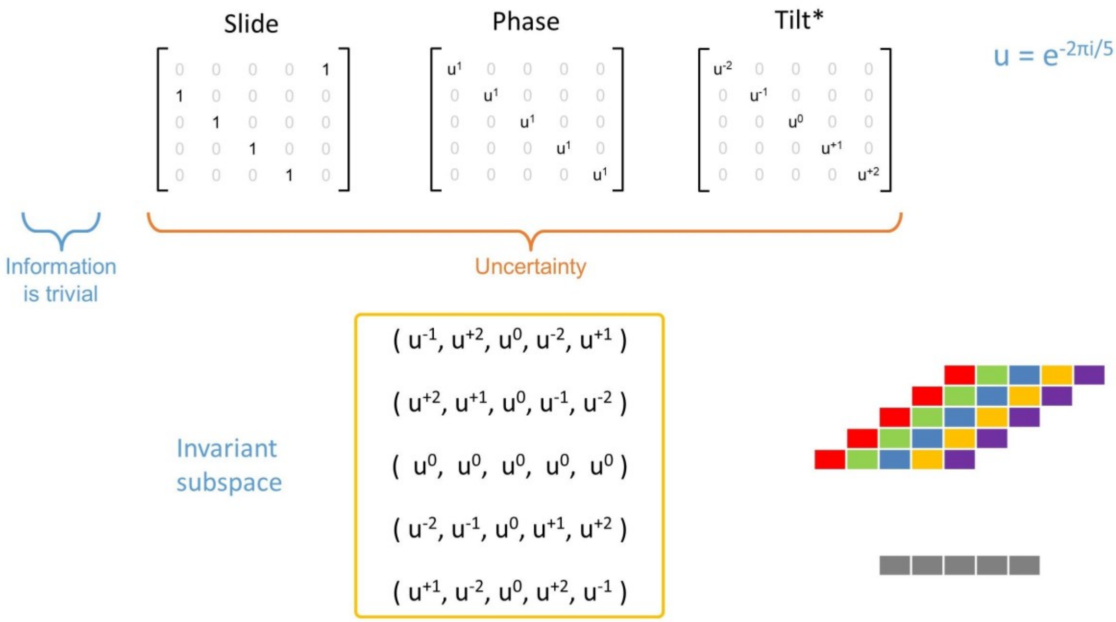
3.6 Invariant subspaces, front flat



At the front flat, the invariant subspaces are *angle states*. The beam state lies within just one of these subspaces, although it may be any one of the set.

All elements of the uncertainty group slide-phase keep the angle state invariant within itself, but they never change the angle into another angle.

3.7 Invariant subspaces, elbow / lens



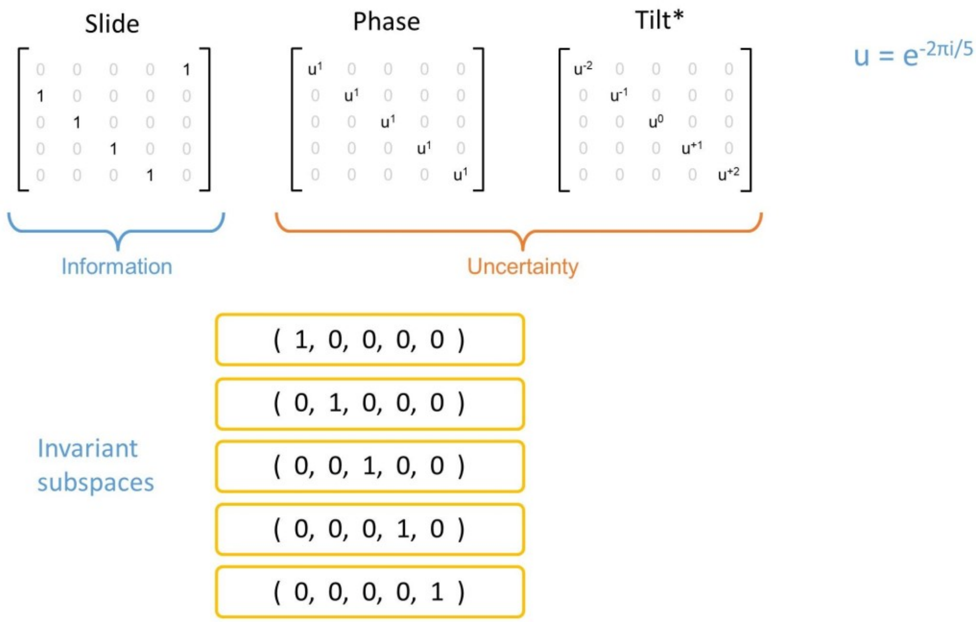
At the elbow, each target patch receives light from all source patches in the beam. In other words, the light contains 5 possible low-ranking angles, which all together form a single large macrostate. Each angle corresponds to one of the colors in this picture of the corresponding source-target grid.

Slide and phase keep each individual angle invariant. However, tilt transforms one angle into to the next. So, the invariant subspace encompasses the entire state space shown inside this box.

Note that these 5 angle states are *not in a coherent superposition. That is a very different case which does often occur in quantum mechanics, and which would be equivalent to a single position state. This is rather a 5-dimensional space, the direct sum of 5 different individual states.

But while we have written the space as the direct sum of angle states, we could also have written it as the direct sum of *position states*. Angle and position are two different sets of basis vectors that span the exact same space.

3.8 Invariant subspaces, rear flat

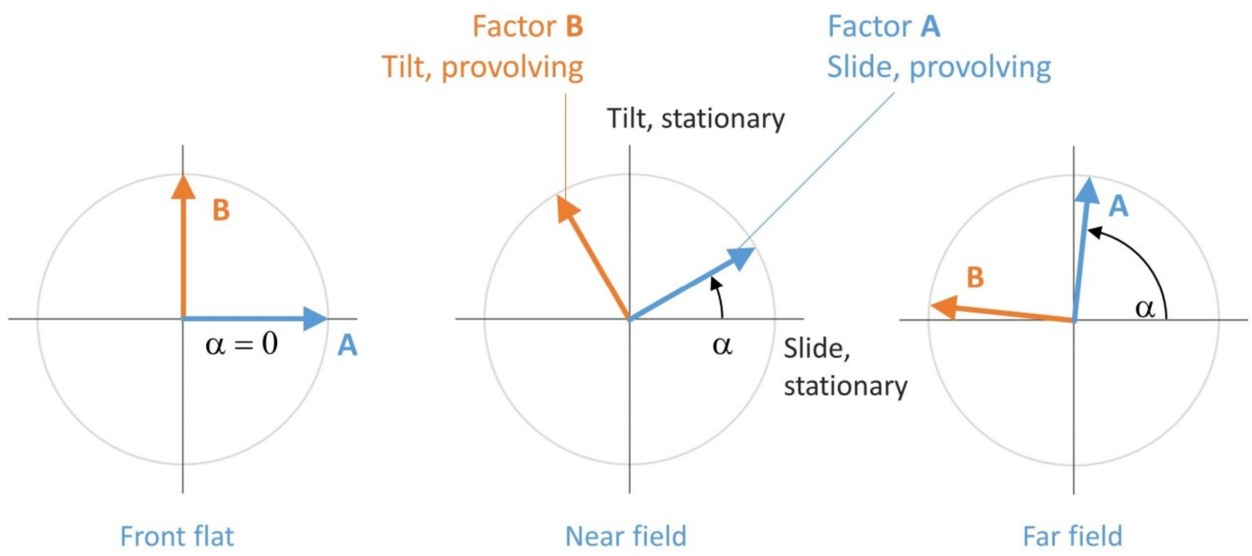


And indeed, the *position states* are the invariant subspaces at the rear flat. Here, all elements of the uncertainty group tilt-phase keep each position state invariant within itself, but they do not change one position into another position.

This is particularly easy to see because Tilt is a block-diagonal matrix, and each diagonal block takes only one subspace as an input and gives the same subspace as an output. This was also true at the front flat. But, it was not nearly as apparent there because Slide was the uncertainty transformation, and slide is not actually block-diagonal when it is expressed in the basis of positions, which we have been using.

Earlier, we noted the asterisk on the label 'Tilt', and we stated that it could be interpreted another way. Let's return to that topic now.

3.9 Stationary and provolving coordinates



We need to consider two different coordinate systems. First, there is the stationary coordinate system, which is fixed in space. This is by far the more intuitive one, because we observers live and perceive in stationary space.

We can visualize stationary slide and tilt as a set of 2 orthogonal axes. In this case, orthogonality signifies that slide and tilt are like a function and its Fourier transform – they are dual to one another, like 2 complementary observables.

Second, there is a provolving coordinate systems. Rather than being fixed to space, it is fixed to the factors of the light itself.

We can visualize optical factors A and B as two arrows oriented at 90 degrees to one another, which can lie at some angle alpha relative to the stationary axes. Alpha is not actually an angle in space that you could measure with a protractor, but rather an amount of provolution.

By convention, we assign provolving slide to factor A, and provolving tilt to factor B. The way this visualization works, the lengths of these vectors stay the same, even when factors A and B get larger and smaller.

At the front flat alpha is equal to zero, so factor A has slide uncertainty in both stationary and provolving coordinates. But alpha increases as the light provolves into the near field. In provolving coordinates, A always remains tied to slide, but in stationary coordinates it is something in between slide and tilt.

When the beam reaches the distant far field, alpha approaches 90 degrees. Then, B corresponds to stationary slide, and A corresponds to stationary tilt, which is the opposite of how it began at the front flat.

But in provolving coordinates, A and B always remain slide and tilt, respectively.

3.10 Different planes, different coordinate types

Provolving coordinates (fixed to factors)

Front flat

$$(u^{+2}, u^{+1}, u^{0}, u^{-1}, u^{-2})$$

A

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Slide

Rear flat

$$(0, 0, 0, 0, 1)$$

B

$$\begin{bmatrix} u^{-2} & 0 & 0 & 0 & 0 \\ 0 & u^{-1} & 0 & 0 & 0 \\ 0 & 0 & u^{0} & 0 & 0 \\ 0 & 0 & 0 & u^{+1} & 0 \\ 0 & 0 & 0 & 0 & u^{+2} \end{bmatrix}$$

Tilt

Stationary coordinates (fixed to space)

Front flat

$$(u^{+2}, u^{+1}, u^{0}, u^{-1}, u^{-2})$$

A

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Slide

Rear flat

$$(u^{+1}, u^{-2}, u^{0}, u^{+2}, u^{-1})$$

B

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Slide

So far, this lecture has implicitly been using provolving coordinates. The left side of the slide repeats what we have already said – that slide is an uncertainty transform at the front flat, and tilt is an uncertainty transformation at the rear flat. The invariant subspaces also change type, from an angle state at the front, to a position state at the rear. This describes the shift of uncertainty, which begins as factor A and changes to factor B as the factors grow and shrink. This is an internal change within the beam. From the perspective of the beam itself, it goes from one orientation to another.

The right side of the slide repeats the same information, describing the same physical states, but in stationary coordinates. Here, slide is an uncertainty transformation at both the front and rear flats. Likewise, invariant subspaces at both flats are angle states. So, to an actual observer in stationary space, the beam never appears as a position state, which is a single patch. Rather, it is always an angle state – in other words, a broad stripe with its intensity spread equally over many different positions.

Naively, it almost appears as though nothing changes between the two flats. But in fact, two different changes occur which tend to cancel one another out. First, the beam undergoes an internal change from factor A to factor B. Second, the beam provolves so that the factors are oriented differently relative to the stationary coordinates. The net result appears – misleadingly – to be just like the initial state.

Also, note that phase is not shown here, even though it is an uncertainty generator in all planes. Phase is written exactly the same, regardless of which coordinate system it's expressed in.

4 Conclusions

4.1 Reviewing key points

That's all for this lecture, so let's review the key points:

- Observable macrostates are invariant subspaces of the uncertainty group.
- When the uncertainty group becomes smaller, the invariant subspaces break up into smaller subspaces. A common example is the Zeeman effect.
- In the lens-balanced beam, the elbow has the largest uncertainty of any plane: < Tilt, Slide, Phase >
- The uncertainties at the flats are subgroups: < Slide, Phase > or < Tilt, Phase >
- If we use stationary coordinates rather than provolving coordinates, then the uncertainty is < Slide, Phase > at both flats.

4.2 Outro

I hope you've found this class informative and interesting. To learn more about symmetry optics, please check out www.symmetryoptics.com If you have specific questions about this or other lectures, please post them on Reddit, at www.reddit.com/r/symmetryOptics/, and I'll try to answer them.

This is a new field, and there's a lot of opportunity to discover new science and develop new applications. I hope you'll take advantage and make your own contributions to the field.

I'm Paul Mirsky, thanks for listening.