Diffrent functional equations of Riemann zeta function and its derivatives.

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I repeated the calculations for functional equation Riemann zeta function for diffrent integral which I found in Gradshtein I.S., Ryzhik I.M. Tables of integrals, series and products (7 ed., AP, 2007).Chapter 4.35 – 4.36 :

$$\int_0^\infty x^{z-1} e^{-nx} \ln(x)\, dx = \frac{\Gamma(z)}{n^z}\left[\psi(z) - \ln(n)\right],$$

$$\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)},\ \operatorname{Re}(z) > 0,\ \operatorname{Re}(n) > 0,\ z \in \mathbb{C},\ n \in \mathbb{N}.$$

Hence

$$\int_0^\infty x^{\frac{z}{2}-1} e^{-n^2\pi x} \ln(x)\, dx = \frac{\Gamma\left(\frac{z}{2}\right)}{\left(n^2\pi\right)^{\frac{z}{2}}}\left[\psi\left(\frac{z}{2}\right) - \ln\left(n^2\pi\right)\right] =$$

$$\frac{\Gamma\left(\frac{z}{2}\right)}{\left(n^2\right)^{\frac{z}{2}}}\pi^{-\frac{z}{2}}\left[\psi\left(\frac{z}{2}\right) - \ln\left(n^2\pi\right)\right] =$$

$$\frac{\Gamma\left(\frac{z}{2}\right)}{n^z}\pi^{-\frac{z}{2}}\left[\psi\left(\frac{z}{2}\right) - 2\ln(n) - \ln(\pi)\right] =$$

$$\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\psi\left(\frac{z}{2}\right) - 2\ln(n) - \ln(\pi)\right] =$$

$$\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left(\psi\left(\frac{z}{2}\right) - \ln(\pi)\right) - \Gamma\left(\frac{z}{2}\right)\pi^{\frac{-z}{2}}\, 2\,\frac{\ln(n)}{n^z} =$$

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\frac{1}{n^z}\left(\psi\left(\frac{z}{2}\right) - \ln(\pi)\right) - 2\,\frac{\ln(n)}{n^z}\right]$$

writing $\theta(x) = \sum_{n=1}^{\infty} e^{-n^2\pi x}$

hence $\sum_{n=1}^{\infty}\int_0^{\infty} x^{\frac{z}{2}-1}\, e^{-n^2 \pi x} \ln(x)\, dx = \int_0^{\infty} x^{\frac{z}{2}-1}\, \theta(x) \ln(x)\, dx =$

$$\sum_{n=1}^{\infty}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\frac{1}{n^z}\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)-2\,\frac{\ln(n)}{n^z}\right]=$$

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+2\left(-\sum_{n=1}^{\infty}\frac{\ln(n)}{n^z}\right)\right]=$$

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+2\,\zeta'(z)\right]$$

because $\theta(x) = \frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}$

hence $\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+2\,\zeta'(z)\right] = \int_0^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$

$$\int_0^{1} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx + \int_1^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$$

$$\int_0^{1} x^{\frac{z}{2}-1}\left(\frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}\right)\ln(x)\,dx + \int_1^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$$

$$-\frac{2}{(z-1)^2}+\frac{2}{z^2}+\int_0^{1} x^{\frac{z}{2}-\frac{3}{2}}\,\theta\left(\frac{1}{x}\right)\ln(x)\,dx + \int_1^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$$

$$\frac{2}{z^2}-\frac{2}{(1-z)^2}+\int_0^{1}\left(\frac{1}{x}\right)^{-\frac{z}{2}+\frac{1}{2}}\left(\frac{1}{x}\right)\theta\left(\frac{1}{x}\right)\left(-\ln\left(\frac{1}{x}\right)\right)dx + \int_1^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$$

$$\frac{2}{z^2}-\frac{2}{(1-z)^2}+\int_{\infty}^{1}(t)^{-\frac{z}{2}+\frac{1}{2}}\,(t)\,\theta(t)\left(-\ln(t)\right)\left(\frac{-1}{t^2}\right)dt + \int_1^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$$

$$\frac{2}{z^2}-\frac{2}{(1-z)^2}+\int_{\infty}^{1} x^{-\frac{z}{2}+\frac{1}{2}}\,\theta(x)\ln(x)\left(\frac{1}{x}\right)dx + \int_1^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$$

$$\frac{2}{z^2}-\frac{2}{(1-z)^2}-\int_1^{\infty} x^{-\frac{z}{2}-\frac{1}{2}}\,\theta(x)\ln(x)\,dx + \int_1^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$$

$$\frac{2}{z^2}-\frac{2}{(1-z)^2}-\int_1^{\infty} x^{-\frac{z}{2}-\frac{1}{2}}\,\theta(x)\ln(x)\,dx + \int_1^{\infty} x^{\frac{z}{2}-1}\,\theta(x)\ln(x)\,dx =$$

$$\frac{2}{z^2}-\frac{2}{(1-z)^2}+\int_1^{\infty}\left(x^{\frac{z}{2}-1}-x^{-\frac{z}{2}-\frac{1}{2}}\right)\theta(x)\ln(x)\,dx$$

The right − hand side is changed only by minus if z is replaced by 1 − z.

Hence

$$\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\left[\zeta(1-z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)+2\,\zeta'(1-z)\right]=$$

$$\int_0^{\infty} x^{\frac{1-z}{2}-1}\,\theta(x)\ln(x)\,dx = \int_0^{\infty} x^{-\frac{z}{2}-\frac{1}{2}}\,\theta(x)\ln(x)\,dx =$$

$$\frac{2}{(z-1)^2}-\frac{2}{z^2}+\int_1^{\infty}\left(x^{\frac{1-z}{2}-1}-x^{-\frac{(1-z)}{2}-\frac{1}{2}}\right)\theta(x)\ln(x)\,dx =$$

$$-\left[\frac{2}{z^2}-\frac{2}{(z-1)^2}+\int_1^{\infty}\left(x^{\frac{z}{2}-1}-x^{-\frac{z}{2}-\frac{1}{2}}\right)\theta(x)\ln(x)\,dx\right]=$$
$$-\int_0^{\infty}x^{\frac{z}{2}-1}\theta(x)\ln(x)\,dx=-\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+2\zeta'(z)\right]$$

Therefore

$$-\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+2\zeta'(z)\right]=$$
$$\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\left[\zeta(1-z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)+2\zeta'(1-z)\right]$$

is a form of the functional equation dzeta function and its derivative of 1-th order.

For z that $\zeta'(1-z)=0$ and $\zeta(z)\neq 0$ and $\zeta(1-z)\neq 0$ and $\zeta'(z)\neq 0$ and $\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)\neq 0$ and $\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)\neq 0$

functional equation 1-th derivative is reduced to :

$$-\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+2\zeta'(z)\right]=\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\zeta(1-z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right).$$

Writing

$$\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\zeta(1-z)=\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\zeta(z)$$

hence

$$-\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+2\zeta'(z)\right]=\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\zeta(z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)$$

$$-\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)-2\zeta'(z)=\zeta(z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)$$

$$-2\zeta'(z)=\zeta(z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)+\zeta(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)$$

$$\zeta'(z)=-\frac{1}{2}\zeta(z)\left(\psi\left(\frac{z}{2}\right)+\psi\left(\frac{1-z}{2}\right)-2\ln(\pi)\right)$$

For z that $\zeta'(z)\neq 0$ and $\zeta'(1-z)\neq 0$ and $\zeta(z)=0$ and $\zeta(1-z)=0$ functional equation is reduced to :

$$-\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\zeta'(z)=\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\zeta'(1-z)$$

If Riemann hypothesis is true then for $z=\frac{1}{2}+yi$

$$-\Gamma\left(\frac{\frac{1}{2}+yi}{2}\right)\pi^{-\frac{\frac{1}{2}+yi}{2}}\zeta'\left(\frac{1}{2}+yi\right)=\Gamma\left(\frac{\frac{1}{2}-yi}{2}\right)\pi^{-\frac{\frac{1}{2}-yi}{2}}\zeta'\left(\frac{1}{2}-yi\right)$$

$$-\Gamma\left(\frac{\frac{1}{2}+yi}{2}\right)\pi^{-\frac{\frac{1}{2}+yi}{2}}\zeta'\left(\frac{1}{2}+yi\right)=\overline{\Gamma\left(\frac{\frac{1}{2}+yi}{2}\right)\pi^{-\frac{\frac{1}{2}+yi}{2}}\zeta'\left(\frac{1}{2}+yi\right)}$$

this equals only if $\zeta'\left(\frac{1}{2}+yi\right)=0$ or $\Gamma\left(\frac{\frac{1}{2}+yi}{2}\right)\pi^{-\frac{\frac{1}{2}+yi}{2}}\zeta'\left(\frac{1}{2}+yi\right)=ci,\ c\in\mathbb{R}$

If Riemann hypothesis is false then

$$-\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\zeta'(z)=\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\zeta'(1-z)$$

this equals for every z that $\zeta(z)=0$, $\zeta(1-z)=0$.

It is possible that for z that
$\zeta(z)=0$ and $\zeta(1-z)=0$ and
$\zeta'(z)=0$ and $\zeta'(1-z)=0$ functional equation 1 – th derivative is reduced to

$$0=0$$

but in Mathematica 13 I received that

$$\int_0^\infty x^{\frac{z}{2}-1}e^{-n^2\pi x}(\ln(x))^2\,dx=\left(n^2\right)^{-z/2}\pi^{-z/2}\Gamma\left(\frac{z}{2}\right)\left[\left(\ln\left(n^2\pi\right)-\psi\left(\frac{z}{2}\right)\right)^2+\psi'\left(\frac{z}{2}\right)\right]$$

hence

$$\int_0^\infty x^{\frac{z}{2}-1}e^{-n^2\pi x}(\ln(x))^2\,dx=$$

$$\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\left(\ln\left(\pi n^2\right)-\psi\left(\frac{z}{2}\right)\right)^2+\psi'\left(\frac{z}{2}\right)\right]=\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\left(\psi\left(\frac{z}{2}\right)-\ln\left(\pi n^2\right)\right)^2+\psi'\left(\frac{z}{2}\right)\right]=$$

$$\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\left(\psi^2\left(\frac{z}{2}\right)-2\psi\left(\frac{z}{2}\right)\ln\left(\pi n^2\right)+\left(\ln\left(\pi n^2\right)\right)^2\right)+\psi'\left(\frac{z}{2}\right)\right]=$$

$$\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[\psi^2\left(\frac{z}{2}\right)-4\psi\left(\frac{z}{2}\right)\ln(n)-2\psi\left(\frac{z}{2}\right)\ln(\pi)+\left(\ln(\pi)+2\ln(n)\right)^2+\psi'\left(\frac{z}{2}\right)\right]=$$

$$\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\Big[$$
$$\psi^2\left(\frac{z}{2}\right)-4\psi\left(\frac{z}{2}\right)\ln(n)-2\psi\left(\frac{z}{2}\right)\ln(\pi)+\left(\ln(\pi)\right)^2+4\ln(\pi)\ln(n)+4\left(\ln(n)\right)^2+\psi'\left(\frac{z}{2}\right)\Big]=$$

$$\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[4\left(\ln(n)\right)^2-4\ln(n)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right]$$

and

$$\sum_{n=1}^{\infty}\int_0^\infty x^{\frac{z}{2}-1}e^{-n^2\pi x}(\ln(x))^2\,dx=\int_0^\infty x^{\frac{z}{2}-1}\theta(x)(\ln(x))^2\,dx=$$

$$\sum_{n=1}^{\infty}\frac{1}{n^z}\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[4\left(\ln(n)\right)^2-4\ln(n)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right]=$$

$$\Gamma\left(\frac{z}{2}\right)$$
$$\pi^{-\frac{z}{2}}\left[4\sum_{n=1}^{\infty}\frac{(\ln(n))^2}{n^z}-4\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)\sum_{n=1}^{\infty}\frac{\ln(n)}{n^z}+\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2\sum_{n=1}^{\infty}\frac{1}{n^z}+\psi'\left(\frac{z}{2}\right)\sum_{n=1}^{\infty}\frac{1}{n^z}\right]=$$

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[4\,\zeta''(z)+4\,\zeta'(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\zeta(z)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}\right]$$

because $\theta(x)=\frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}$

hence $\int_0^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=\int_0^1 x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=$

$$\int_0^1 x^{\frac{z}{2}-1}\left(\frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}\right)\left(\ln(x)\right)^2 dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=$$

$$-\frac{8}{(1-z)^3}-\frac{8}{z^3}+\int_0^1 x^{\frac{z}{2}-\frac{3}{2}}\,\theta\left(\frac{1}{x}\right)\left(\ln(x)\right)^2 dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=$$

$$-\frac{8}{z^3}-\frac{8}{(1-z)^3}+\int_0^1\left(\frac{1}{x}\right)^{-\frac{z}{2}+\frac{1}{2}}\left(\frac{1}{x}\right)\theta\left(\frac{1}{x}\right)\left(-\ln\left(\frac{1}{x}\right)\right)^2 dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=$$

$$-\frac{8}{z^3}-\frac{8}{(1-z)^3}+\int_\infty^1 (t)^{-\frac{z}{2}+\frac{1}{2}}\,(t)\,\theta(t)\left(-\ln(t)\right)^2\left(\frac{-1}{t^2}\right)dt+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=$$

$$-\frac{8}{z^3}-\frac{8}{(1-z)^3}-\int_\infty^1 x^{-\frac{z}{2}+\frac{1}{2}}\,\theta(x)\left(\ln(x)\right)^2\left(\frac{1}{x}\right)dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=$$

$$-\frac{8}{z^3}-\frac{8}{(1-z)^3}+\int_1^\infty x^{-\frac{z}{2}-\frac{1}{2}}\,\theta(x)\left(\ln(x)\right)^2 dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=$$

$$-\frac{8}{z^3}-\frac{8}{(1-z)^3}+\int_1^\infty\left(x^{\frac{z}{2}-1}+x^{-\frac{z}{2}-\frac{1}{2}}\right)\theta(x)\left(\ln(x)\right)^2 dx$$

The right − hand side is unchanged if z is replaced by 1 − z.

Hence

$$\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\Big[$$

$$4\,\zeta''(1-z)+4\,\zeta'(1-z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)+\zeta(1-z)\left\{\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{1-z}{2}\right)\right\}\Big]=$$

$$\int_0^\infty x^{\frac{1-z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=\int_0^\infty x^{-\frac{z}{2}-\frac{1}{2}}\,\theta(x)\left(\ln(x)\right)^2 dx=-\frac{8}{(1-z)^3}-\frac{8}{(z)^3}+$$

$$\int_1^\infty\left(x^{\frac{1-z}{2}-1}+x^{\frac{-(1-z)}{2}-\frac{1}{2}}\right)\theta(x)\left(\ln(x)\right)^2 dx=-\frac{8}{z^3}-\frac{8}{(1-z)^3}+\int_1^\infty\left(x^{\frac{z}{2}-1}+x^{-\frac{z}{2}-\frac{1}{2}}\right)\theta(x)\left(\ln(x)\right)^2 dx=$$

$$\int_0^\infty x^{\frac{z}{2}-1}\,\theta(x)\left(\ln(x)\right)^2 dx=$$

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[4\,\zeta''(z)+4\,\zeta'(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\zeta(z)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}\right]$$

Therefore

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\left[4\,\zeta''(z)+4\,\zeta'(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\zeta(z)\left(\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right)\right]=$$

$$\Gamma\left(\frac{1-z}{2}\right)$$

$$\pi^{-\frac{1-z}{2}}\left[4\,\zeta''(1-z)+4\,\zeta'(1-z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)+\zeta(1-z)\left(\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{1-z}{2}\right)\right)\right]$$

is a form of the functional equation zeta function and its derivative of 2-th order.

For z that $\zeta(z)=0$ and $\zeta(1-z)=0$ and
$\zeta'(z)=0$ and $\zeta'(1-z)=0$ functional equation $2-$th order is reduced to

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\,4\,\zeta''(z)=\Gamma\left(\frac{1-z}{2}\right)\pi^{-\frac{1-z}{2}}\,4\,\zeta''(1-z)$$

If Riemann hypothesis is true then $z=\frac{1}{2}+yi$

$$\Gamma\left(\frac{\frac{1}{2}+yi}{2}\right)\pi^{-\frac{\frac{1}{2}+yi}{2}}\,\zeta''\left(\frac{1}{2}+yi\right)=\Gamma\left(\frac{\frac{1}{2}-yi}{2}\right)\pi^{-\frac{\frac{1}{2}-yi}{2}}\,\zeta''\left(\frac{1}{2}-yi\right)$$

$$\Gamma\left(\frac{\frac{1}{2}+yi}{2}\right)\pi^{-\frac{\frac{1}{2}+yi}{2}}\,\zeta''\left(\frac{1}{2}+yi\right)=\overline{\Gamma\left(\frac{\frac{1}{2}+yi}{2}\right)\pi^{-\frac{\frac{1}{2}+yi}{2}}\,\zeta''\left(\frac{1}{2}+yi\right)}$$

this equals only if $\zeta''\left(\frac{1}{2}+yi\right)=0$ or $\Gamma\left(\frac{\frac{1}{2}+yi}{2}\right)\pi^{-\frac{\frac{1}{2}+yi}{2}}\,\zeta''\left(\frac{1}{2}+yi\right)=c,\ c\in\mathbb{R}$.

If Riemann hypothesis is false then

$$\Gamma\left(\frac{z}{2}\right)\pi^{-\frac{z}{2}}\,\zeta''(z)=\Gamma\left(\frac{1\text{-}z}{2}\right)\pi^{-\frac{1\text{-}z}{2}}\,\zeta''(1\text{-}z)$$

for z that $\zeta(z)=0$ and $\zeta(1\text{-}z)=0$ and $\zeta'(z)=0$ and $\zeta'(1\text{-}z)=0$.

It is possible that for z that
$\zeta(z)=0$ and $\zeta(1-z)=0$ and $\zeta'(z)=0$ and $\zeta'(1-z)=$
0 and $\zeta''(z)=0$ and $\zeta''(1-z)=0$ functional equation $2-$th derivative is reduced to

$$0=0$$

but in Mathematica 13 I received that

$$\int_0^\infty x^{\frac{z}{2}-1}\,e^{-n^2\pi x}\,(\ln(x))^3\,dx=-\left(n^2\right)^{-z/2}\pi^{-z/2}\,\Gamma$$
$$\left(\frac{z}{2}\right)\left[\left(\ln\left(n^2\right)\right)^3+3\left(\ln\left(n^2\right)\right)^2\ln(\pi)+3\ln\left(n^2\right)(\ln(\pi))^2+(\ln(\pi))^3+3\ln\left(n^2\pi\right)\left(\psi\left(\frac{z}{2}\right)\right)^2-\right.$$
$$\left.\left(\psi\left(\frac{z}{2}\right)\right)^3+3\ln\left(n^2\pi\right)\psi'\left(\frac{z}{2}\right)-3\,\psi\left(\frac{z}{2}\right)\left\{\left(\ln\left(n^2\pi\right)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}-\psi''\left(\frac{z}{2}\right)\right]$$

hence

$$\int_0^\infty x^{\frac{z}{2}-1}\,e^{-n^2\pi x}\,(\ln(x))^3\,dx=$$

$$-\left(n^2\right)^{-\frac{z}{2}}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\Big[\left(\ln\left(n^2\right)\right)^3+3\left(\ln\left(n^2\right)\right)^2\ln(\pi)+3\ln\left(n^2\right)\left(\ln(\pi)\right)^2+\left(\ln(\pi)\right)^3+3\ln\left(n^2\pi\right)$$

$$\left(\psi\left(\frac{z}{2}\right)\right)^2-\left(\psi\left(\frac{z}{2}\right)\right)^3+3\ln\left(n^2\pi\right)\psi'\left(\frac{z}{2}\right)-3\psi\left(\frac{z}{2}\right)\left\{\left(\ln\left(n^2\pi\right)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}-\psi''\left(\frac{z}{2}\right)\Big]=$$

$$-\frac{1}{n^z}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\Big[\left(2\ln(n)\right)^3+3\left(2\ln(n)\right)^2\ln(\pi)+6\ln(n)\left(\ln(\pi)\right)^2+\left(\ln(\pi)\right)^3+$$

$$3\left(2\ln(n)+\ln(\pi)\right)\left(\psi\left(\frac{z}{2}\right)\right)^2-\left(\psi\left(\frac{z}{2}\right)\right)^3+3\left(2\ln(n)+\ln(\pi)\right)\psi'\left(\frac{z}{2}\right)-$$

$$3\psi\left(\frac{z}{2}\right)\left\{\left(2\ln(n)+\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}-\psi''\left(\frac{z}{2}\right)\Big]=$$

$$-\frac{1}{n^z}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\Big[8\left(\ln(n)\right)^3+12\left(\ln(n)\right)^2\ln(\pi)+6\ln(n)\left(\ln(\pi)\right)^2+\left(\ln(\pi)\right)^3$$

$$+6\ln(n)\left(\psi\left(\frac{z}{2}\right)\right)^2+3\ln(\pi)\left(\psi\left(\frac{z}{2}\right)\right)^2-\left(\psi\left(\frac{z}{2}\right)\right)^3+6\ln(n)\psi'\left(\frac{z}{2}\right)+3\ln(\pi)\psi'\left(\frac{z}{2}\right)$$

$$-3\psi\left(\frac{z}{2}\right)\left\{4\left(\ln(n)\right)^2+4\ln(n)\ln(\pi)+\left(\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}-\psi''\left(\frac{z}{2}\right)\Big]=$$

$$-\frac{1}{n^z}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\Big[8\left(\ln(n)\right)^3+12\left(\ln(n)\right)^2\ln(\pi)+6\ln(n)\left(\ln(\pi)\right)^2+\left(\ln(\pi)\right)^3+6\ln(n)\left(\psi\left(\frac{z}{2}\right)\right)^2$$

$$+3\ln(\pi)\left(\psi\left(\frac{z}{2}\right)\right)^2-\left(\psi\left(\frac{z}{2}\right)\right)^3+6\ln(n)\psi'\left(\frac{z}{2}\right)+3\ln(\pi)\psi'\left(\frac{z}{2}\right)-12\psi\left(\frac{z}{2}\right)\left(\ln(n)\right)^2$$

$$-12\psi\left(\frac{z}{2}\right)\ln(n)\ln(\pi)-3\psi\left(\frac{z}{2}\right)\left(\ln(\pi)\right)^2-3\psi\left(\frac{z}{2}\right)\psi'\left(\frac{z}{2}\right)\Big\}-\psi''\left(\frac{z}{2}\right)\Big]$$

$$=-\frac{1}{n^z}\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\Big[8\left(\ln(n)\right)^3-12\left(\ln(n)\right)^2\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+6\ln(n)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}-$$

$$\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^3+3\psi'\left(\frac{z}{2}\right)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\psi''\left(\frac{z}{2}\right)\right\}\Big]=$$

$$-\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\Big[8\frac{\left(\ln(n)\right)^3}{n^z}-12\frac{\left(\ln(n)\right)^2}{n^z}\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+6\frac{\ln(n)}{n^z}\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}-$$

$$\frac{1}{n^z}\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^3+3\psi'\left(\frac{z}{2}\right)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\psi''\left(\frac{z}{2}\right)\right\}\Big]$$

and

$$\sum_{n=1}^{\infty}\int_0^{\infty}x^{\frac{z}{2}-1}e^{-n^2\pi x}\left(\ln(x)\right)^3dx=-\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\Big[$$

$$8\sum_{n=1}^{\infty}\frac{\left(\ln(n)\right)^3}{n^z}-12\sum_{n=1}^{\infty}\frac{\left(\ln(n)\right)^2}{n^z}\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+6\sum_{n=1}^{\infty}\frac{\ln(n)}{n^z}\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}-$$

$$\sum_{n=1}^{\infty}\frac{1}{n^z}\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^3+3\psi'\left(\frac{z}{2}\right)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\psi''\left(\frac{z}{2}\right)\right\}\Big]=$$

$$-\pi^{-\frac{z}{2}}\Gamma\left(\frac{z}{2}\right)\Big[-8\zeta'''(z)-12\zeta''(z)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)-6\zeta'(z)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}-$$

$$\zeta(z)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^3+3\,\psi'\left(\frac{z}{2}\right)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\psi''\left(\frac{z}{2}\right)\right\}\Big] =$$

$$\pi^{-\frac{z}{2}}\,\Gamma\left(\frac{z}{2}\right)\Big[8\,\zeta'''(z)+12\,\zeta''(z)\,\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+6\,\zeta'(z)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}\; +$$

$$\zeta(z)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^3+3\,\psi'\left(\frac{z}{2}\right)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\psi''\left(\frac{z}{2}\right)\right\}\Big]$$

because $\theta(x)=\frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right)+\frac{1}{2\,\sqrt{x}}-\frac{1}{2}$

$$\int_0^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=\int_0^1 x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=$$

$$\int_0^1 x^{\frac{z}{2}-1}\left(\frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right)+\frac{1}{2\,\sqrt{x}}-\frac{1}{2}\right)(\ln(x))^3\,dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=$$

$$-\frac{48}{(1-z)^4}+\frac{48}{z^4}+\int_0^1 x^{\frac{z}{2}-\frac{3}{2}}\,\theta\left(\frac{1}{x}\right)(\ln(x))^3\,dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=$$

$$\frac{48}{z^4}-\frac{48}{(1-z)^4}+\int_0^1\left(\frac{1}{x}\right)^{-\frac{z}{2}+\frac{1}{2}}\left(\frac{1}{x}\right)\theta\left(\frac{1}{x}\right)\left(-\ln\left(\frac{1}{x}\right)\right)^3 dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=$$

$$\frac{48}{z^4}-\frac{48}{(1-z)^4}+\int_\infty^1 (t)^{-\frac{z}{2}+\frac{1}{2}}\,(t)\,\theta(t)\,(-\ln(t))^3\left(\frac{-1}{t^2}\right)dt+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=$$

$$\frac{48}{z^4}-\frac{48}{(1-z)^4}+\int_\infty^1 x^{-\frac{z}{2}+\frac{1}{2}}\,\theta(x)\,(\ln(x))^3\left(\frac{1}{x}\right)dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=$$

$$\frac{48}{z^4}-\frac{48}{(1-z)^4}-\int_1^\infty x^{-\frac{z}{2}-\frac{1}{2}}\,\theta(x)\,(\ln(x))^3\,dx+\int_1^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=$$

$$\frac{48}{z^4}-\frac{48}{(1-z)^4}+\int_1^\infty\left(x^{\frac{z}{2}-1}-x^{-\frac{z}{2}-\frac{1}{2}}\right)\theta(x)\,(\ln(x))^3\,dx$$

The right – hand side is changed only by minus if z is replaced by 1 – z.

Hence

$$\int_0^\infty x^{\frac{1-z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx=$$

$$\int_0^\infty x^{-\frac{z}{2}-\frac{1}{2}}\,\theta(x)\,(\ln(x))^3\,dx=\frac{48}{(1-z)^4}-\frac{48}{(z)^4}+\int_1^\infty\left(x^{\frac{1-z}{2}-1}-x^{-\frac{(1-z)}{2}-\frac{1}{2}}\right)\theta(x)\,(\ln(x))^3\,dx=$$

$$-\left[\frac{48}{z^4}-\frac{48}{(1-z)^4}+\int_1^\infty\left(x^{\frac{z}{2}-1}-x^{-\frac{z}{2}-\frac{1}{2}}\right)\theta(x)\,(\ln(x))^3\,dx\right]=-\int_0^\infty x^{\frac{z}{2}-1}\,\theta(x)\,(\ln(x))^3\,dx$$

Therefore

$$-\pi^{-\frac{z}{2}}\,\Gamma\left(\frac{z}{2}\right)\Big[8\,\zeta'''(z)+12\,\zeta''(z)\,\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+6\,\zeta'(z)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{z}{2}\right)\right\}\; +$$

$$\zeta(z)\left\{\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)^3+3\,\psi'\left(\frac{z}{2}\right)\left(\psi\left(\frac{z}{2}\right)-\ln(\pi)\right)+\psi''\left(\frac{z}{2}\right)\right\}\Big] =$$

$$\pi^{-\frac{1-z}{2}}\,\Gamma\left(\frac{1-z}{2}\right)\Bigg[8\,\zeta'''(1-z)+12\,\zeta''(1-z)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)+$$
$$6\,\zeta'(1-z)\left\{\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)^2+\psi'\left(\frac{1-z}{2}\right)\right\}+$$
$$\zeta(1-z)\left\{\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)^3+3\,\psi'\left(\frac{1-z}{2}\right)\left(\psi\left(\frac{1-z}{2}\right)-\ln(\pi)\right)+\psi''\left(\frac{1-z}{2}\right)\right\}\Bigg]$$

is a form of the functional equation zeta function and its derivative of 3-th order.

Exists many integrals $\int_0^\infty x^{\frac{z}{2}-1}\,e^{-n^2\pi x}\,(\ln(x))^k\,dx$ for k= 0,1,2,3,... so there is many functional equations of Riemann zeta function and its derivative of k-th order.

References

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