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FM-Frequency Modulation bandwidth and the proof of the BESSEL's Series

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Abstract: here is a proof of the Bessel's Series used to figure out the bandwidth in the Frequency Modulation-FM.

Contents:

GENERAL VIEW

FM-Frequency Modulation:

Carrier: $v(t) = V_P \cos\varphi = V_P \cos(\omega t + \theta)$, Modulating signal: $v_m(t) = V_m \cos\omega_m t$

The modulated pulsation is: $\omega = \omega_P + K_f V_m \cos\omega_m t$, so:

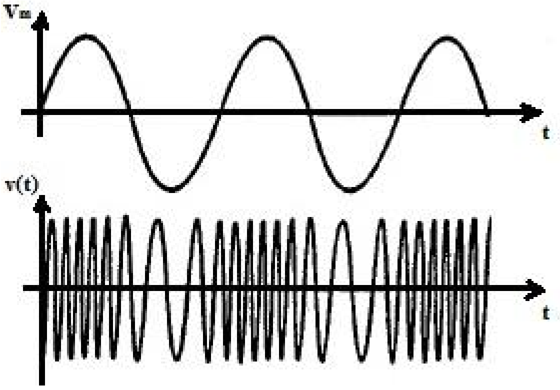
$f = f_P + \frac{K_f}{2\pi} V_m \cos\omega_m t = f_P + \Delta f \cos\omega_m t$; Δf is the frequency deviation. In general:

$\omega = \frac{d\varphi}{dt} = \omega_P + K_f v_m(t)$, so:

$\varphi = \int \omega dt = 2\pi \int f dt = 2\pi \int (f_P + \Delta f \cos\omega_m t) dt = 2\pi [f_P t + \frac{\Delta f}{2\pi f_m} \sin 2\pi f_m t]$, and:

$$v(t) = V_P \cos\varphi = V_P \cos[2\pi f_P t + \frac{\Delta f}{f_m} \sin 2\pi f_m t] . \qquad (1.1)$$

$m_f = \frac{\Delta f}{f_m}$ is the Modulation Index.



We develop the above (1.1) in Bessel's Series (see (4.3) and (4.4) in Appendix 4, here reported):

$$\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + ...$$
$$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + ...$$

($\cos(a+b) = \cos a\cos b - \sin a\sin b$)

Well, $v(t) = V_P\cos(\omega_P t + m_f\sin\omega_m t) = V_P[\cos\omega_P t\cos(m_f\sin\omega_m t) - \sin\omega_P t\sin(m_f\sin\omega_m t)] =$

$= V_P\cos\omega_P t[J_0(m_f) + 2J_2(m_f)\cos 2\omega_m t + 2J_4(m_f)\cos 4\omega_m t + ...] -$

$- V_P\sin\omega_P t[2J_1(m_f)\sin\omega_m t + 2J_3(m_f)\sin 3\omega_m t + ...]$

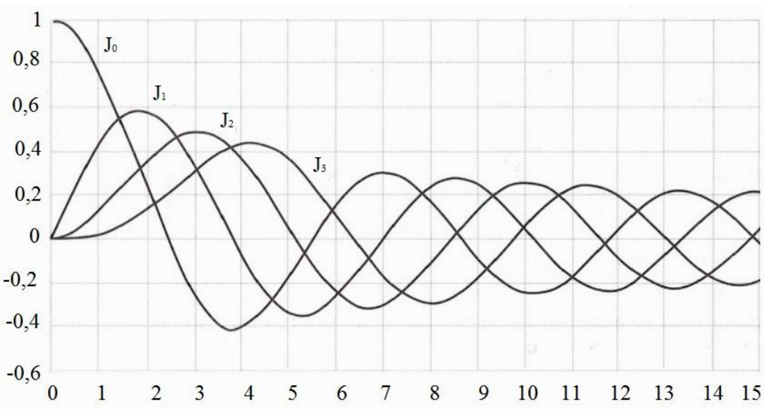
Now, as: $\cos a\cdot\cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$ and $\sin a\cdot\sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$,

it follows that:

$$v(t) = V_P[J_0(m_f)\cos\omega_P t - J_1(m_f)\cos(\omega_P - \omega_m)t + J_1(m_f)\cos(\omega_P + \omega_m)t + J_2(m_f)\cos(\omega_P - 2\omega_m)t +$$
$$+ J_2(m_f)\cos(\omega_P + 2\omega_m)t - J_3(m_f)\cos(\omega_P - 3\omega_m)t + J_3(m_f)\cos(\omega_P + 3\omega_m)t + ...] \quad (1.2)$$

This equation (1.2) gives us the spectrum of the modulated signal; it has the carrier and the side components, spaced by ω_m and whose amplitudes are given by the Bessel functions which depend on m_f. As usually small m_f (<0,5) give small values of the Bessel functions, then only the side rows at freq. $f_P - f_m$ and $f_P + f_m$ are considered. As $J_0(m_f) = 0$ when $m_f = 2{,}40$, there the carrier disappears.

On the basis of all that, the bandwidth is approximately $B = 2(\Delta f + f_m)$.



The expressions for the Bessel Functions $J_n(x)$ are given by the following (see (2.2) below):

$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}}{\Gamma(n+k+1)k!}$ and are represented in the above image. For instance:

$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + ...$. In order to figure out the value of such a $J_0(x)$ by adding terms of a series, do not worry, because as long as k gets high enough, the terms will start being negligible.

Appendix 1: EULER GAMMA FUNCTION

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Of course $\Gamma(1) = 1$. Then, $\Gamma(n+1) = n\Gamma(n)$; in fact, after an integration by parts:

$\Gamma(z) = e^{-t} \frac{t^z}{z}\Big|_0^\infty + \frac{1}{z}\int_0^\infty e^{-t} t^z dt = 0 + \frac{1}{z}\Gamma(z+1)$ and by iterating the $\Gamma(n+1) = n\Gamma(n)$, we get: $\Gamma(n+1) = n!$.

<u>Stirling's Formula:</u>

$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = \int_0^\infty e^{n \ln x - x} dx$ and the function $n \ln x - x$ has a maximum in x=n, as you can see by putting to zero its derivative. Now, after saying x=n+y, we have:

$$\Gamma(n+1) = e^{-n} \int_{-n}^\infty e^{n \ln(n+y) - y} dy = e^{-n} \int_{-n}^\infty e^{n \ln n + n \ln(1 + y/n) - y} dy = n^n e^{-n} \int_{-n}^\infty e^{n \ln(1 + y/n) - y} dy$$

Now, according to Taylor, we have that: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ....$; we put first x=y/n and later $y = \sqrt{n}v$, so: $\Gamma(n+1) = n^n e^{-n} \int_{-n}^\infty e^{n \ln(1 + y/n) - y} dy = n^n e^{-n} \int_{-n}^\infty e^{-y^2/2n + y^3/3n^2 - ......} dy =$

$= n^n e^{-n} \sqrt{n} \int_{-\sqrt{n}}^\infty e^{-v^2/2 + v^3/3\sqrt{n} - ......} dv$, and if n is large enough: $\Gamma(n+1) = \sqrt{2\pi n} n^n e^{-n}$ (Stirling's Formula), as the last Gauss integral was $\sqrt{2\pi}$.

Appendix 2: BESSEL'S DIFFERENTIAL EQUATION

When you face a differential equation like this:

$$x^2 y'' + xy' + (x^2 - n^2) y = 0 \tag{2.1}$$

you have a Bessel Differential Equation.

If we look for a solution as a series $J_n(x)$, then the differential equation becomes: $x^2 J''_n + xJ'_n + (x^2 - n^2) J_n = 0$ and the solution series is: $J_n(x) = \sum_{k=0}^\infty \frac{(-1)^k (x/2)^{n+2k}}{\Gamma(n+k+1)k!}$, (2.2)

where Γ is the Gamma Euler Function. In fact, if you insert the (2.2) into the (2.1), you get 0=0:

$$y = \sum_{k=0}^\infty \frac{(-1)^k (x)^{n+2k}}{(2)^{n+2k}(n+k)!k!} \quad , y' = \sum_{k=0}^\infty \frac{(-1)^k (n+2k)(x)^{n+2k-1}}{(2)^{n+2k}(n+k)!k!} \quad , \quad y'' = \sum_{k=0}^\infty \frac{(-1)^k (n+2k)(n+2k-1)(x)^{n+2k-2}}{(2)^{n+2k}(n+k)!k!}$$

$$(x^2 - n^2) y = \sum_{k=0}^\infty \frac{(-1)^k (x)^{n+2k+2}}{(2)^{n+2k}(n+k)!k!} - \sum_{k=0}^\infty \frac{(-1)^k n^2 (x)^{n+2k}}{(2)^{n+2k}(n+k)!k!} \quad , \quad xy' = \sum_{k=0}^\infty \frac{(-1)^k (n+2k)(x)^{n+2k}}{(2)^{n+2k}(n+k)!k!} ,$$

$$x^2 y'' = \sum_{k=0}^\infty \frac{(-1)^k (n+2k)(n+2k-1)(x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}$$

$$x^2y''+xy'+(x^2-n^2)y=\sum_{k=0}^{\infty}\frac{(-1)^k(n+2k)(n+2k-1)(x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}+\sum_{k=0}^{\infty}\frac{(-1)^k(n+2k)(x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}+$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k(x)^{n+2k+2}}{(2)^{n+2k}(n+k)!k!}-\sum_{k=0}^{\infty}\frac{(-1)^k n^2(x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}=\sum_{k=0}^{\infty}\frac{(-1)^k(x)^{n+2k+2}}{(2)^{n+2k}(n+k)!k!}+\sum_{k=0}^{\infty}\frac{(-1)^k[4k(k+n)](x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}=$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(x)^{n+2k}}{(2)^{n+2k-2}(n+k-1)!(k-1)!}+\sum_{k=0}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}\frac{(n+k)!}{(n+k)}\frac{k!}{k}}=-\sum_{k=1}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}+$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}$$, so:

$$x^2y''+xy'+(x^2-n^2)y=-\sum_{k=1}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}+\sum_{k=0}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}$$, but we can also say that: $x^2y''+xy'+(x^2-n^2)y=-\sum_{k=1}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}+\sum_{k=1}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}=0$, where also the latter summation can start from k=1, as its term in k=0 has in its denominator: $(k-1)!=\Gamma(k)=(-1)!=\Gamma(0)=\infty$.

We see that: $J_{-n}(x)=(-1)^n J_n(x)$; in fact:

$$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)}=\sum_{r=0}^{n-1}\frac{(-1)^r(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)}+\sum_{r=n}^{\infty}\frac{(-1)^r(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)}$$

but as $\Gamma(-n+r+1)=\infty$ for every r = 0,1,…..,n-1, the first summation is zero. Moreover, in the second summation we put r = n+k , so:

$$\sum_{k=0}^{\infty}\frac{(-1)^{n+k}(x/2)^{n+2k}}{(n+k)!\Gamma(k+1)}=(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}}{\Gamma(n+k+1)k!}=(-1)^n J_n(x)$$, qed.

The denominators in both summations are the same after considering that: $\Gamma(n+1)=n\Gamma(n)$.

We also have that: $J'_n(x)J_{-n}(x)-J'_{-n}(x)J_n(x)=\frac{2}{x\Gamma(n)\Gamma(1-n)}$ (2.3)

In fact, as both $J_n(x)$ and $J_{-n}(x)$ satisfy the Bessel Equation, then:

$x^2J''_n+xJ'_n+(x^2-n^2)J_n=0$ and $x^2J''_{-n}+xJ'_{-n}+(x^2-n^2)J_{-n}=0$ and by multiplying the former by J_{-n} and the latter by J_n and finally after subtracting side from side, we get:

$x^2(J''_n J_{-n}-J''_{-n}J_n)+x(J'_n J_{-n}-J'_{-n}J_n)=0$, or:

$x\frac{d}{dx}(J'_n J_{-n}-J'_{-n}J_n)+(J'_n J_{-n}-J'_{-n}J_n)=0$, or again: $\frac{d}{dx}[x(J'_n J_{-n}-J'_{-n}J_n)]=0 \rightarrow$

$J'_n J_{-n}-J'_{-n}J_n=\frac{c}{x}$. (2.4)

Now, in order to calculate c, we remind that: $J_n=\frac{x^n}{2^n\Gamma(n+1)}-.....$, $J'_n=\frac{x^{n-1}}{2^n\Gamma(n)}-.....$,

$J'_{-n}=\frac{x^{-n-1}}{2^{-n}\Gamma(-n)}-.....$, $J_{-n}=\frac{x^{-n}}{2^{-n}\Gamma(-n+1)}-.....$, and all these in (2.4) yield:

$(((.......\Gamma(n+1)=n\Gamma(n).......)))$

$$c=\frac{1}{\Gamma(n)\Gamma(1-n)}-\frac{1}{\Gamma(n+1)\Gamma(-n)}=\frac{1}{\Gamma(n)\Gamma(1-n)}-\frac{1}{[n\Gamma(n)]\Gamma(-n)}=\frac{1}{\Gamma(n)\Gamma(1-n)}-\frac{1}{\Gamma(n)[n\Gamma(-n)]}$$

$$=\frac{1}{\Gamma(n)\Gamma(1-n)}-\frac{-1}{\Gamma(n)[-n\Gamma(-n)]}=\frac{1}{\Gamma(n)\Gamma(1-n)}+\frac{1}{\Gamma(n)\Gamma(1-n)}=\frac{2}{\Gamma(n)\Gamma(1-n)}\text{, so:}$$

$c=\dfrac{2}{\Gamma(n)\Gamma(1-n)}$. So: $J'_n J_{-n}-J'_{-n}J_n=\dfrac{2}{x\Gamma(n)\Gamma(1-n)}$.

Appendix 3: REFLECTION FORMULA

Here it is: $\dfrac{2}{\Gamma(n)\Gamma(1-n)}=\dfrac{2\sin n\pi}{\pi}$. (3.1)

In fact, we know that $\dfrac{\sin x}{x}$ is zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, so:

$\dfrac{\sin x}{x}=(1-\dfrac{x}{\pi})(1+\dfrac{x}{\pi})(1-\dfrac{x}{2\pi})(1+\dfrac{x}{2\pi}).........=(1-\dfrac{x^2}{\pi^2})(1-\dfrac{x^2}{4\pi^2}).....=\prod_{k=1}^{\infty}[1-(\dfrac{x}{k\pi})^2]$, and so:

$$\frac{\pi x}{\sin\pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} \qquad (3.2)$$

Moreover, we can see a factorial like this:

$n!=\lim\limits_{k\to\infty}\dfrac{k!k^n}{(n+1)...(n+k)}\cdot\lim\limits_{k\to\infty}\dfrac{(k+1)(k+2)...(k+n)}{k^n}$; in fact, the denominator of the first limit is $(n+1)...(n+k)=\dfrac{(n+k)!}{n!}$, while from both numerators we can collect: $k!(k+1)...(k+n)=(k+n)!$, so just n! is left. Furthermore, we see that the second limit is 1: $\lim\limits_{k\to\infty}\dfrac{(k+1)(k+2)...(k+n)}{k^n}=1$, as it gets of the form $\lim\limits_{k\to\infty}\dfrac{k^n}{k^n}$. Therefore: $n!=\lim\limits_{k\to\infty}\dfrac{k!k^n}{(n+1)...(n+k)}$. As $\Gamma(x+1)=x!$, we have: $\Gamma(x+1)=x!=\lim\limits_{k\to\infty}\dfrac{k!k^x}{(x+1)...(x+k)}$ and after dividing both numerator and denominator by k!:

$$\Gamma(x+1)=\lim_{k\to\infty}\frac{k^x}{(1+x)(1+x/2)...(1+x/k)} \quad . \qquad (3.3)$$

Now, let's introduce the constant $\gamma_k=1+\dfrac{1}{2}+\dfrac{1}{3}+...+\dfrac{1}{k}-\ln k$, and let's say: $\gamma=\lim\limits_{k\to\infty}\gamma_k$; on the basis of that, the (3.3) becomes: $\Gamma(x+1)=e^{-\gamma x}\lim\limits_{k\to\infty}\dfrac{e^x}{(1+x)}\dfrac{e^{x/2}}{(1+x/2)}....\dfrac{e^{x/k}}{(1+x/k)}$. (3.4)

In fact, all the e^x, $e^{x/2}$,…, $e^{x/k}$ cancel with terms in $e^{-\gamma x}$ and the term $k^x=e^{(\ln k)x}$ still comes from $e^{-\gamma x}$. Moreover, as $\Gamma(x+1)=x\Gamma(x)$, then the (3.4) becomes:

$$\frac{1}{\Gamma(x)}=xe^{\gamma x}\lim_{k\to\infty}\frac{(1+x)}{e^x}\frac{(1+x/2)}{e^{x/2}}....\frac{(1+x/k)}{e^{x/k}}=xe^{\gamma x}\prod_{k=1}^{\infty}[1+(\frac{x}{k})]e^{-x/k} \qquad (3.5)$$

Now, as $\Gamma(y+1)=y\Gamma(y)$, then $\Gamma(y)=\dfrac{1}{y}\Gamma(y+1)$ and after replacing y by –x:

$\Gamma(-x)=-\dfrac{1}{x}\Gamma(1-x)$ and so: $\Gamma(1-x)=-x\Gamma(-x)$. Let's figure out $\Gamma(x)\Gamma(1-x)$:

$$\Gamma(x)\Gamma(1-x)=[x^{-1}e^{-\gamma x}\prod_{k=1}^{\infty}(1+\frac{x}{k})^{-1}e^{x/k}]\cdot[e^{\gamma x}\prod_{k=1}^{\infty}(1-\frac{x}{k})^{-1}e^{-x/k}]=[A]\cdot[B]=\frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}, \qquad (3.6)$$

where A is an inverted (3.5) and B is the (3.4) with x changed into –x. So, (3.6) is:

$\Gamma(x)\Gamma(1-x)=\frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$, and for the (3.2) (($\frac{\pi x}{\sin \pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$)) we have:

$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}$ which is the (3.1) indeed.

Appendix 4: BESSEL'S SERIES

Now, as according to (2.3): $J'_n(x)J_{-n}(x)-J'_{-n}(x)J_n(x)=\frac{2}{x\Gamma(n)\Gamma(1-n)}$ and using in it the (3.1),

we get: $J'_n(x)J_{-n}(x)-J'_{-n}(x)J_n(x)=\frac{2\sin n\pi}{\pi x}$. (4.1)

The following equality holds: $e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x)t^n$; (4.2)

in fact: $e^{\frac{x}{2}(t-\frac{1}{t})}=e^{\frac{xt}{2}}e^{-\frac{x}{2t}}=[\sum_{r=0}^{\infty}\frac{(xt/2)^r}{r!}][\sum_{k=0}^{\infty}\frac{(-x/2t)^k}{k!}]=\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{(r+k)}t^{(r-k)}}{r!k!}$

and defining n=r-k, it follows that n will go from $-\infty$ to $+\infty$, so:

$$e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{(n+2k)}t^n}{(n+k)!k!}=\sum_{n=-\infty}^{\infty}[\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{(n+2k)}}{k!(n+k)!}]t^n=\sum_{n=-\infty}^{\infty}J_n(x)t^n \text{ qed.}$$

The following equalities (**<u>Bessel's Series</u>**) hold:

$\cos(x\sin\theta)=J_0(x)+2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta+......$ (4.3)

$\sin(x\sin\theta)=2J_1(x)\sin\theta+2J_3(x)\sin 3\theta+2J_5(x)\sin 5\theta+......$ (4.4)

In fact, if we put $t=e^{i\theta}$ into the (4.2), we get:

$$e^{\frac{1}{2}x(e^{i\theta}-e^{-i\theta})}=e^{ix\sin x}=\sum_{-\infty}^{\infty}J_n(x)e^{in\theta}=$$

$$=\sum_{-\infty}^{\infty}J_n(x)[\cos n\theta+i\sin n\theta]=\{J_0(x)+[J_{-1}(x)+J_1(x)]\cos\theta+[J_{-2}(x)+J_2(x)]\cos 2\theta+....\}+$$

$$+\mathrm{i}\{[J_1(x)-J_{-1}(x)]\sin\theta+[J_2(x)-J_{-2}(x)]\sin 2\theta+....\}=$$

$=\{J_0(x)+2J_2(x)\cos 2\theta+....\}+\mathrm{i}\{2J_1(x)\sin\theta+2J_3(x)\sin 3\theta+....\}$, and making equal real and imaginary parts just obtained with those from $e^{ix\sin x}$, we have (4.3) and (4.4) indeed.

FM-Modulazione di Frequenza-larghezza di banda e dimostrazione della Serie di BESSEL

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Abstract: ecco una dimostrazione della Serie di Bessel utilizzata per valutare la larghezza di banda nella Modulazione di Frequenza-FM.

Indice:

CONCETTI GENERALI

FM-Modulazione di Frequenza:

Portante: $v(t) = V_P \cos\varphi = V_P \cos(\omega t + \theta)$, Segnale modulante: $v_m(t) = V_m \cos\omega_m t$

La pulsazione modulata è: $\omega = \omega_P + K_f V_m \cos\omega_m t$, dunque:

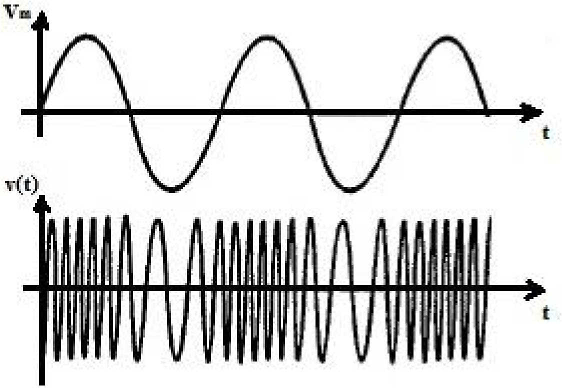
$f = f_P + \frac{K_f}{2\pi} V_m \cos\omega_m t = f_P + \Delta f \cos\omega_m t$; Δf è la deviazione di frequenza. In generale:

$\omega = \frac{d\varphi}{dt} = \omega_P + K_f v_m(t)$, così:

$\varphi = \int \omega dt = 2\pi \int f dt = 2\pi \int (f_P + \Delta f \cos\omega_m t) dt = 2\pi [f_P t + \frac{\Delta f}{2\pi f_m} \sin 2\pi f_m t]$, e:

$$v(t) = V_P \cos\varphi = V_P \cos[2\pi f_P t + \frac{\Delta f}{f_m} \sin 2\pi f_m t] . \tag{1.1}$$

$m_f = \frac{\Delta f}{f_m}$ è l'Indice di Modulazione.



Sviluppiamo la (1.1) qui sopra in Serie di Bessel (vedi le (4.3) e (4.4) in Appendce 4, qui riportate):

$$\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + ...$$
$$\sin(x\sin\theta) = 2J_1(x)\sin\theta + 2J_3(x)\sin 3\theta + 2J_5(x)\sin 5\theta + ...$$

($\cos(a+b) = \cos a\cos b - \sin a\sin b$)

Bene, $v(t) = V_P\cos(\omega_P t + m_f\sin\omega_m t) = V_P[\cos\omega_P t\cos(m_f\sin\omega_m t) - \sin\omega_P t\sin(m_f\sin\omega_m t)] =$

$= V_P\cos\omega_P t[J_0(m_f) + 2J_2(m_f)\cos 2\omega_m t + 2J_4(m_f)\cos 4\omega_m t + ...] -$

$- V_P\sin\omega_P t[2J_1(m_f)\sin\omega_m t + 2J_3(m_f)\sin 3\omega_m t + ...]$

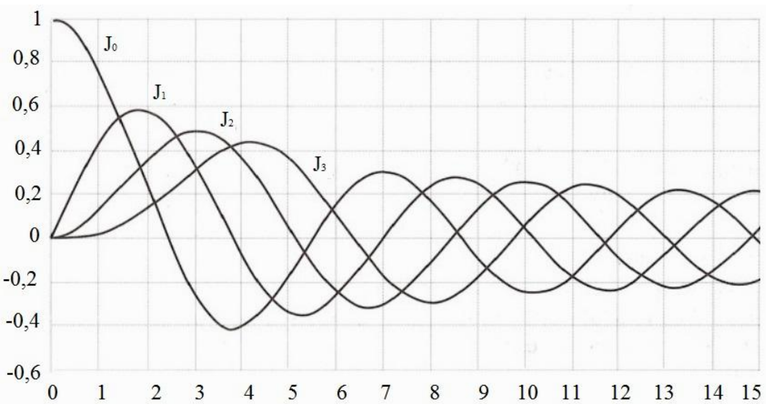
Ora, siccome: $\cos a\cdot\cos b = \frac{1}{2}\cos(a-b) + \frac{1}{2}\cos(a+b)$ e $\sin a\cdot\sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$,

segue che:

$$v(t) = V_P[J_0(m_f)\cos\omega_P t - J_1(m_f)\cos(\omega_P - \omega_m)t + J_1(m_f)\cos(\omega_P + \omega_m)t + J_2(m_f)\cos(\omega_P - 2\omega_m)t +$$
$$+ J_2(m_f)\cos(\omega_P + 2\omega_m)t - J_3(m_f)\cos(\omega_P - 3\omega_m)t + J_3(m_f)\cos(\omega_P + 3\omega_m)t + ...] \quad (1.2)$$

Questa equazione (1.2) ci fornisce lo spettro del segnale modulato; essa contiene la portante e le component laterali, spaziate di ω_m e le ampiezze delle quali sono date dalle funzioni di Bessel, che dipendono da m_f. Siccome, di solito, piccoli m_f (<0,5) danno piccolo valori delle funzioni di Bessel, allora solo le righe laterali a frequenze $f_P - f_m$ e $f_P + f_m$ sono considerate. Siccome $J_0(m_f) = 0$ quando $m_f = 2{,}40$, lì la portante scompare.

Sulla base di tutto ciò, la larghezza di banda è approssimativamente $B = 2(\Delta f + f_m)$.



Le espressioni per le Funzioni di Bessel $J_n(x)$ sono date dalla seguente (vedi la (2.2) più sotto):

$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}}{\Gamma(n+k+1)k!}$ e sono rappresentate nella figura qui sopra. Per esempio:

$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + ...$. Nel calcolare il valore di tale $J_0(x)$ sommando termini di una serie, non preoccupatevi, poichè appena k cresce un po', i termini iniziano ad essere trascurabili.

Appendice 1: FUNZIONE GAMMA DI EULERO

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Naturalmente, $\Gamma(1) = 1$. Poi, $\Gamma(n+1) = n\Gamma(n)$; infatti, dopo una integrazione per parti:

$\Gamma(z) = e^{-t} \left. \frac{t^z}{z} \right|_0^\infty + \frac{1}{z} \int_0^\infty e^{-t} t^z dt = 0 + \frac{1}{z} \Gamma(z+1)$ ed iterando la $\Gamma(n+1) = n\Gamma(n)$, otteniamo: $\Gamma(n+1) = n!$.

<u>Formula di Stirling:</u>

$\Gamma(n+1) = \int_0^\infty x^n e^{-x} dx = \int_0^\infty e^{n \ln x - x} dx$ e la funzione $n \ln x - x$ ha un massimo in x=n, come si può vedere ponendo a zero la sua derivata. Ora, ponendo x=n+y, abbiamo:

$$\Gamma(n+1) = e^{-n} \int_{-n}^\infty e^{n \ln(n+y) - y} dy = e^{-n} \int_{-n}^\infty e^{n \ln n + n \ln(1 + y/n) - y} dy = n^n e^{-n} \int_{-n}^\infty e^{n \ln(1 + y/n) - y} dy$$

Ora, in accordo con Taylor, abbiamo che: $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - ....$; poniamo dapprima x=y/n e poi $y = \sqrt{n}v$, così: $\Gamma(n+1) = n^n e^{-n} \int_{-n}^\infty e^{n \ln(1 + y/n) - y} dy = n^n e^{-n} \int_{-n}^\infty e^{-y^2/2n + y^3/3n^2 - ......} dy =$

$= n^n e^{-n} \sqrt{n} \int_{-\sqrt{n}}^\infty e^{-v^2/2 + v^3/3\sqrt{n} - ......} dv$,e se n è grande abbastanza: $\Gamma(n+1) = \sqrt{2\pi n} n^n e^{-n}$ (Formula di Stirling), siccome l'integrale di Gauss valeva $\sqrt{2\pi}$.

Appendice 2: EQUAZIONE DIFFERENZIALE DI BESSEL

Quando vi imbattete in un'equazione differenziale come questa:

$$x^2 y'' + xy' + (x^2 - n^2) y = 0 \tag{2.1}$$

avete una Equazione Differenziale di Bessel.
Se cerchiamo una soluzione in forma di serie $J_n(x)$, allora l'equazione differenziale diventa:

$x^2 J''_n + xJ'_n + (x^2 - n^2) J_n = 0$ e la soluzione in forma di serie è: $J_n(x) = \sum_{k=0}^\infty \frac{(-1)^k (x/2)^{n+2k}}{\Gamma(n+k+1)k!}$, (2.2)

dove Γ è la Funzione Gama di Eulero. Infatti, se inserite la (2.2) nella (2.1), otterrete 0=0:

$y = \sum_{k=0}^\infty \frac{(-1)^k (x)^{n+2k}}{(2)^{n+2k} (n+k)! k!}$, $y' = \sum_{k=0}^\infty \frac{(-1)^k (n+2k)(x)^{n+2k-1}}{(2)^{n+2k} (n+k)! k!}$, $y'' = \sum_{k=0}^\infty \frac{(-1)^k (n+2k)(n+2k-1)(x)^{n+2k-2}}{(2)^{n+2k} (n+k)! k!}$

$(x^2 - n^2) y = \sum_{k=0}^\infty \frac{(-1)^k (x)^{n+2k+2}}{(2)^{n+2k} (n+k)! k!} - \sum_{k=0}^\infty \frac{(-1)^k n^2 (x)^{n+2k}}{(2)^{n+2k} (n+k)! k!}$, $xy' = \sum_{k=0}^\infty \frac{(-1)^k (n+2k)(x)^{n+2k}}{(2)^{n+2k} (n+k)! k!}$,

$x^2 y'' = \sum_{k=0}^\infty \frac{(-1)^k (n+2k)(n+2k-1)(x)^{n+2k}}{(2)^{n+2k} (n+k)! k!}$

$$x^2y''+xy'+(x^2-n^2)y=\sum_{k=0}^{\infty}\frac{(-1)^k(n+2k)(n+2k-1)(x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}+\sum_{k=0}^{\infty}\frac{(-1)^k(n+2k)(x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}+$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k(x)^{n+2k+2}}{(2)^{n+2k}(n+k)!k!}-\sum_{k=0}^{\infty}\frac{(-1)^k n^2(x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}=\sum_{k=0}^{\infty}\frac{(-1)^k(x)^{n+2k+2}}{(2)^{n+2k}(n+k)!k!}+\sum_{k=0}^{\infty}\frac{(-1)^k[4k(k+n)](x)^{n+2k}}{(2)^{n+2k}(n+k)!k!}=$$

$$=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}(x)^{n+2k}}{(2)^{n+2k-2}(n+k-1)!(k-1)!}+\sum_{k=0}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}\dfrac{(n+k)!}{(n+k)}\dfrac{k!}{k}}=-\sum_{k=1}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}+$$

$$+\sum_{k=0}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}$$, così:

$$x^2y''+xy'+(x^2-n^2)y=-\sum_{k=1}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}+\sum_{k=0}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}$$, ma possiamo anche dire che:

$$x^2y''+xy'+(x^2-n^2)y=-\sum_{k=1}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}+\sum_{k=1}^{\infty}\frac{(-1)^k 4(x)^{n+2k}}{(2)^{n+2k}(n+k-1)!(k-1)!}=0$$, dove anche l'ultima sommatoria può iniziare da k=1, visto che il suo termine in k=0 ha nel suo denominatore: $(k-1)!=\Gamma(k)=(-1)!=\Gamma(0)=\infty$.

Vediamo che: $J_{-n}(x)=(-1)^n J_n(x)$; infatti:

$$J_{-n}(x)=\sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)}=\sum_{r=0}^{n-1}\frac{(-1)^r(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)}+\sum_{r=n}^{\infty}\frac{(-1)^r(x/2)^{-n+2r}}{r!\Gamma(-n+r+1)}$$

ma essendo che $\Gamma(-n+r+1)=\infty$ per ogni r = 0,1,…..,n-1, la prima sommatoria è zero. Inoltre, nella seconda sommatoria poniamo r = n+k , sicchè:

$$\sum_{k=0}^{\infty}\frac{(-1)^{n+k}(x/2)^{n+2k}}{(n+k)!\Gamma(k+1)}=(-1)^n\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{n+2k}}{\Gamma(n+k+1)k!}=(-1)^n J_n(x)$$, cvd.

I denominatori in entrambe le sommatorie sono uguali, visto che: $\Gamma(n+1)=n\Gamma(n)$.

Abbiamo anche che: $J'_n(x)J_{-n}(x)-J'_{-n}(x)J_n(x)=\dfrac{2}{x\Gamma(n)\Gamma(1-n)}$ (2.3)

Infatti, siccome entrambi $J_n(x)$ e $J_{-n}(x)$ soddisfano l'Equazione di Bessel, allora:

$x^2J''_n+xJ'_n+(x^2-n^2)J_n=0$ e $x^2J''_{-n}+xJ'_{-n}+(x^2-n^2)J_{-n}=0$ e moltiplicando la prima per J_{-n} e la seconda per J_n e, infine, dopo aver sottratto membro a membro, otteniamo:

$x^2(J''_n J_{-n}-J''_{-n}J_n)+x(J'_n J_{-n}-J'_{-n}J_n)=0$, o:

$x\dfrac{d}{dx}(J'_n J_{-n}-J'_{-n}J_n)+(J'_n J_{-n}-J'_{-n}J_n)=0$, o ancora: $\dfrac{d}{dx}[x(J'_n J_{-n}-J'_{-n}J_n)]=0$ $\rightarrow$

$J'_n J_{-n}-J'_{-n}J_n=\dfrac{c}{x}$. (2.4)

Ora, nel calcolare c, ricordiamo che: $J_n=\dfrac{x^n}{2^n\Gamma(n+1)}-.....$, $J'_n=\dfrac{x^{n-1}}{2^n\Gamma(n)}-.....$,

$J'_{-n}=\dfrac{x^{-n-1}}{2^{-n}\Gamma(-n)}-.....$, $J_{-n}=\dfrac{x^{-n}}{2^{-n}\Gamma(-n+1)}-.....$, e tutte queste dentro la (2.4) danno:

$(((.......\Gamma(n+1)=n\Gamma(n).......)))$

$$c=\frac{1}{\Gamma(n)\Gamma(1-n)}-\frac{1}{\Gamma(n+1)\Gamma(-n)}=\frac{1}{\Gamma(n)\Gamma(1-n)}-\frac{1}{[n\Gamma(n)]\Gamma(-n)}=\frac{1}{\Gamma(n)\Gamma(1-n)}-\frac{1}{\Gamma(n)[n\Gamma(-n)]}$$
$$=\frac{1}{\Gamma(n)\Gamma(1-n)}-\frac{-1}{\Gamma(n)[-n\Gamma(-n)]}=\frac{1}{\Gamma(n)\Gamma(1-n)}+\frac{1}{\Gamma(n)\Gamma(1-n)}=\frac{2}{\Gamma(n)\Gamma(1-n)}$$, dunque:

$c=\dfrac{2}{\Gamma(n)\Gamma(1-n)}$. Così: $J'_n J_{-n}-J'_{-n}J_n=\dfrac{2}{x\Gamma(n)\Gamma(1-n)}$.

Appendice 3: FORMULA DI RIFLESSIONE

Eccola: $\dfrac{2}{\Gamma(n)\Gamma(1-n)}=\dfrac{2\sin n\pi}{\pi}$. (3.1)

Infatti, sappiamo che $\dfrac{\sin x}{x}$ vale zero in $-\pi,+\pi,-2\pi,+2\pi,....,-n\pi,+n\pi$, sicchè:

$\dfrac{\sin x}{x}=(1-\dfrac{x}{\pi})(1+\dfrac{x}{\pi})(1-\dfrac{x}{2\pi})(1+\dfrac{x}{2\pi}).........=(1-\dfrac{x^2}{\pi^2})(1-\dfrac{x^2}{4\pi^2}).....=\prod_{k=1}^{\infty}[1-(\dfrac{x}{k\pi})^2]$, e dunque:

$$\frac{\pi x}{\sin\pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1} \qquad (3.2)$$

Inoltre, un fattoriale possiamo anche vederlo così:

$n!=\lim\limits_{k\to\infty}\dfrac{k!k^n}{(n+1)...(n+k)}\cdot\lim\limits_{k\to\infty}\dfrac{(k+1)(k+2)...(k+n)}{k^n}$; infatti, il denominatore del primo limite è:

$(n+1)...(n+k)=\dfrac{(n+k)!}{n!}$, mentre da entrambi i numeratori possiamo raccogliere:

$k!(k+1)...(k+n)=(k+n)!$, così resta solo n! . Poi, vediamo che il secondo limite vale 1:

$\lim\limits_{k\to\infty}\dfrac{(k+1)(k+2)...(k+n)}{k^n}=1$, poichè si riduce alla forma $\lim\limits_{k\to\infty}\dfrac{k^n}{k^n}$. Perciò: $n!=\lim\limits_{k\to\infty}\dfrac{k!k^n}{(n+1)...(n+k)}$.

Siccome $\Gamma(x+1)=x!$, si ha: $\Gamma(x+1)=x!=\lim\limits_{k\to\infty}\dfrac{k!k^x}{(x+1)...(x+k)}$ e dopo aver diviso sia numeratore che denominatore per k!:

$$\Gamma(x+1)=\lim_{k\to\infty}\frac{k^x}{(1+x)(1+x/2)...(1+x/k)} \quad . \qquad (3.3)$$

Ora, introduciamo la costante $\gamma_k=1+\dfrac{1}{2}+\dfrac{1}{3}+...+\dfrac{1}{k}-\ln k$, e poniamo: $\gamma=\lim\limits_{k\to\infty}\gamma_k$; sulla base di ciò, la (3.3) diventa: $\Gamma(x+1)=e^{-\gamma x}\lim\limits_{k\to\infty}\dfrac{e^x}{(1+x)}\dfrac{e^{x/2}}{(1+x/2)}....\dfrac{e^{x/k}}{(1+x/k)}$. (3.4)

Infatti, tutti gli e^x, $e^{x/2}$,…, $e^{x/k}$ si elidono con i termini in $e^{-\gamma x}$ ed il termine $k^x=e^{(\ln k)x}$ pure proviene da $e^{-\gamma x}$. Inoltre, siccome $\Gamma(x+1)=x\Gamma(x)$, la (3.4) diventa:

$$\frac{1}{\Gamma(x)}=xe^{\gamma x}\lim_{k\to\infty}\frac{(1+x)}{e^x}\frac{(1+x/2)}{e^{x/2}}....\frac{(1+x/k)}{e^{x/k}}=xe^{\gamma x}\prod_{k=1}^{\infty}[1+(\frac{x}{k})]e^{-x/k} \qquad (3.5)$$

Ora, poichè $\Gamma(y+1)=y\Gamma(y)$, segue che $\Gamma(y)=\dfrac{1}{y}\Gamma(y+1)$ e dopo aver sostituito y con –x:

$\Gamma(-x)=-\dfrac{1}{x}\Gamma(1-x)$ e dunque: $\Gamma(1-x)=-x\Gamma(-x)$. Valutiamo $\Gamma(x)\Gamma(1-x)$:

$$\Gamma(x)\Gamma(1-x)=[x^{-1}e^{-\gamma x}\prod_{k=1}^{\infty}(1+\frac{x}{k})^{-1}e^{x/k}]\cdot[e^{\gamma x}\prod_{k=1}^{\infty}(1-\frac{x}{k})^{-1}e^{-x/k}]=[A]\cdot[B]=\frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}\,, \qquad (3.6)$$

dove A è la (3.5) invertita e B è la (3.4) con x scambiato con –x. Perciò, la (3.6) è:

$\Gamma(x)\Gamma(1-x)=\frac{1}{x}\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$, e per la (3.2) (($\frac{\pi x}{\sin \pi x}=\prod_{k=1}^{\infty}[1-(\frac{x}{k})^2]^{-1}$)) abbiamo:

$\frac{2}{\Gamma(n)\Gamma(1-n)}=\frac{2\sin n\pi}{\pi}$ che è proprio la (3.1).

Appendice 4: SERIE DI BESSEL

Adesso, siccome, in accordo con la (2.3): $J'_n(x)J_{-n}(x)-J'_{-n}(x)J_n(x)=\frac{2}{x\Gamma(n)\Gamma(1-n)}$, usando in essa la (3.1), otteniamo: $J'_n(x)J_{-n}(x)-J'_{-n}(x)J_n(x)=\frac{2\sin n\pi}{\pi x}$. (4.1)

Vale la seguente eguaglianza: $e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}J_n(x)t^n$; (4.2)

infatti: $e^{\frac{x}{2}(t-\frac{1}{t})}=e^{\frac{xt}{2}}e^{-\frac{x}{2t}}=[\sum_{r=0}^{\infty}\frac{(xt/2)^r}{r!}][\sum_{k=0}^{\infty}\frac{(-x/2t)^k}{k!}]=\sum_{r=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{(r+k)}t^{(r-k)}}{r!k!}$

e definendo n=r-k, segue che n andrà da $-\infty$ a $+\infty$, così:

$$e^{\frac{x}{2}(t-\frac{1}{t})}=\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{(n+2k)}t^n}{(n+k)!k!}=\sum_{n=-\infty}^{\infty}[\sum_{k=0}^{\infty}\frac{(-1)^k(x/2)^{(n+2k)}}{k!(n+k)!}]t^n=\sum_{n=-\infty}^{\infty}J_n(x)t^n \text{ cvd.}$$

Valgono le seguenti eguaglianze (**<u>Serie di Bessel</u>**):

$$\cos(x\sin\theta)=J_0(x)+2J_2(x)\cos 2\theta+2J_4(x)\cos 4\theta+...... \qquad (4.3)$$
$$\sin(x\sin\theta)=2J_1(x)\sin\theta+2J_3(x)\sin 3\theta+2J_5(x)\sin 5\theta+...... \qquad (4.4)$$

Infatti, se poniamo $t=e^{i\theta}$ nella (4.2), otteniamo:

$$e^{\frac{1}{2}x(e^{i\theta}-e^{-i\theta})}=e^{ix\sin x}=\sum_{-\infty}^{\infty}J_n(x)e^{in\theta}=$$

$$=\sum_{-\infty}^{\infty}J_n(x)[\cos n\theta+i\sin n\theta]=\{J_0(x)+[J_{-1}(x)+J_1(x)]\cos\theta+[J_{-2}(x)+J_2(x)]\cos 2\theta+....\}+$$
$$+\mathrm{i}\{[J_1(x)-J_{-1}(x)]\sin\theta+[J_2(x)-J_{-2}(x)]\sin 2\theta+....\}=$$
$=\{J_0(x)+2J_2(x)\cos 2\theta+....\}+\mathrm{i}\{2J_1(x)\sin\theta+2J_3(x)\sin 3\theta+....\}$, ed eguagliando parte reale e parte immaginaria appena ottenute con quelle scaturenti da $e^{ix\sin x}$, otteniamo appunto le (4.3) e (4.4) .