**Tutorial: The Galilean Transformations' Conflict with Electrodynamics, and its Resolution Using the Four-Potentials of Constant-Velocity Point Charges**

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**Abstract** Acceleration is invariant under the Galilean transformations, which implies that a system moving at a nonzero constant velocity doesn't undergo acceleration it isn't already subject to when it is at rest. However a charged particle moving at a nonzero constant velocity in a static magnetic field undergoes acceleration it isn't subject to when it is at rest in that field (Faraday's Law or the Lorentz Force Law), and the needle of a magnetic compass moving at a nonzero constant velocity in a static electric field undergoes deflection it isn't subject to when it is at rest in that field (Maxwell's Law). The Galilean transformations therefore conflict with electrodynamics, and must be modified. Einstein obtained the modified Galilean transformations via postulating that the speed of light in empty space is fixed at the value $c$, which in fact is a consequence of electrodynamics rather than a postulate. Here we instead read off the space part of a modified constant-velocity Galilean transformation from the four-potential of a point charge moving at that constant velocity; its time part then follows from its space part plus either relativistic reciprocity (a fundamental property of the unmodified Galilean transformations) or the fixed speed $c$ of light.

## 1. Invariance of acceleration under Galilean transformations conflicts with electrodynamics

The time $t$ and space $\mathbf{r}$ Galilean transformation of a system *due to its travel at a constant velocity* $\mathbf{v}$ is,

$$t' = t \quad \text{and} \quad \mathbf{r}' = \mathbf{r} - \mathbf{v}t, \tag{1.1a}$$

which implies that transformation's effect on the system's velocity $d\mathbf{r}/dt$ *is to merely subtract* $\mathbf{v}$ *from* $d\mathbf{r}/dt$,

$$d\mathbf{r}'/dt' = d(\mathbf{r} - \mathbf{v}t)/dt = d\mathbf{r}/dt - \mathbf{v}, \tag{1.1b}$$

and that transformation *leaves the system's acceleration* $d^2\mathbf{r}/dt^2$ *invariant,*

$$d^2\mathbf{r}'/d(t')^2 = d(d\mathbf{r}'/dt')/dt' = d(d\mathbf{r}/dt - \mathbf{v})/dt = d^2\mathbf{r}/dt^2. \tag{1.1c}$$

However Faraday's Law or the magnetic-field Lorentz Force Law,

$$\nabla \times \mathbf{E} = -(1/c)(d\mathbf{B}/dt) \quad \text{or} \quad \mathbf{F} = q(\mathbf{v}/c) \times \mathbf{B}, \tag{1.2a}$$

implies that a charged particle *moving at nonzero constant velocity* in a static magnetic field *undergoes acceleration* which *it isn't subject to* when *that particle is at rest* in that field. Furthermore, Maxwell's Law,

$$\nabla \times \mathbf{B} = (1/c)(d\mathbf{E}/dt), \tag{1.2b}$$

implies that the needle of a magnetic compass *moving at nonzero constant velocity* in a static electric field *undergoes deflection* which *it isn't subject to* when *that compass is at rest* in that field.

Faraday's Law is the dynamical electromagnetic principle which underlies the functioning of electric generators, and the Biot-Savart-Maxwell Law is the dynamical electromagnetic principle which underlies the functioning of electric motors, so the *violation* of those two Laws of electrodynamics by the Eq. (1.1a) constant-velocity-$\mathbf{v}$ Galilean transformation makes *modification* of that transformation *imperative.*

Einstein obtained the *modified* constant-velocity-$\mathbf{v}$ Galilean transformation via *postulating* that the speed of light in empty space *is fixed at the value c* (which in fact *is a consequence of electrodynamics* rather than a postulate), *and keeping those features of the* Eq. (1.1a) *unmodified Galilean transformation which are compatible with that postulate.*

Here *we instead study the scalar and vector potential pair* $(\phi^q_{\mathbf{v}}(\mathbf{r},t), \mathbf{A}^q_{\mathbf{v}}(\mathbf{r},t))$ *which are produced by a charge-q point charge moving at the transformation's constant velocity* $\mathbf{v}$. The charge-density/current-density pair $\rho^q_{\mathbf{v}}(\mathbf{r},t)/\mathbf{j}^q_{\mathbf{v}}(\mathbf{r},t)$ of such a charge-$q$ point charge moving at constant velocity $\mathbf{v}$ is,

$$\rho^q_{\mathbf{v}}(\mathbf{r},t)/\mathbf{j}^q_{\mathbf{v}}(\mathbf{r},t) = (q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t))/(\mathbf{v}q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t)). \tag{1.3}$$

In Section **2** we obtain *the decoupled pair of equations* for the scalar and vector potential pair $(\phi(\mathbf{r},t), \mathbf{A}(\mathbf{r},t))$ which are produced by *any* charge-density/current-density pair $\rho(\mathbf{r},t)/\mathbf{j}(\mathbf{r},t)$ *that locally conserves charge by satisfying the equation of continuity,* $d\rho/dt + \nabla\cdot\mathbf{j} = 0$. In Section **3** we use a specialized Fourier representation *to solve those equations for* $(\phi^q_{\mathbf{v}}(\mathbf{r},t), \mathbf{A}^q_{\mathbf{v}}(\mathbf{r},t))$ *in the special case that their charge-density/current-density pair* $\rho^q_{\mathbf{v}}(\mathbf{r},t)/\mathbf{j}^q_{\mathbf{v}}(\mathbf{r},t)$ *is that given by* Eq. (1.3) *for a charge-q point charge moving at constant velocity* $\mathbf{v}$.

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## 2. The equations for the four-potentials of locally conserved charge and current densities

In addition to the *dynamical* Faraday's Law, $\nabla \times \mathbf{E} = -(1/c)(d\mathbf{B}/dt)$, and Biot-Savart-Maxwell Law, $\nabla \times \mathbf{B} = (1/c)(4\pi\mathbf{j} + d\mathbf{E}/dt)$, the $\mathbf{E}$ and $\mathbf{B}$ fields are governed by the *non-dynamical* Coulomb's Law,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{2.1a}$$

and Gauss' Law,

$$\nabla \cdot \mathbf{B} = 0. \tag{2.1b}$$

Gauss' Law implies that the $\mathbf{B}$ field can be expressed as,

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{2.2a}$$

where the vector potential $\mathbf{A}(\mathbf{r}, t)$ *is determined only up to the addition of the gradient of an arbitrary scalar function* $\chi(\mathbf{r}, t)$ *because,*

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \textit{also implies that} \quad \mathbf{B} = \nabla \times (\mathbf{A} + \nabla\chi). \tag{2.2b}$$

Insertion of $\mathbf{B} = \nabla \times \mathbf{A}$ into Faraday's Law, $\nabla \times \mathbf{E} = -(1/c)(d\mathbf{B}/dt)$, yields $\nabla \times (\mathbf{E} + (1/c)(d\mathbf{A}/dt)) = \mathbf{0}$, which implies that $\mathbf{E} + (1/c)(d\mathbf{A}/dt) = -\nabla\phi$, where $\phi(\mathbf{r}, t)$ *is the scalar potential.* Therefore,

$$\mathbf{E} = -\nabla\phi - (1/c)(d\mathbf{A}/dt). \tag{2.2c}$$

Notice, however, that Eq. (2.2a), i.e., $\mathbf{B} = \nabla \times \mathbf{A}$, and Eq. (2.2c), i.e., $\mathbf{E} = -\nabla\phi - (1/c)(d\mathbf{A}/dt)$, *fail to uniquely determine the four-potential* $(\phi(\mathbf{r}, t), \mathbf{A}(\mathbf{r}, t))$ *because it is also true that,*

$$\mathbf{B} = \nabla \times (\mathbf{A} + \nabla\chi) \text{ and } \mathbf{E} = -\nabla(\phi - (1/c)(d\chi/dt)) - (1/c)(d(\mathbf{A} + \nabla\chi)/dt), \tag{2.2d}$$

where $\chi(\mathbf{r}, t)$ is an arbitrary scalar function. In other words, the four-potential $(\phi, \mathbf{A})$ isn't unique *because* $(\phi', \mathbf{A}')$, *where* $\phi' = \phi - (1/c)(d\chi/dt)$ *and* $\mathbf{A}' = \mathbf{A} + \nabla\chi$, $\chi(\mathbf{r}, t)$ *being an arbitrary scalar function, also satisfies* $\nabla \times \mathbf{A}' = \mathbf{B}$ *and* $-\nabla\phi' - (1/c)(d\mathbf{A}'/dt) = \mathbf{E}$. This scalar "gauge ambiguity" of the four-potential $(\phi, \mathbf{A})$ will enable us *to advantageously simplify the results of inserting* $\mathbf{B} = \nabla \times \mathbf{A}$ *and* $\mathbf{E} = -\nabla\phi - (1/c)(d\mathbf{A}/dt)$ *into the Biot-Savart-Maxwell Law,* $\nabla \times \mathbf{B} = (1/c)(4\pi\mathbf{j} + (d\mathbf{E}/dt))$, *and into Coulomb's Law,* $\nabla \cdot \mathbf{E} = 4\pi\rho$. Carrying out those two insertions produces,

$$\nabla \times (\nabla \times \mathbf{A}) = 4\pi(\mathbf{j}/c) - \nabla((1/c)(d\phi/dt)) - (1/c)^2(d^2\mathbf{A}/dt^2) \text{ and } -\nabla^2\phi = 4\pi\rho + (1/c)(d(\nabla \cdot \mathbf{A})/dt), \tag{2.3a}$$

which, after noting that $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, is readily algebraically manipulated to read,

$$(1/c)^2(d^2\phi/dt^2) - \nabla^2\phi = 4\pi\rho + (1/c)(d((1/c)(d\phi/dt) + \nabla \cdot \mathbf{A})/dt) \quad \text{and}$$
$$(1/c)^2(d^2\mathbf{A}/dt^2) - \nabla^2\mathbf{A} = 4\pi(\mathbf{j}/c) - \nabla((1/c)(d\phi/dt) + \nabla \cdot \mathbf{A}). \tag{2.3b}$$

On the basis of the scalar "gauge ambiguity" of the scalar and vector potential pair $(\phi, \mathbf{A})$ we are now permitted to *stipulate* that it satisfies the scalar "Lorentz condition" equation,

$$(1/c)(d\phi/dt) + \nabla \cdot \mathbf{A} = 0, \tag{2.3c}$$

which when inserted into the two equations of Eq. (2.3b) *simplifies* them into the following *decoupled pair of equations* for the scalar and vector potential pair $(\phi, \mathbf{A})$, which is called *the four-potential,*

$$(1/c)^2(d^2\phi/dt^2) - \nabla^2\phi = 4\pi\rho \quad \text{and} \quad (1/c)^2(d^2\mathbf{A}/dt^2) - \nabla^2\mathbf{A} = 4\pi(\mathbf{j}/c). \tag{2.3d}$$

For Eqs. (2.3c) and (2.3d) to *both* hold, the charge-density/current-density pair $\rho(\mathbf{r}, t)/\mathbf{j}(\mathbf{r}, t)$ of Eq. (2.3d) *is obliged to satisfy the equation of continuity,*

$$d\rho/dt + \nabla \cdot \mathbf{j} = 0, \tag{2.3e}$$

*which ensures that charge is locally conserved.* For *physically sensible* source functions $\rho$ and $\mathbf{j}$ which *satisfy* the equation of continuity given by Eq. (2.3e), attention can be focused *solely* on solving the Eq. (2.3d) decoupled equations for the four-potential $(\phi, \mathbf{A})$. We next undertake solving the Eq. (2.3d) decoupled equations for the four-potential $(\phi, \mathbf{A})$ *when the charge-density/current-density pair* $\rho(\mathbf{r}, t)/\mathbf{j}(\mathbf{r}, t)$ *is* $(q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t))/(\mathbf{v}q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t))$ *of* Eq. (1.3) *for a charge-q point charge moving at constant velocity* $\mathbf{v}$.

## 3. The constant-velocity point-charge's four-potential and contracted four-potential

Before we undertake solving Eq. (2.3d) in the special case of the Eq. (1.3) charge-q point charge moving at constant velocity $\mathbf{v}$, for which $\rho^q_{\mathbf{v}}(\mathbf{r},t)/\mathbf{j}^q_{\mathbf{v}}(\mathbf{r},t)$ is $(q\,\delta^{(3)}(\mathbf{r}-\mathbf{v}t))/(\mathbf{v}q\,\delta^{(3)}(\mathbf{r}-\mathbf{v}t))$, we need to verify that the Eq. (2.3e) local charge-conservation condition,

$$d\rho^q_{\mathbf{v}}/dt + \nabla\cdot\mathbf{j}^q_{\mathbf{v}} = 0, \tag{3.1a}$$

holds. Doing so requires writing out in detail that $\delta^{(3)}(\mathbf{r}-\mathbf{v}t) = \delta(x-v_xt)\delta(y-v_yt)\delta(z-v_zt)$, which implies,

$$\rho^q_{\mathbf{v}}(\mathbf{r},t) = q\,\delta^{(3)}(\mathbf{r}-\mathbf{v}t) = q\,\delta(x-v_xt)\delta(y-v_yt)\delta(z-v_zt), \tag{3.1b}$$

and,

$$\mathbf{j}^q_{\mathbf{v}}(\mathbf{r},t) = \mathbf{v}q\,\delta^{(3)}(\mathbf{r}-\mathbf{v}t) = \mathbf{v}\rho^q_{\mathbf{v}}(\mathbf{r},t) = (v_x,v_y,v_z)q\,\delta(x-v_xt)\delta(y-v_yt)\delta(z-v_zt). \tag{3.1c}$$

Therefore,

$$\begin{aligned} d\rho^q_{\mathbf{v}}/dt = &-qv_x\delta'(x-v_xt)\delta(y-v_yt)\delta(z-v_zt) - qv_y\delta(x-v_xt)\delta'(y-v_yt)\delta(z-v_zt) \\ &-qv_z\delta(x-v_xt)\delta(y-v_yt)\delta'(z-v_zt), \end{aligned} \tag{3.1d}$$

and,

$$\begin{aligned} \nabla\cdot\mathbf{j}^q_{\mathbf{v}} = &\,qv_x\delta'(x-v_xt)\delta(y-v_yt)\delta(z-v_zt) + qv_y\delta(x-v_xt)\delta'(y-v_yt)\delta(z-v_zt) \\ &+qv_z\delta(x-v_xt)\delta(y-v_yt)\delta'(z-v_zt) = -d\rho^q_{\mathbf{v}}/dt, \end{aligned} \tag{3.1e}$$

which implies the Eq. (3.1a) local charge-conservation condition $d\rho^q_{\mathbf{v}}/dt + \nabla\cdot\mathbf{j}^q_{\mathbf{v}} = 0$ indeed holds.

Also, since $\mathbf{j}^q_{\mathbf{v}} = \mathbf{v}\rho^q_{\mathbf{v}}$, as has been noted in Eq. (3.1c), we see from inspection of Eq. (2.3d) that,

$$\mathbf{A}^q_{\mathbf{v}}(\mathbf{r},t) = (\mathbf{v}/c)\phi^q_{\mathbf{v}}(\mathbf{r},t), \tag{3.2a}$$

so we only need to solve the Eq. (2.3d) partial differential equation which pertains to $\phi^q_{\mathbf{v}}(\mathbf{r},t)$, namely,

$$(1/c)^2(d^2\phi^q_{\mathbf{v}}(\mathbf{r},t)/dt^2) - \nabla^2\phi^q_{\mathbf{v}}(\mathbf{r},t) = 4\pi\rho^q_{\mathbf{v}}(\mathbf{r},t) = 4\pi q\delta^{(3)}(\mathbf{r}-\mathbf{v}t). \tag{3.2b}$$

We note that a particularly simple *specialized* Fourier representation of the right side of Eq. (3.2b) is,

$$4\pi q\delta^{(3)}(\mathbf{r}-\mathbf{v}t) = (q/(2\pi^2))\int d^3\mathbf{k}\,\exp(i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)). \tag{3.2c}$$

We now *assume* that the as yet unsolved-for potential $\phi^q_{\mathbf{v}}(\mathbf{r},t)$ on the left side of Eq. (3.2b) *has the same specialized Fourier representation as we have adopted in* Eq. (3.2c) *for the right side of* Eq. (3.2b),

$$\phi^q_{\mathbf{v}}(\mathbf{r},t) = \int d^3\mathbf{k}\,\exp(i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t))\,\widetilde{\phi}^q_{\mathbf{v}}(\mathbf{k}). \tag{3.2d}$$

Inserting Eqs. (3.2d) and (3.2c) into the Eq. (3.2b) partial differential equation for $\phi^q_{\mathbf{v}}(\mathbf{r},t)$ produces,

$$(-(\mathbf{k}\cdot(\mathbf{v}/c))^2 + |\mathbf{k}|^2)\widetilde{\phi}^q_{\mathbf{v}}(\mathbf{k}) = (q/(2\pi^2)), \tag{3.2e}$$

which implies that,

$$\widetilde{\phi}^q_{\mathbf{v}}(\mathbf{k}) = (q/(2\pi^2))/(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2), \tag{3.2f}$$

and therefore from Eq. (3.2d),

$$\phi^q_{\mathbf{v}}(\mathbf{r},t) = (q/(2\pi^2))\int d^3\mathbf{k}\,\exp(i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t))/(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2). \tag{3.2g}$$

When $\mathbf{v}=\mathbf{0}$, so that *the point charge is stationary,* $\phi^q_{\mathbf{v}=\mathbf{0}}(\mathbf{r},t)$ *is independent of the time* $t$,

$$\begin{aligned} \phi^q_{\mathbf{v}=\mathbf{0}}(\mathbf{r}) = (q/(2\pi^2))\int d^3\mathbf{k}\,\exp(i\mathbf{k}\cdot\mathbf{r})/(|\mathbf{k}|^2) = (q/(2\pi^2))\int_0^\infty k^2\,dk\,(2\pi)\int_0^\pi \sin\theta\,d\theta\,\exp(ik|\mathbf{r}|\cos\theta)/(k^2) = \\ (q/\pi)\int_0^\infty dk\,[2\sin(k|\mathbf{r}|)/(k|\mathbf{r}|)] = q(2/\pi)(1/|\mathbf{r}|)\int_0^\infty du\,[\sin(u)/(u)] = q/|\mathbf{r}|, \end{aligned} \tag{3.2h}$$

the familiar *time-independent Coulomb potential* of a charge-$q$ stationary point charge. To evaluate Eq. (3.2g) when $0 < |\mathbf{v}| < c$, we define $r_\parallel$ as $(\mathbf{r}\cdot\mathbf{v})/|\mathbf{v}|$, the *component* of the vector $\mathbf{r}$ *in the direction of* $\mathbf{v}$, and likewise, $k_\parallel \stackrel{\text{def}}{=} (\mathbf{k}\cdot\mathbf{v})/|\mathbf{v}|$. We also define $\mathbf{r}_\parallel$ as $r_\parallel\mathbf{v}/|\mathbf{v}| = (\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2$, the *part* of the vector $\mathbf{r}$ *in the direction of* $\mathbf{v}$, and likewise, $\mathbf{k}_\parallel \stackrel{\text{def}}{=} k_\parallel\mathbf{v}/|\mathbf{v}| = (\mathbf{k}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2$. We as well define $\mathbf{r}_\perp$ as $\mathbf{r}-\mathbf{r}_\parallel = \mathbf{r}-r_\parallel\mathbf{v}/|\mathbf{v}| = \mathbf{r}-(\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2$, the *part* of the vector $\mathbf{r}$ *perpendicular to* $\mathbf{v}$, and likewise, $\mathbf{k}_\perp \stackrel{\text{def}}{=} \mathbf{k} - \mathbf{k}_\| = \mathbf{k} - k_\|\mathbf{v}/|\mathbf{v}| = \mathbf{k} - (\mathbf{k}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2$. The *following identities* now greatly aid the evaluation of Eq. (3.2g),

$$\mathbf{k} = \mathbf{k}_\perp + \mathbf{k}_\|, \quad d^3\mathbf{k} = d^2\mathbf{k}_\perp dk_\|, \quad \mathbf{r} = \mathbf{r}_\perp + \mathbf{r}_\|, \quad \mathbf{k}_\perp\cdot\mathbf{v} = \mathbf{r}_\perp\cdot\mathbf{v} = \mathbf{k}_\perp\cdot\mathbf{r}_\| = \mathbf{k}_\|\cdot\mathbf{r}_\perp = 0,$$
$$\mathbf{k}\cdot\mathbf{r} = \mathbf{k}_\perp\cdot\mathbf{r}_\perp + \mathbf{k}_\|\cdot\mathbf{r}_\| = \mathbf{k}_\perp\cdot\mathbf{r}_\perp + k_\| r_\|, \quad \mathbf{k}\cdot\mathbf{v} = (k_\|)|\mathbf{v}| \quad \text{and} \quad |\mathbf{k}|^2 = |\mathbf{k}_\perp|^2 + (k_\|)^2. \tag{3.3a}$$

Applying the foregoing definitions and identities to Eq. (3.2g), we obtain,

$$\phi^q_{\mathbf{v}}(\mathbf{r},t) = (q/(2\pi^2))\int d^3\mathbf{k}\ \exp(i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t))/(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2) =$$
$$(q/(2\pi^2))\int d^2\mathbf{k}_\perp dk_\|\ \exp[i[\mathbf{k}_\perp\cdot\mathbf{r}_\perp + k_\|(r_\| - |\mathbf{v}|t)]]/[|\mathbf{k}_\perp|^2 + (k_\|)^2(1-|\mathbf{v}/c|^2)] =$$
$$(q/(2\pi^2))\int d^2\mathbf{k}_\perp dk_\|\ \exp[i[\mathbf{k}_\perp\cdot\mathbf{r}_\perp + k_\|(r_\| - |\mathbf{v}|t)]]/[|\mathbf{k}_\perp|^2 + (k_\|/\gamma)^2], \tag{3.3b}$$

where,

$$\gamma \stackrel{\text{def}}{=} 1/\sqrt{1-|\mathbf{v}/c|^2}. \tag{3.3c}$$

We now change the vector variable of integration in Eq. (3.3b) from $\mathbf{k} = (\mathbf{k}_\perp, k_\|)$ to $\mathbf{l} = (\mathbf{l}_\perp, l_\|)$, where $\mathbf{l}_\perp = \mathbf{k}_\perp$ and $l_\| = (k_\|/\gamma)$, which implies that $\mathbf{k}_\perp = \mathbf{l}_\perp$ and $k_\| = \gamma\, l_\|$, so Eq. (3.3b) becomes,

$$\phi^q_{\mathbf{v}}(\mathbf{r},t) = \gamma(q/(2\pi^2))\int d^2\mathbf{l}_\perp dl_\|\ \exp[i[\mathbf{l}_\perp\cdot\mathbf{r}_\perp + l_\|(\gamma(r_\| - |\mathbf{v}|t))]]/[|\mathbf{l}_\perp|^2 + (l_\|)^2] =$$
$$\gamma(q/(2\pi^2))\int d^3\mathbf{l}\ \exp(i\mathbf{l}\cdot(\mathbf{r}_\perp + \gamma(\mathbf{r}_\| - \mathbf{v}t)))/(|\mathbf{l}|^2). \tag{3.3d}$$

Comparing Eq. (3.3d) with Eq. (3.2h), and then noting that $\mathbf{r}_\perp = \mathbf{r} - \mathbf{r}_\|$, yields closed forms for $\phi^q_{\mathbf{v}}(\mathbf{r},t)$,

$$\phi^q_{\mathbf{v}}(\mathbf{r},t) = \gamma\,\phi^q_{\mathbf{v}=\mathbf{0}}(\mathbf{r}_\perp + \gamma(\mathbf{r}_\| - \mathbf{v}t)) = \gamma q/|\mathbf{r}_\perp + \gamma(\mathbf{r}_\| - \mathbf{v}t)| = \gamma q/|\mathbf{r} + (\gamma-1)\mathbf{r}_\| - \gamma\mathbf{v}t|. \tag{3.3e}$$

Since from Eq. (3.2a) the vector potential $\mathbf{A}^q_{\mathbf{v}}(\mathbf{r},t) = (\mathbf{v}/c)\phi^q_{\mathbf{v}}(\mathbf{r},t)$, we obtain from Eq. (3.3e) that *the scalar and vector potential pair* or *four-potential* is,

$$(\phi^q_{\mathbf{v}}(\mathbf{r},t), \mathbf{A}^q_{\mathbf{v}}(\mathbf{r},t)) = q(\gamma, \gamma(\mathbf{v}/c))/|\mathbf{r}_\perp + \gamma(\mathbf{r}_\| - \mathbf{v}t)| = q(\gamma,\gamma(\mathbf{v}/c))/|\mathbf{r} + (\gamma-1)\mathbf{r}_\| - \gamma\mathbf{v}t|. \tag{3.3f}$$

Since from Eq. (3.3c), $\gamma = 1/\sqrt{1-|\mathbf{v}/c|^2}$, Eq. (3.3f) yields the *contracted four-potential* result,

$$\sqrt{(\phi^q_{\mathbf{v}}(\mathbf{r},t))^2 - |\mathbf{A}^q_{\mathbf{v}}(\mathbf{r},t)|^2} = |q|/|\mathbf{r}_\perp + \gamma(\mathbf{r}_\| - \mathbf{v}t)| = |q|/|\mathbf{r} + (\gamma-1)\mathbf{r}_\| - \gamma\mathbf{v}t|, \tag{3.3g}$$

which has *a simpler form* than that *of any of the* Eq. (3.3f) *four-potential's components.* From *this simple form* we read off *the space part* of the *modified* constant-velocity-$\mathbf{v}$ Galilean transformation,

$$\mathbf{r}' \;=\; \mathbf{r}_\perp + \gamma(\mathbf{r}_\| - \mathbf{v}t) \;=\; \mathbf{r} + (\gamma-1)\mathbf{r}_\| - \gamma\mathbf{v}t \;=\; \mathbf{r} + (\gamma-1)(\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t. \tag{3.4}$$

## 4. Resolution of the Galilean transformations' conflict with electrodynamics

Note that when $c\to\infty$, $\gamma\to 1$ and Eq. (3.4) becomes $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, *the space part of the* Eq. (1.1a) *unmodified* constant-velocity-$\mathbf{v}$ Galilean transformation. That is *a necessary property* of the *modified* constant-velocity-$\mathbf{v}$ Galilean transformation because when $c\to\infty$, the electromagnetic Laws become $\nabla\cdot\mathbf{E} = 4\pi\rho$, $\nabla\times\mathbf{E} = \mathbf{0}$, $\nabla\times\mathbf{B} = \mathbf{0}$, $\nabla\cdot\mathbf{B} = 0$ and $\mathbf{F} = q\mathbf{E}$, *which no longer describe the velocity-dependent forces* that *conflict* with the *unmodified* Galilean transformations. We *also* note that the *part* of Eq. (3.4) which is *parallel* to $\mathbf{v}$ is,

$$\mathbf{r}'_\| = \gamma(\mathbf{r}_\| - \mathbf{v}t) \;\text{ or }\; r'_\| = \gamma(r_\| - |\mathbf{v}|t), \tag{4.1}$$

while the *part* of Eq. (3.4) which is *perpendicular* to $\mathbf{v}$ is,

$$\mathbf{r}'_\perp = \mathbf{r}_\perp. \tag{4.2}$$

The Eq. (3.3f) four-potential *led us to* Eqs. (4.1) *and* (4.2), *the space part* of the *modified* constant-velocity-$\mathbf{v}$ Galilean transformation, but that four-potential *doesn't by itself lead us to the time part* of that *modified* Galilean transformation. The Eq. (1.1a) *unmodified* constant-velocity-$\mathbf{v}$ Galilean transformation, $t' = t$ and $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, however, has the attribute *that reversing the sign of* $\mathbf{v}$ *inverts the transformation,* i.e.,

$$t' = t \text{ and } \mathbf{r}' = \mathbf{r} - (-\mathbf{v})t \quad \textit{is equivalent to} \quad t = t' \text{ and } \mathbf{r} = \mathbf{r}' - \mathbf{v}t'. \tag{4.3}$$

This attribute of the Eq. (1.1a) *unmodified* constant-velocity-$\mathbf{v}$ Galilean transformation implies that observing the "rest" system from the "moving" system *is indistinguishable from observing the "moving" system* *from the "rest" system with the sign of the the velocity* $\mathbf{v}$ *of the "moving" system reversed,* an *effective equivalence of the two systems* which we call *their relativistic reciprocity.*

The *modified* constant-velocity-$\mathbf{v}$ Galilean transformation, *whose space part is given by* Eqs. (4.1) *and* (4.2), *also* must be such *that reversing the sign of* $\mathbf{v}$ *inverts that transformation,* in order to ensure the relativistic reciprocity of the two systems it relates. But *combining that with its* Eqs. (4.1) *and* (4.2) *space part* causes its time part *to counterintuitively depend on the velocity-*$\mathbf{v}$ *space component* $r_\parallel$ in addition to depending on time $t$. Consequently the *modified* constant-velocity-$\mathbf{v}$ Galilean transformation has the form,

$$t' = \kappa(t - \lambda(r_\parallel/|\mathbf{v}|)), \quad r'_\parallel = \gamma(r_\parallel - |\mathbf{v}|t) \quad \text{and} \quad \mathbf{r}'_\perp = \mathbf{r}_\perp, \tag{4.4a}$$

where $\kappa$ and $\lambda$ are dimensionless entities *whose values are determined by the requirement that reversing the sign of* $\mathbf{v}$ *inverts the* Eq. (4.4a) *transformation. To be able to proceed,* we tentatively *assume that* $\kappa$ *and* $\lambda$ *are functions of* $\gamma$ *only, which must be confirmed when the values of* $\kappa$ *and* $\lambda$ *have been obtained.* Since,

$$r_\parallel = (\mathbf{r}\cdot\mathbf{v})/|\mathbf{v}| \quad \text{and} \quad \mathbf{r}_\parallel = r_\parallel\mathbf{v}/|\mathbf{v}| = (\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2 \quad \text{and} \quad \mathbf{r}_\perp = \mathbf{r} - \mathbf{r}_\parallel = \mathbf{r} - (\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2, \tag{4.4b}$$

*reversing the sign of* $\mathbf{v}$ reverses the sign of $r_\parallel$, has no effect on $\mathbf{r}_\parallel$ or $\mathbf{r}_\perp$, *and changes* Eq. (4.4a) *to,*

$$t' = \kappa(t + \lambda(r_\parallel/|\mathbf{v}|)), \quad r'_\parallel = \gamma(r_\parallel + |\mathbf{v}|t) \quad \text{and} \quad \mathbf{r}'_\perp = \mathbf{r}_\perp, \tag{4.4c}$$

which we *next* must solve for $t$, $r_\parallel$ and $\mathbf{r}_\perp$ in terms of $t'$, $r'_\parallel$ and $\mathbf{r}'_\perp$. *Then* we must determine *the values of* $\kappa$ *and* $\lambda$ *which make that result the inverse of the* Eq. (4.4a) *transformation.* To solve Eq. (4.4c) for $t$, $r_\parallel$ and $\mathbf{r}_\perp$ in terms of $t'$, $r'_\parallel$ and $\mathbf{r}'_\perp$, *we first rearrange each one of the three equations of* Eq. (4.4c) *as follows,*

$$t = (1/\kappa)t' - \lambda(r_\parallel/|\mathbf{v}|), \quad r_\parallel = (1/\gamma)r'_\parallel - |\mathbf{v}|t \quad \text{and} \quad \mathbf{r}_\perp = \mathbf{r}'_\perp. \tag{4.4d}$$

We then *substitute the first two equations of* Eq. (4.4d) *into each other* to produce,

$$t = (1/\kappa)t' - (\lambda/\gamma)(r'_\parallel/|\mathbf{v}|) + \lambda t, \quad r_\parallel = (1/\gamma)r'_\parallel - (1/\kappa)|\mathbf{v}|t' + \lambda r_\parallel \quad \text{and} \quad \mathbf{r}_\perp = \mathbf{r}'_\perp. \tag{4.4e}$$

The first and second equations of Eq. (4.4e) now readily yield $t$ and $r_\parallel$ respectively in terms of $t'$ and $r'_\parallel$,

$$t = (1/(1-\lambda))[(1/\kappa)t' - (\lambda/\gamma)(r'_\parallel/|\mathbf{v}|)], \quad r_\parallel = (1/(1-\lambda))[(1/\gamma)r'_\parallel - (1/\kappa)|\mathbf{v}|t'] \quad \text{and} \quad \mathbf{r}_\perp = \mathbf{r}'_\perp. \tag{4.4f}$$

For Eq. (4.4f) to be the *inverse* of Eq. (4.4a), $\kappa$ *and* $\lambda$ *must be such that* Eq. (4.4f) *has the form of* Eq. (4.4a) *with* $t'$ *interchanged with* $t$, $r'_\parallel$ *interchanged with* $r_\parallel$ *and* $\mathbf{r}'_\perp$ *interchanged with* $\mathbf{r}_\perp$. Therefore $\kappa$ *and* $\lambda$ *must satisfy the following four equalities,*

$$(1/(1-\lambda))(1/\kappa) = \kappa, \quad (1/(1-\lambda))(\lambda/\gamma) = \kappa\lambda, \quad (1/(1-\lambda))(1/\gamma) = \gamma \quad \text{and} \quad (1/(1-\lambda))(1/\kappa) = \gamma. \tag{4.4g}$$

The third equality of Eq. (4.4g) immediately yields that $(1/(1-\lambda)) = \gamma^2$. Putting this result into the fourth equality of Eq. (4.4g) then yields $\kappa = \gamma$, which, together with $(1/(1-\lambda)) = \gamma^2$ is consistent with *both* the first *and* second equalities of Eq. (4.4g). Below Eq. (4.4a) we tentatively *assumed,* in order to be able to proceed, that $\kappa$ *and* $\lambda$ *are functions of* $\gamma$ *only; the validity of that assumption is now confirmed.* The result $(1/(1-\lambda)) = \gamma^2$ implies that $\lambda = (1 - (1/\gamma^2)) = |\mathbf{v}/c|^2$, since $\gamma = 1/\sqrt{1 - |\mathbf{v}/c|^2}$. We now insert the results $\lambda = |\mathbf{v}/c|^2$ and $\kappa = \gamma$, along with Eq. (4.4b), into the *time part* of the Eq. (4.4a) transformation to obtain,

$$t' = \kappa(t - \lambda(r_\parallel/|\mathbf{v}|)) = \gamma(t - |\mathbf{v}/c|^2(r_\parallel/|\mathbf{v}|)) = \gamma(t - |\mathbf{v}/c|^2((\mathbf{r}\cdot\mathbf{v})/|\mathbf{v}|^2)) = \gamma(t - ((\mathbf{r}\cdot\mathbf{v})/c^2)). \tag{4.4h}$$

The Eq. (4.4a) transformation's *space part* is given by Eqs. (4.1) and (4.2), *which are equivalent to* Eq. (3.4),

$$\mathbf{r}' = \mathbf{r}_\perp + \gamma(\mathbf{r}_\parallel - \mathbf{v}t) = \mathbf{r} + (\gamma - 1)\mathbf{r}_\parallel - \gamma\mathbf{v}t = \mathbf{r} + (\gamma - 1)(\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t. \tag{4.4i}$$

Eqs. (4.4h) and (4.4i) *combined* comprise the Eq. (4.4a) *modified* constant-velocity-$\mathbf{v}$ Galilean transformation,

$$t' = \gamma(t - ((\mathbf{r}\cdot\mathbf{v})/c^2)) \quad \text{and} \quad \mathbf{r}' = \mathbf{r} + (\gamma - 1)(\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t. \tag{4.4j}$$

When $c \to \infty$, $\gamma \to 1$ and the Eq. (4.4j) *modified* constant-velocity-$\mathbf{v}$ Galilean transformation becomes,

$$t' = t \quad \text{and} \quad \mathbf{r}' = \mathbf{r} - \mathbf{v}t, \tag{4.4k}$$

the Eq. (1.1a) *unmodified* Galilean transformation, *which is expected;* see the first paragraph of this section.

A fascinating *highly counterintuitive feature* of the Eq. (4.4j) *modified* transformation is that the evolution of a spherical-shell light-wave front *is completely insensitive to the transformation's constant velocity* $\mathbf{v}$, which of course *is far from the case* for the Eq. (1.1a) *unmodified* constant-velocity-$\mathbf{v}$ Galilean transformation.

The locus of a spherical-shell light-wave front which is centered on $\mathbf{r} = \mathbf{0}$ is $|\mathbf{r}|^2 = (ct)^2$, or $(ct)^2 - |\mathbf{r}|^2 = 0$. We next show that *all* Eq. (4.4j) *modified* constant-velocity-$\mathbf{v}$ Galilean transformations *preserve the quadratic form* $(ct)^2 - |\mathbf{r}|^2$ *regardless of the value of the transformation's constant velocity* $\mathbf{v}$. We show that by *calculating* $(ct')^2 - |\mathbf{r}'|^2$, where $t' = \gamma(t - ((\mathbf{r}\cdot\mathbf{v})/c^2))$ and $\mathbf{r}' = \mathbf{r} + (\gamma-1)(\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t$ in accord with the Eq. (4.4j) *modified* constant-velocity-$\mathbf{v}$ Galilean transformation. Thus,

$$(ct')^2 - |\mathbf{r}'|^2 = (\gamma(ct - ((\mathbf{r}\cdot\mathbf{v})/c)))^2 - |\mathbf{r} + (\gamma-1)(\mathbf{r}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2 - \gamma\mathbf{v}t|^2 =$$

$$(ct)^2[\gamma^2(1 - |\mathbf{v}/c|^2)] - |\mathbf{r}|^2 - 2(\mathbf{r}\cdot\mathbf{v})t[\gamma^2 - \gamma - \gamma(\gamma-1)] - ((\mathbf{r}\cdot\mathbf{v})/|\mathbf{v}|)^2[(\gamma-1)^2 + 2(\gamma-1) - \gamma^2|\mathbf{v}/c|^2] =$$

$$(ct)^2 - |\mathbf{r}|^2 \ \textit{because,}$$

$$[\gamma^2(1 - |\mathbf{v}/c|^2)] = 1, \quad [\gamma^2 - \gamma - \gamma(\gamma-1)] = 0 \quad \textit{and} \quad [(\gamma-1)^2 + 2(\gamma-1) - \gamma^2|\mathbf{v}/c|^2] = 0. \tag{4.4l}$$

Therefore $(ct')^2 - |\mathbf{r}'|^2 = (ct)^2 - |\mathbf{r}|^2$ *regardless of the value of the transformation's constant velocity* $\mathbf{v}$.

Einstein obtained the Eq. (4.4j) *modified* constant-velocity-$\mathbf{v}$ Galilean transformation via *postulating* the *highly counterintuitive* Eq. (4.4l) *independence* of the speed-$c$ evolution of a spherical-shell light-wave front *of the velocity of its observer, and keeping those features of the* Eq. (1.1a) *unmodified Galilean transformation which are compatible with that postulate.*

Einstein's *highly counterintuitive postulate* of the *independence* of the speed $c$ of light in empty space of the velocity of the observer can understandably elicit *skepticism or rejection.* A minority who *rejected* Einstein's postulate *and demanded reinstatement of the* Eq. (1.1a) *unmodified Galilean transformation arose immediately upon the* 1905 *dissemination of Einstein's paper, and exists to this day.*

In fact, the thesis that the speed of light in empty space is $c$ for any observer *is a consequence of electrodynamics rather than merely a postulate.* In empty space the charge and current densities $\rho$ and $\mathbf{j}$ *vanish,* so the Eq. (2.3d) pair of equations governing the four-potential $(\phi, \mathbf{A})$ are *the pure wave equations,*

$$(1/c)^2(d^2\phi/dt^2) - \nabla^2\phi = 0 \quad \text{and} \quad (1/c)^2(d^2\mathbf{A}/dt^2) - \nabla^2\mathbf{A} = \mathbf{0}, \tag{4.5}$$

*which admit only the wave speed c.* Thus the highly counterintuitive fixed-speed-$c$ propagation of light in empty space *is a theorem of electrodynamics rather than merely a postulate.* No matter how highly counterintuitive it is, *a theorem of electrodynamics is far less likely to elicit skepticism or rejection than is a highly counterintuitive mere postulate.* Of *even greater importance,* the Eq. (4.4j) *modified* Galilean transformation *has been experimentally verified to very high accuracy.*

In Eq. (4.4h) the *time part* of the *modified* constant-velocity-$\mathbf{v}$ Galilean transformation was obtained from *its space part* plus *its relativistic reciprocity.* That *time part* can *also* be obtained from *its* Eq. (3.4) *space part,* $\mathbf{r}' = \mathbf{r}_\perp + \gamma(\mathbf{r}_\parallel - \mathbf{v}t)$, plus *the* Eq. (4.4l) *independence of the spherical-shell light-wave front of the transformation's constant velocity* $\mathbf{v}$, i.e., $(ct')^2 - |\mathbf{r}'|^2 = (ct)^2 - |\mathbf{r}|^2$. The *foregoing two equalities* yield,

$$t' = \sqrt{t^2 + (|\mathbf{r}'|^2 - |\mathbf{r}|^2)/c^2} = \sqrt{t^2 + ((\gamma^2-1)|\mathbf{r}_\parallel|^2 - 2\gamma^2(\mathbf{r}_\parallel\cdot\mathbf{v})t + \gamma^2|\mathbf{v}|^2t^2)/c^2} =$$

$$\sqrt{(1 + |\mathbf{v}/c|^2\gamma^2)t^2 - 2\gamma^2((\mathbf{r}\cdot\mathbf{v})/c^2)t + (\gamma^2-1)((\mathbf{r}\cdot\mathbf{v})/(|\mathbf{v}|c))^2} =$$

$$\sqrt{\gamma^2(t^2 - 2((\mathbf{r}\cdot\mathbf{v})/c^2)t + ((\mathbf{r}\cdot\mathbf{v})/c^2)^2)} = \gamma(t - ((\mathbf{r}\cdot\mathbf{v})/c^2)), \tag{4.6}$$

which *agrees* with Eq. (4.4h). *Therefore both the space and time parts of the* Eq. (4.4j) *modified constant-velocity-*$\mathbf{v}$ *Galilean transformation follow from special cases of electrodynamics,* namely the special case of electrodynamics where $\rho(\mathbf{r},t)/\mathbf{j}(\mathbf{r},t)$ is $(q\,\delta^{(3)}(\mathbf{r}-\mathbf{v}t))/(\mathbf{v}q\,\delta^{(3)}(\mathbf{r}-\mathbf{v}t))$ (see Section **3**) plus the special case of electrodynamics where $\rho(\mathbf{r},t)/\mathbf{j}(\mathbf{r},t)$ is $0/\mathbf{0}$ (see the foregoing paragraph containing Eq. (4.5)).

*Electrodynamics alone thus suffices to work out the modified constant-velocity-*$\mathbf{v}$ *Galilean transformation of* Eq. (4.4j), *so the sole independent postulate of Einstein's 1905 paper* "On the Electrodynamics of Moving Bodies" *is his assertion that Maxwell's equations of electrodynamics are valid in all inertial frames of reference,* which he called the "Principle of Relativity". Maxwell had *contrariwise* asserted that applying those equations in an arbitrary inertial frame of reference *requires a Galilean transformation to that inertial frame from the rest frame of the "luminiferous aether",* a *hypothesized* light-wave supporting medium *whose physical existence became implausible after the null result of the 1887 Michelson-Morley experiment.*