About $2^{3/4} F(-1/8, 1/4, 5/4, 1)$

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abstract

We give some formulas related to $2^{3/4} F(-1/8, 1/4, 5/4, 1)$, where F is the Gauss hypergeometric function.

keywords: Integrals, hypergeometric series, number Pi

I. Introduction

Recall that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots \tag{1}$$

the Gauss hypergeometric function is defined by

$$F(a, b, c, x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n \quad , \quad |x| < 1 \tag{2}$$

where

$$(a)_n = a(a+1)(a+2)(a+3)\ldots(a+n-1) \ , \ (a)_0 = 1 \tag{3}$$

the gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} \, dt \quad , \quad x > 0 \tag{4}$$

$$\Gamma(x+1) = x\,\Gamma(x) \tag{5}$$

In this note we give some formulas related to $2^{3/4} F(-1/8, 1/4, 5/4, 1)$.

II. Formulas

Entry 1.

$$\int_0^{\infty} \frac{1}{\sqrt{1+\sqrt{1+x^8}}} \, dx = \frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = 2^{3/4} F(-1/8, 1/4, 5/4, 1) \tag{6}$$

Entry 2.

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \frac{1}{4}\int_0^1 x^{-3/4}(1-x)^{-7/8}(1+x)^{-11/8}\, dx = \frac{1}{4}\int_0^1 x^{-7/8}(1-x)^{-3/4}(2-x)^{-11/8}\, dx \tag{7}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \sqrt[4]{8}\int_0^1 \sqrt[8]{1-x^4}\,dx = \sqrt[4]{8}\int_0^1 \sqrt[4]{1-x^8}\,dx \tag{8}$$

Entry 3.

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \frac{1}{4\sqrt[4]{2}}\int_0^\infty \left(\sqrt[4]{\frac{\cosh x}{(\sinh x)^3}} + \sqrt[4]{\frac{(\sinh x)^5}{(\cosh x)^7}}\right) dx \tag{9}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \frac{1}{2\sqrt{2}}\int_0^\infty \frac{\cosh(2x)}{(\sinh(2x))^{3/4}\cosh x}\,dx \tag{10}$$

Entry 4. for $u \geq 1$, we have

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \int_0^u \frac{1}{\sqrt{1+\sqrt{1+x^8}}}\,dx + \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{-2n}\, u^{-4n-1}}{4n+1} \binom{2n}{n} F\left(\frac{2n+1}{4}, \frac{4n+1}{8}, \frac{4n+9}{8}, -\frac{1}{u^8}\right) \tag{11}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \int_0^u \frac{1}{\sqrt{1+\sqrt{1+x^8}}}\,dx + \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{-2n}\, u}{(4n+1)\left(1+u^8\right)^{(2n+1)/4}} \binom{2n}{n} F\left(\frac{2n+1}{4}, 1, \frac{4n+9}{8}, \frac{1}{1+u^8}\right) \tag{12}$$

Entry 5. for $0 < u \leq 1$, we have

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \int_u^\infty \frac{1}{\sqrt{1+\sqrt{1+x^8}}}\,dx + \sum_{n=0}^{\infty} \binom{2n}{n} 2^{-3n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} F\left(-\frac{k}{2}, \frac{1}{8}, \frac{9}{8}, -u^8\right) \tag{13}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \int_u^\infty \sqrt{\sqrt{1+x^8} - x^4}\,dx + \sum_{n=0}^{\infty} \binom{2n}{n} \frac{2^{-2n}\, u^{4n+1}}{(1-2n)(4n+1)} F\left(\frac{2n-1}{4}, \frac{4n+1}{8}, \frac{4n+9}{8}, -u^8\right) \tag{14}$$

Entry 6. for $u \geq 0$, we have

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \frac{u}{\sqrt{1+\sqrt{1+u^8}}} + \frac{2\sqrt[4]{8}}{9}\left(\frac{\sqrt{1+u^8}-1}{\sqrt{1+u^8}+1}\right)^{9/8} F\left(\frac{3}{4}, \frac{9}{8}, \frac{17}{8}, \frac{\sqrt{1+u^8}-1}{\sqrt{1+u^8}+1}\right) + \int_u^\infty \frac{1}{\sqrt{1+\sqrt{1+x^8}}}\,dx \tag{15}$$

Entry 7.

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \int_0^\infty \frac{1}{\sqrt{x^4+\sqrt{1+x^8}}}\,dx = \int_0^\infty \sqrt{\sqrt{1+x^8} - x^4}\,dx \tag{16}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = 2\sqrt{2}\int_0^1 \frac{\sqrt[4]{x\left(1+x^2\right)}}{\sqrt{1-x^2}\ (1+x)^2}\,dx \tag{17}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \frac{1}{4}\int_0^\infty \frac{e^{-x/2}\cosh x}{(\sinh x)^{3/4}}\,dx \tag{18}$$

Entry 8. for $0 \leq u \leq 1$, we have

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = 2^{3/4}\, u^{1/4} F\left(-\frac{1}{8}, \frac{1}{4}, \frac{5}{4}, u\right) + \frac{2\cdot 2^{3/4}}{9}(1-u)^{9/8} F\left(\frac{3}{4}, \frac{9}{8}, \frac{17}{8}, 1-u\right) \tag{19}$$

Entry 9. for $u \geq 1$, we have

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} =$$

$$\int_0^u \frac{1}{\sqrt{1+\sqrt{1+x^8}}}\,dx + \left(\frac{2}{u^4+\sqrt{1+u^8}}\right)^{1/4} \sum_{n=0}^{\infty}\left(-\frac{1}{u^4+\sqrt{1+u^8}}\right)^n \left(\frac{1}{4n+1}+\frac{\left(u^4+\sqrt{1+u^8}\right)^{-2}}{4n+9}\right)\sum_{k=0}^{[n/2]}\frac{(3/4)_k}{k!} \tag{20}$$

Entry 10.

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \int_0^1 \frac{1}{\sqrt{x^4+\sqrt{x^8+(1-x)^8}}}\,dx = \int_0^1 \frac{1}{\sqrt{(1-x)^4+\sqrt{x^8+(1-x)^8}}}\,dx \tag{21}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \int_0^\infty \frac{x^2}{\sqrt{x^4+\sqrt{x^8+\left(x^2-1\right)^8}}}\,dx \tag{22}$$

Entry 11.

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \frac{1}{\sqrt[4]{2}}\int_0^1 \sqrt[4]{\frac{1}{x^2}-x^2}\,dx \tag{23}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \int_0^\infty e^{-x}\sqrt[4]{\sinh(2x)}\,dx = \sqrt[4]{2}\int_0^\infty\left(\sqrt[4]{\sinh x\,(\cosh x)^5} - \sqrt[4]{\cosh x\,(\sinh x)^5}\right)dx \tag{24}$$

Endnote

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = 2\cdot 2^{3/8}\int_0^1\left(\sqrt[8]{1+\sqrt{1-x^8}} - \sqrt[8]{1-\sqrt{1-x^8}}\right)dx \tag{25}$$

$$\frac{(\Gamma(1/4))^2}{6\sqrt{2-\sqrt{2}}\ \sqrt{\pi}} = \frac{1}{2^{5/4}}\int_0^\infty\left(\left(\frac{2}{3}\right)^{1/3}\left(9x^4+\sqrt{3}\ \sqrt{27x^8-4}\right)^{-1/3} + \frac{1}{2^{1/3}\cdot 3^{2/3}}\left(9x^4+\sqrt{3}\ \sqrt{27x^8-4}\right)^{1/3}\right)^{-1}dx \tag{26}$$

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