Relation between Mass and Angular Momentum for a Covariant Charge Distribution

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Abstract

A relation is found between the proper time derivative of mass and angular momentum for a covariant charge distribution. This is based on the rest frame equations of motion of a relativistic rotating charge distribution.

I. INTRODUCTION

This paper is an extension of articles written by the author and gives a relation between the proper time derivative of the mass and angular momentum based on equations from the previous articles.

II. REVIEW OF EQUATIONS OF PREVIOUS ARTICLES

From previous articles, French[1,2,3], we consider a classical mechanical stress-energy distribution $S^{\alpha\beta}$, four-current j^{α}, and Electromagnetic field tensor $F^{\alpha}{}_{\beta}$. Based on the relation $S^{\alpha\beta}{}_{,\beta} = F^{\alpha}{}_{\beta} j^{\beta}$ (for example see Misner, Thorne and Wheeler (herein MTW)[4])

and its corresponding rotational equation, we integrate over a space-time volume to obtain the following equations in the rest frame

$$\frac{dm_0}{d\tau} + a_i p^i = \int (1 + a_k r^k) F^0{}_\beta j^\beta dv \tag{1}$$

$$\frac{dp^i}{d\tau} + m_0 a^i = \int (1 + a_k r^k) F^i{}_\beta j^\beta dv \tag{2}$$

$$\frac{dm^i}{d\tau} - p^i + a_j L^{ij} = \int (1 + a_k r^k) r^i F^0{}_\beta j^\beta dv \tag{3}$$

$$\frac{dL^{ij}}{d\tau} + a^j m^i - a^i m^j = \int (1 + a_k r^k) \{ r^i F^j{}_\beta j^\beta - r^j F^i{}_\beta j^\beta \} dv \tag{4}$$

where $m_0 = \int S^{00} dv$, $p^i = \int S^{i0} dv$, $m^i = \int r^i S^{00} dv$ and $L^{ij} = \int (r^i S^{j0} - r^j S^{i0}) dv$. The acceleration a^i is based on an arbitrary world line with r^i being the distance from the world line and v represents the volume of the 3-space normal to the world line. The speed of light has also been set to one and repeated indices indicate a summation. Greek indices represent space-time coordinates, Latin indices represent three-space coordinates, and a zero index represents time.

To put a restriction on the world line, we set the mechanical momentum p^i to zero in this frame. If another restriction is added, that of the center of mass coinciding with the world line, then m^i is also set to zero. If these restrictions are applied to eqs. (1-4) they become

$$\frac{dm_0}{d\tau} = \int (1 + a_k r^k) F^0{}_\beta j^\beta dv \qquad (5)$$

$$m_0 a^i = \int (1 + a_k r^k) F^i{}_\beta j^\beta dv \qquad (6)$$

$$a_j L^{ij} = \int (1 + a_k r^k) r^i F^0{}_\beta j^\beta dv \qquad (7)$$

$$\frac{dL^{ij}}{d\tau} = \int (1 + a_k r^k)\{r^i F^j{}_\beta j^\beta - r^j F^i{}_\beta j^\beta\} dv \qquad (8)$$

Eq. (7) is a constraint equation and from French[1,2,3] if the self-field terms are ignored it yields the condition g = 2 if we ignore second order acceleration terms and take the net current in the rest frame to be zero. If the self-field terms are not ignored, and we use a non-relativistic shell of charge we also obtain the condition g = 2 if we ignore the rotational contribution to the mass. and again ignore second order acceleration terms. Both these results are based on the idea that the object is small enough that the external fields can be taken as constant over the size of it.

III. COVARIANT CHARGE DISTRIBUTION

To include an angular velocity ω^i, we will follow Nodvik[5] and use the covariant charge distribution

$$j^\alpha = \int d\tau \sigma((x^\mu - x^\mu{}_0)^2)[u^\alpha(1 + a_\beta(x^\beta - x^\beta{}_0)) - \varepsilon_{\gamma\delta}{}^\alpha{}_\beta u^\gamma \omega^\delta (x^\beta - x^\beta{}_0)]\delta(u_\nu(x^\nu - x^\nu{}_0)) \qquad (9)$$

where u^α is the 4-velocity, a_β is the 4-acceleration, ω^δ is the 4-angular velocity, and $\varepsilon_{\gamma\delta}{}^{\alpha}{}_{\beta}$ is the antisymmetric Levi-Cevita tensor (for example see MTW[4]). σ is an arbitrary function representing the charge density and $x^\beta{}_0$ represents the world line of the system. The 4-vectors u^α, a_β, ω^δ and $x^\beta{}_0$ are functions of the proper time τ. Other authors have also used this distribution, for example see Appel and Kiessling[6]. In the rest frame eq. (9) reduces to the relations

$$j^0(r^i) = \sigma(r) \tag{10}$$

$$j^i(r^i) = \sigma(r)\frac{\varepsilon^i{}_{jk}\omega^j r^k}{1 + a_m r^m} \tag{11}$$

where $r = \left| r^i \right|$.

When eq. (10) and eq. (11) are put into eqs. (5-8) they become

$$\frac{dm_0}{d\tau} = \int \sigma(r) F^0{}_i\, \varepsilon^i{}_{jk}\omega^j r^k dv \tag{12}$$

$$m_0 a^i = \int \sigma(r)((1 + a_k r^k)F^i{}_0 + F^i{}_j \varepsilon^j{}_{j'k}\omega^{j'} r^k)dv \tag{13}$$

$$a_j \varepsilon^{ij}{}_k L^k = \int \sigma(r) r^i F^0{}_j \varepsilon^j{}_{j'k}\omega^{j'} r^k dv \tag{14}$$

$$\frac{dL^k}{d\tau} = \varepsilon^k{}_{ij}\int \sigma(r)\{(1 + a_l r^l)r^i F^j{}_0 + r^i F^j{}_{k''}\varepsilon^{k''}{}_{j'k'}\omega^{j'} r^{k'}\}dv \tag{15}$$

where we have used the relation $L^{ij} = \varepsilon^{ij}{}_k L^k$.

IV. RELATION BETWEEN MASS AND ANGULAR MOMENTUM

If we add the non-relativistic rotational kinetic energy to a non-rotating rest mass m_{00} we obtain

$$m_0 = m_{00} + \frac{1}{2} I\omega^2 \tag{16}$$

where I is the moment of inertia (for example see Goldstein[7]). We want to try to obtain a similar equation based on eqs. (12-15). If we multiply eq. (15) by ω_k it becomes

$$\omega_k \frac{dL^k}{d\tau} = \omega_k \varepsilon^k{}_{ij}\int \sigma(r)(1 + a_l r^l)r^i F^j{}_0 dv + \omega_k \varepsilon^k{}_{ij}\int \sigma(r) r^i F^j{}_{k''}\varepsilon^{k''}{}_{j'k'}\omega^{j'} r^{k'} dv \tag{17}$$

Now consider the second term of eq. (17) which can be written as

$$\omega_k \varepsilon^k{}_{ij}\int \sigma(r) r^i F^j{}_{k''}\varepsilon^{k''}{}_{j'k'}\omega^{j'} r^{k'} dv = \int \sigma(r) r^i \varepsilon_{kij}\omega^k F^{jk''} r^{k'}\varepsilon_{k''j'k'}\omega^{j'} dv$$

$$= \int \sigma(r)\varepsilon_{jki}\omega^k r^i F^{jk''}\varepsilon_{k''j'k'}\omega^{j'} r^{k'} dv = \int \sigma(r) f_j F^{jk''} f_{k''} dv \tag{18}$$

where $f_i = \varepsilon_{ijk}\omega^j r^k$. Since $F^{jk"}$ is antisymmetric under the exchange of j and k", eq. (18) is the negative of itself and thus zero. Thus the second part of eq. (17) is zero, and if we combine eq. (17) with eq. (12) we obtain

$$\frac{dm_0}{d\tau} = \omega_k \frac{dL^k}{d\tau} - a_l \int \sigma(r) r^l \varepsilon^i{}_{jk} \omega^j r^k F^0{}_i dv \qquad (19)$$

Now multiply the constraint eq. (14) by a_i to obtain

$$a_i a_j \varepsilon^{ij}{}_k L^k = a_i \int \sigma(r) r^i F^0{}_j \varepsilon^j{}_{j'k} \omega^{j'} r^k dv = a_l \int \sigma(r) r^l \varepsilon^i{}_{jk} \omega^j r^k F^0{}_i dv \qquad (20)$$

This is zero since $a_i a_j \varepsilon^{ij}{}_k$ is zero. Using eq. (20) in eq. (19) it becomes

$$\frac{dm_0}{d\tau} = \omega_k \frac{dL^k}{d\tau} \qquad (21)$$

which gives us a relation between the mass and angular momentum in the rest frame.

V. ANGULAR MOMENTUM IN TERMS OF ANGULAR VELOCITY AND NEW EQUATIONS

If we can express L^k in the form of

$$L^k = I(\omega^2)\omega^k \qquad (22)$$

for some function $I(\omega^2)$, then eq. (21) reduces to

$$m_0 = m_{00} + F(\omega^2) \tag{23}$$

where the integration constant m_{00} is again taken as the non-rotating rest mass and

$$F(\omega^2) = \frac{1}{2}\int (2\omega^2 \frac{dI}{d\omega^2} + I)d\omega^2 \tag{24}$$

If I is a constant then we obtain eq. (16).

If eq. (22) is valid, then taking the constraint eq. (14) as a condition on the distribution of charge in the system, eqs. (12-15) reduce to the two equations of motion

$$(m_{00} + F(\omega^2))a^i = \int \sigma(r)((1 + a_k r^k)F^i{}_0 + F^i{}_j \varepsilon^j{}_{j'k}\omega^{j'} r^k)dv \tag{25}$$

and

$$\frac{d}{d\tau}\{I(\omega^2)\omega^k\} = \varepsilon^k{}_{ij}\int \sigma(r)\{(1 + a_l r^l)r^i F^j{}_0 + r^i F^j{}_{k''}\varepsilon^{k''}{}_{j'k'}\omega^{j'} r^{k'}\}dv \tag{26}$$

If eqs. (25) and (26) are looked at in a general frame of reference and a Lagrangian is found for them along with the electromagnetic field then we might be able to quantize the system in a way similar to a non-relativistic quantization, French[8], and perhaps compare this to a Foldy-Wouthuysen type expansion of relativistic Quantum Electrodynamics, (for example see Lin[9]).

VI. LAGRANGIAN FOR FREE PARTICLE IN GENERAL FRAME

Consider a free particle with a moment of inertia given by

$$I = \sum_{n=0}^{\infty} I_n \frac{{\omega_0}^{2n}}{c^{2n}} \tag{27}$$

where the I_n are constants, and ω_0 is the angular velocity in the rest frame, and we have set the speed of light to c. Then F, as defined by eq. (24), becomes

$$F({\omega_0}^2) = \frac{1}{2} \sum_{n=0}^{\infty} I_n \frac{1+2n}{1+n} \frac{{\omega_0}^{2(n+1)}}{c^{2n}} \tag{28}$$

Using eq. (12), eq. (23), eq. (25) and eq. (26), the equations of motion of a free particle in a general frame are given by

$$\frac{d}{dt}\left\{\left(m_{00} + \frac{1}{c^2}F\right)\frac{v^i}{(1-\frac{v^2}{c^2})^{\frac{1}{2}}}\right\} = 0 \tag{29}$$

and

$$\frac{d}{dt}\left(I{\omega_0}^i\right) = 0 \tag{30}$$

Now set ${\omega_0}^i = \omega^i(1 - \frac{v^2}{c^2})^{-1/2}$. We use ω^i since this is in a general frame and when we make a variation with respect to the Euler angles we will want to use the time variable in the general frame and ${\omega_0}^i$ is the angular velocity defined in terms of the derivative of the Euler angles with respect to the rest frame time.

A Lagrangian is then given by

$$L = -m_{00}c^2(1 - \frac{v^2}{c^2})^{\frac{1}{2}} + c^2 \sum_{n=1}^{\infty} \frac{1}{2n} I_{n-1} \frac{\omega^{2n}}{c^{2n}} (1 - \frac{v^2}{c^2})^{-n+1/2} \quad (31)$$

Variation of L with respect to the position yields eq. (29), and variation with respect to the Euler angles yields

$$\frac{d}{dt}\frac{\partial L}{\partial \omega^i} = \epsilon_{ijk}\omega^j \frac{\partial L}{\partial \omega^k} \quad (32)$$

which yields eq. (30).

Using L to find the corresponding conjugate momentum and Hamiltonian, we find that The Hamiltonian is given by

$$H = (m_{00}c^2 + F)(1 - \frac{v^2}{c^2})^{-1/2} \quad (33)$$

Note that this is the same as the classical energy of the particle.

VII. CONCLUSIONS

Now that a relation is found between the mass and angular momentum it might be possible to find a classical system of particles interacting with the electromagnetic field which when quantized yields another way to look at relativistic quantum electrodynamics.

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