A Modified Formula of Function Li (x) for Prime Number Counting

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Abstract: To calculate the number of prime numbers, the prime number theorem function x/ln (x) and Gaussian function Li (x) are most commonly used. However, the former is always less than and the latter is always greater than the actual number of prime numbers, and the deviation increases with the increase of the order of magnitude. The function p (x) has been proposed to improve x/ln (x). Now the Gaussian function Li (x) is dynamically modified, and a more ideal prime number estimation function q (x) is obtained. Numerical experiments show that the modified q (x) calculation is simple and accurate compared with the calculation results of p (x) and Riemann function R (x).

Key words: prime number theorem, number of prime numbers, prime number counting function, dynamic correction

To calculate the number of prime numbers, the prime number theorem function x/ln (x) and Gaussian function Li (x) are most commonly used [1]. The Gaussian function Li (x) is expressed as [2]:

$$\mathrm{Li}(x) = \int_0^x \frac{dt}{\ln(t)}$$

Gauss discovered the prime number theorem in 1792 and Legendre discovered it in 1798 [2]. However, there is a large deviation between the two estimates of the number of prime numbers. The previous article has improved x/ln (x) and obtained the improved prime number estimation function p (x), expressed as [3]:

$$p(x) = \frac{x}{\ln(x) - (1 + \frac{8\ln(2)}{15\sum_{y=1}^{x}\frac{1}{y^n}})}$$

where: n=1.2 is the prime function constant.

One representation of Riemann function R (x) is [4]:

$$R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln(x))^k}{k\,k!\ \zeta(k+1)}$$

where: $\zeta(n)$ is Riemann ζ function.

The calculation of Riemann function is complicated, and it deviates from the true value as Li(x) when the given order of magnitude is very large [4]. Therefore, it is necessary to explore more accurate prime number counting function.

The distribution type of prime numbers belongs to deterministic random distribution [5]. Therefore, theoretically, there is a function that can relatively accurately represent the number of prime numbers.

Now, the Gaussian function Li (x) is dynamically modified to arrive at a more ideal prime number estimation formula q (x). Numerical experiments show that the modified q (x) is simpler and more accurate than p (x) and R (x) .

1. The number of prime numbers less than the given order of magnitude

The experimental observation shows that the deviation of calculated value of Gaussian function Li (x) is large and always greater than the actual prime number count. In this paper, Gaussian function Li (x) is dynamically modified, and an appropriate positive function is added to the denominator of its integral formula to reduce the value of the integral and make it closer to the actual prime number count. After a lot of experiments and further optimization, the function expression is determined and the following definition is given.

Define q (x) as the modified prime count function:

$$q(x) = 1 + \int_{2}^{x} \frac{1}{\ln(t) + \frac{1}{t^{n}}}$$

where: $n = 0.27 + \frac{1}{100}\log_{10}(x)$.

2. Experimental verification

Use q (x) to calculate the number of prime numbers smaller than the given number order, and compare with the calculation results of p (x) and R (x). See Table 1 to Table 2 and Figure 1 to Figure 9 for details. The experiment shows that in many cases, q (x) and p (x) and R (x) are very close to the prime number counting function π (x), and with the increase of x, q (x) and p (x) and R (x) and π (x) cross each other, and q (x) shows good approximation performance. The function q (x) is simple in form, convenient in calculation and relatively high in accuracy.

Table 1 Calculation and comparison of the number of prime numbers less than the given magnitude

x	π(x)	p(x)	R(x)	q(x)	p(x)/π(x)	R(x)/π(x)	q(x)/π(x)	(p(x)-π(x))/π(x)	(R(x)-π(x))/π(x)	(q(x)-π(x))/π(x)
10^1	4	9	5	5	2.2500	1.2500	1.2500	1.2500	0.2500	0.2500
10^2	25	29	26	26	1.1600	1.0400	1.0400	0.1600	0.0400	0.0400
10^3	168	172	168	170	1.0238	1.0000	1.0119	0.0238	0.0000	0.0119
10^4	1229	1230	1227	1229	1.0008	0.9984	1.0000	0.0008	-0.0016	0.0000
10^5	9592	9578	9587	9588	0.9985	0.9995	0.9996	-0.0015	-0.0005	-0.0004
10^6	78498	78459	78527	78516	0.9995	1.0004	1.0002	-0.0005	0.0004	0.0002
10^7	664579	664472	664667	664605	0.9998	1.0001	1.0000	-0.0002	0.0001	0.0000

Note: x is an integer, π (x) is the actual number function of prime numbers, p (x) is the improved prime number counting function, R (x) is the Riemann prime number counting function, and q (x) is the modified prime number counting function in this paper.

Table 2 Calculation and comparison of the number of prime numbers within a given integer interval

x	π(x)	p(x)	R(x)	q(x)	(p(x)-π(x)) /π(x)	(R(x)-π(x)) /π(x)	(q(x)-π(x)) /π(x)
[10000,10200]	23	21.5365	21.5658	21.5666	-0.0635	-0.0622	-0.0623
[20000,20200]	22	20.0734	20.1033	20.1003	-0.0877	-0.0864	-0.0864
[30000,30200]	21	19.3023	19.3310	19.3271	-0.0810	-0.0795	-0.0797
[40000,40200]	20	18.7889	18.8164	18.8121	-0.0605	-0.0590	-0.0594
[50000,50200]	20	18.4086	18.4349	18.4306	-0.0795	-0.0785	-0.0785
[60000,60200]	19	18.1088	18.1342	18.1298	-0.0468	-0.0458	-0.0458
[70000,70200]	22	17.8627	17.8871	17.8828	-0.1882	-0.1868	-0.1484
[80000,80200]	15	17.6547	17.6784	17.6741	0.1767	0.1787	0.1783
[90000,90200]	23	17.4752	17.4981	17.4939	-0.2400	-0.2391	-0.2048
[100000,100200]	15	17.3175	17.3399	17.3357	0.1547	0.1560	0.1557
[200000,200200]	17	16.3465	16.3646	16.3610	-0.0382	-0.0376	-0.0376
[300000,300200]	15	15.8267	15.8423	15.8391	0.0553	0.0560	0.0559
[400000,400200]	14	15.4773	15.4912	15.4883	0.1057	0.1064	0.1063
[500000,500200]	16	15.2166	15.2293	15.2266	-0.0488	-0.0481	-0.0483
[600000,600200]	10	15.0100	15.0217	15.0191	0.5010	0.5020	0.5019
[700000,700200]	14	14.8395	14.8504	14.8480	0.0593	0.0607	0.0606
[800000,800200]	16	14.6951	14.7052	14.7029	-0.0813	-0.0806	-0.0811
[900000,900200]	16	14.5698	14.5794	14.5772	-0.0894	-0.0888	-0.0889
[1000000,1000200]	16	14.4596	14.4686	14.4665	-0.0900	-0.0956	-0.0958
Total deviation					2.2475	2.2479	2.1757
average deviation					0.11829	0.11831	0.11451
maximum deviation					0.5010	0.5020	0.5019

Table 3 Comparison of deviations of q (x) and R(x) in calculating the number of prime numbers in given intervals

interval range x	q(x)			R(x)		
	maximum deviation absolute value	average deviation absolute value	standard deviation	maximum deviation absolute value	average deviation absolute value	standard deviation
[1,1000]	3.4268	1.2878	0.7667	2.0294	0.5706	0.4310
[10000,11000]	3.7922	1.1069	0.7897	6.1107	2.1816	1.2874
[50000,51000]	6.6674	2.3664	1.9654	5.0441	1.8486	1.3319

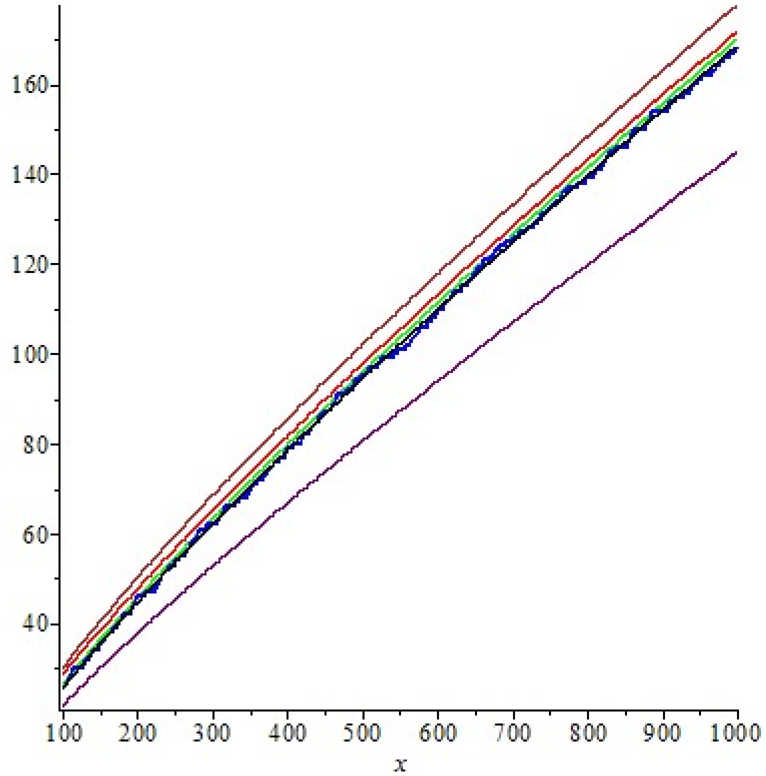

Fig. 1 Brown Li (x), green q (x), red p (x), blue π (x), black R (x), purple x/ln (x).

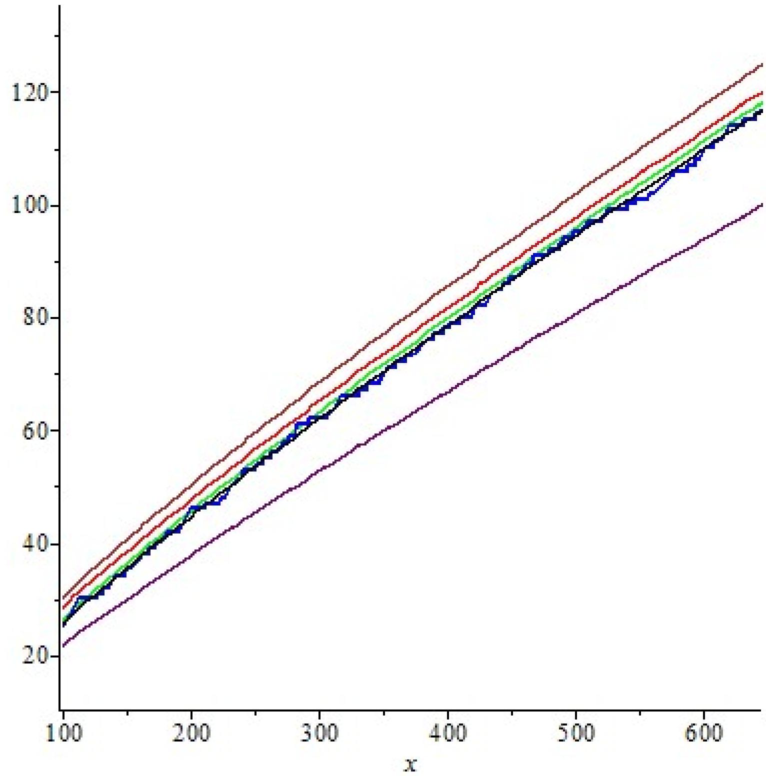

Fig. 2 Brown Li (x), green q (x), red p (x), blue π (x), black R (x), purple x/ln (x).

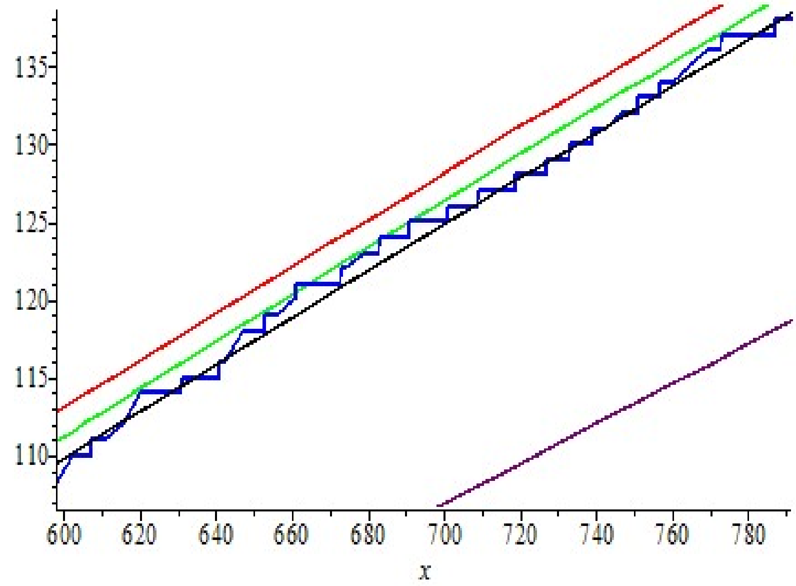

Fig. 3 Green q (x), red p (x), blue π (x), black R (x), purple x/ln (x).

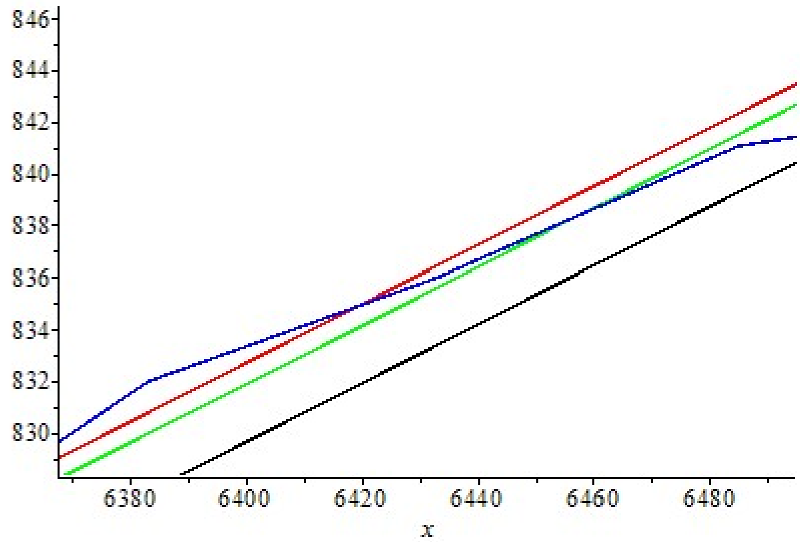

Fig. 4 Green q (x), red p (x), blue π (x), black R (x).

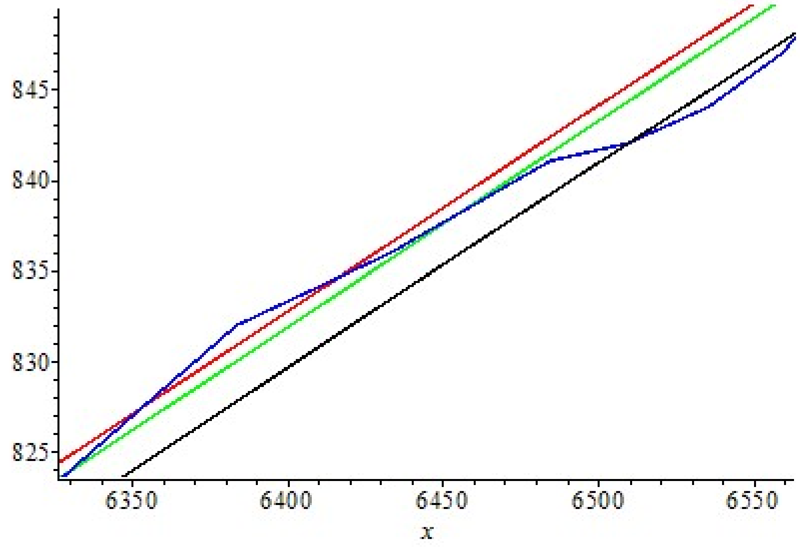

Fig. 5 Green q (x), red p (x), blue π (x), black R (x).

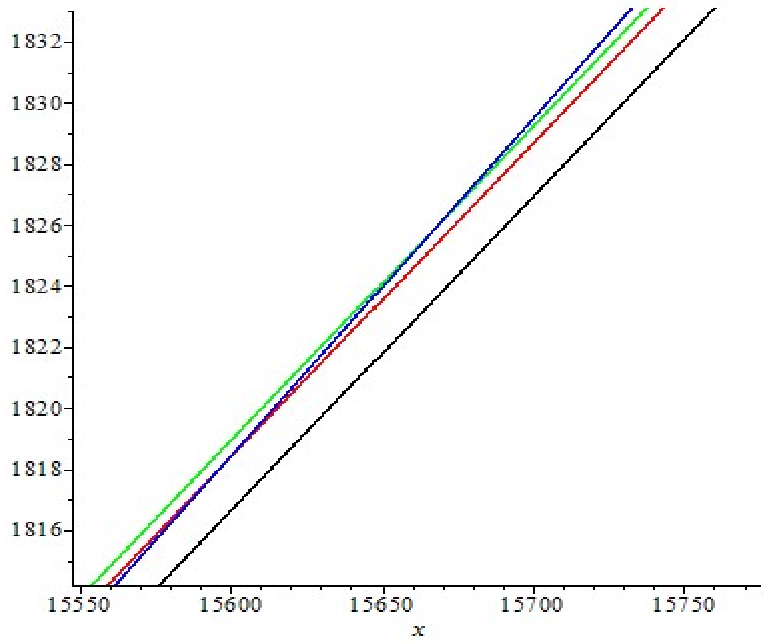

Fig. 6 Green q (x), red p (x), blue π (x), black R (x).

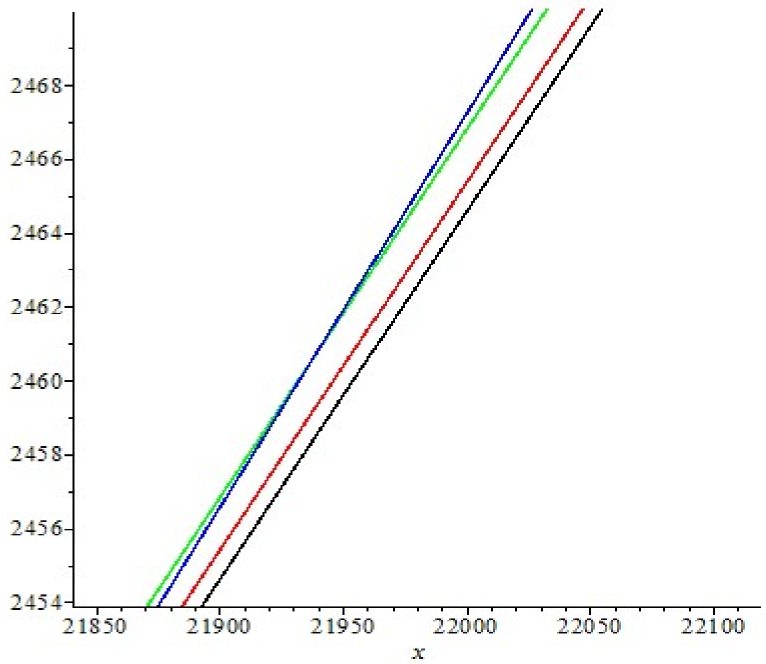

Fig. 7 Green q (x), red p (x), blue π (x), black R (x).

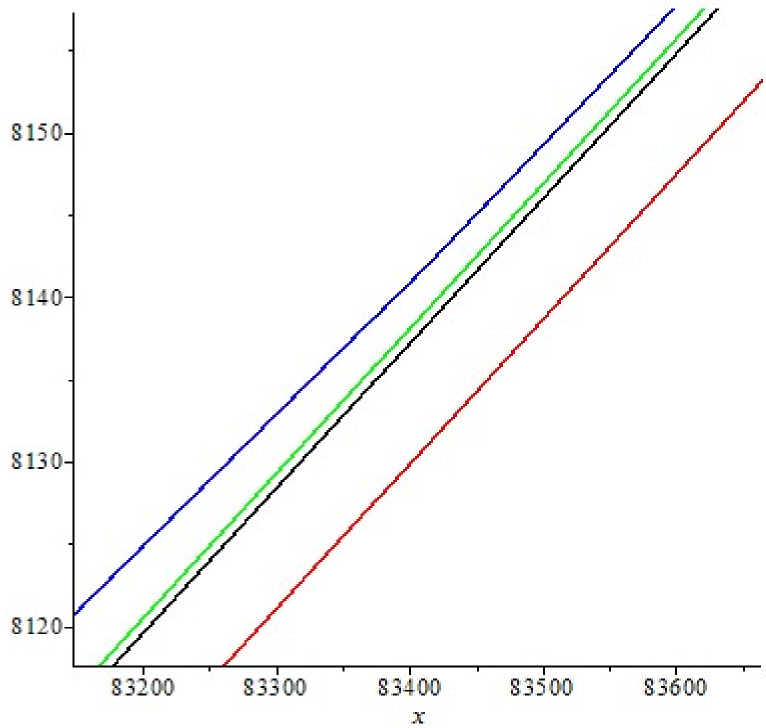

Fig. 8 Green q (x), red p (x), blue π (x), black R (x).

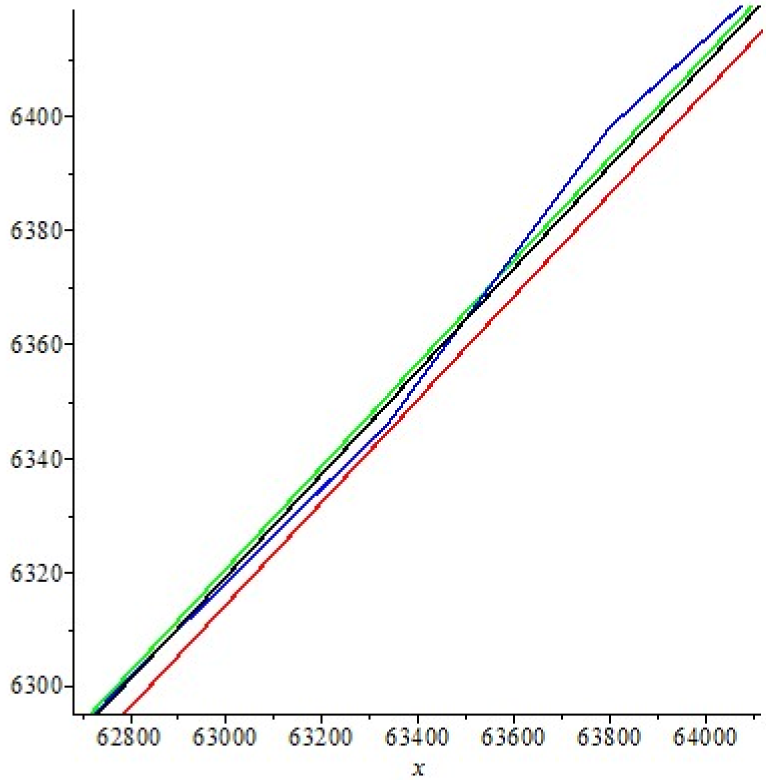

Fig. 9 Green q (x), red p (x), blue π (x), black R (x).

3. Discussion and conclusion

Table 1 shows that using q (x) to calculate the number of prime numbers smaller than the given number order is relatively accurate, which is better than using p (x) and R (x).

Table 2 shows that there are 19 groups of data for q (x), p (x) and R (x) in a given interval. Comparing the calculated values of q (x) and p (x) and R (x) with the calculated values of π (x), the variation direction of q (x) and p (x) and R (x) is completely consistent, and the synchronization rate reaches 100.00%; 6 groups and 13 groups (31.58% and 68.42%) are more than and less than the calculated value of π (x); among them, 7 groups of q (x) are superior to R (x), accounting for 36.84%; q (x) is superior to p (x) in 14 groups, accounting for 73.68%.

The calculated results of q (x) are similar to those of p (x) and R (x), but the maximum deviation and average deviation are slightly better than R (x).

As can be seen from the results in Table 3, there are three groups of data for q(x) and R(x) in a given interval. Among them, there is one group where the the deviation calculated by q(x) is better than that by R(x). The maximum deviation, average deviation and standard deviation in the three groups of data are relatively small.

Figure 1 shows the relationship between the six functions on the interval [10, 1000]. Li (x) is always greater than, x/ln (x) is always less than p (x), q (x), π (x), and R (x); and q (x), π (x) and R (x) are intertwined. The results in Figures 2 to 9 show that the function curve of q (x) and p (x) and R (x) and the function curve of π (x) are entangled and crossed in many places, indicating that when calculating the number of prime numbers in different intervals, there will be a situation where q (x) and p (x) and R (x) are alternately optimal. In addition, the function curves of q (x) and p (x), π (x) and R (x) are very close and fit well. In Figure 4 and after, due to the large deviation of x/ln (x), it no longer appears in the figure.

Because of the randomness of the distribution of prime numbers, the actual number of prime numbers always fluctuates above and below the value of the function q (x) , changing frequently and presenting a fluctuating state. Therefore, the function curve of q (x) can be called prime number curve, and q (x) can be called prime number counting function.

And it is easy to know that the calculation of q (x) is simple and the accuracy is higher than that of p (x) and R (x). The function q (x) is an ideal prime number counting function and may have a wide application prospect.

4. References

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