

Quantum corrections to the Alfvén waves in the Tokamak and ITER plasma

Miroslav Pardy

Department of Physical Electronics,
and
The Laboratory of the Plasma Physics,
Masaryk University,
Kotlářská 2, 611 37 Brno, Czech Republic
e-mail:pamir@physics.muni.cz

January 23, 2023

Abstract

The hydrodynamical model of quantum mechanics based on the Schrödinger equation is combined with the magnetohydrodynamical term to form so called quantum magnetohydrodynamic equation. It is shown that the quantum correction to the Alfvén waves follows from this new equation. The possible generalization is considered for the so called nonlinear Schrödinger equation and for the situation where dissipation is described by the Navier-Stokes equation.

1 Introduction

Tokamak is the fusion reactor invented by Tamm and Sakharov. They formulated the principles of magnetic confinement of high temperature plasmas, that would allow the development of a thermonuclear reactor. Following this, experimental research on plasma initiation and heating in toroidal systems began in 1951 at the Kurchatov Institute (Smirnov, 2010).

ITER - initially the International Thermonuclear Experimental Reactor, meaning "the way" or "the path" in Latin - is an international nuclear fusion research megaproject

creating energy by the fusion processes of the Sun. Upon completion of construction in 2025 in Cadarache facility, it will be the world's largest magnetic confinement plasma physics experiment and the largest experimental tokamak nuclear fusion reactor. ITER will be of more than 100 fusion reactors with ten times the plasma volume of any other tokamak operating today. The theoretical proposal of the ignition reaction in the form of the laser beams was elaborated by author (Pardy, 2005; 2006).

The classical magnetohydrodynamics (MHD) treats on the conductive liquids in magnetic field. It describes the motion of the conductive liquids. On the other hand, it describes also the ionized gases not only in the laboratory conditions, but also in the cosmical space. It involves such liquids as mercury, liquid sodium and so on.

The hydrodynamical motion of the liquid induces the electric and magnetic fields. At the same time the forces act on currents in the magnetic fields and they influence substantially the motion of the liquid. These currents change the original magnetic fields. It means that the very complicated situation arises as a consequence of the interaction between liquid medium and the magnetic and electric fields. The basic classical problem is to formulate the (MHD) equations which describe the complex motion of such plasma medium.

The basic (MHD) equations are well known (Landau et al., 1982) and the question arises what is the quantum description of the motion of the conductive liquid, or in other words what are the quantum magnetohydrodynamical (QMHD) equations. Our goal is to postulate such equation and to solve them for the simple case of the Alfvén waves. We use the approximation where the quantum description can be expressed as the quantum corrections to the classical Alfvén waves. The similar known analogue is the determination of the radiative corrections to the classical synchrotron radiation.

According to Madelung (1926) Bohm and Vigier (1954), Wilhelm (1970), Rosen (1974, 1986) and others, the original Schrödinger equation can be transformed into the hydrodynamical system of equations by using the so called Madelung ansatz:

$$\Psi = \sqrt{n} e^{\frac{i}{\hbar} S}, \quad (1)$$

where n is interpreted as the density of particles and S is the classical action for $\hbar \rightarrow 0$. The mass density is defined by relation $\rho = nm$ where m is mass of a particle.

It is well known that after insertion of the relation (1) into the original Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi, \quad (2)$$

where V is the potential energy, we get, after separating the real and imaginary parts, the following system of equations:

$$\frac{\partial S}{\partial t} + \frac{1}{2m} (\nabla S)^2 + V = \frac{\hbar^2}{2m} \frac{\Delta \sqrt{n}}{\sqrt{n}} \quad (3)$$

$$\frac{\partial n}{\partial t} + \text{div}(n\mathbf{v}) = 0 \quad (4)$$

with

$$\mathbf{v} = \frac{\nabla S}{m}. \quad (5)$$

Equation (3) is the Hamilton-Jacobi equation with the additional term

$$V_q = -\frac{\hbar^2}{2m} \frac{\Delta\sqrt{n}}{\sqrt{n}}, \quad (6)$$

which is called the quantum Bohm potential and equation (4) is the continuity equation. To our knowledge, equation (4) was never solved using the Hartree-Fock method applied on the other hand successfully in the solid state physics.

After application of operator ∇ on eq. (3), it can be cast into the Euler hydrodynamical equation of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{m}\nabla(V + V_q). \quad (7)$$

It is evident that this equation is from the hydrodynamical point of view incomplete as a consequence of the missing term $-\rho^{-1}\nabla p$ where p is hydrodynamical pressure. We complete the equation (7) by adding the pressure term and in such a way we get the total Euler equation in the form:

$$m \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla V_q - \frac{1}{n}\nabla p, \quad (8)$$

where we have put $V = 0$.

In case of the (MHD), it is necessary to add the so called magnetic term

$$\frac{1}{4\pi n}\mathbf{H} \times \text{rot } \mathbf{H} \quad (9)$$

to the equation (8) in order to get

$$m \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) = -\nabla V_q - \frac{1}{n}\nabla p - \frac{1}{4\pi n}\mathbf{H} \times \text{rot } \mathbf{H}. \quad (10)$$

The last equation (10) with the equation (4) and equations

$$\text{div } \mathbf{H} = 0 \quad (11)$$

$$\frac{\partial \mathbf{H}}{\partial t} = \text{rot } (\mathbf{v} \times \mathbf{H}) \quad (12)$$

form the basic equations of the quantum magnetohydrodynamics (QMHD).

Now, let us find the solution of the system (4), (10), (11) and (12) in the form of waves.

2 Alfvén waves with quantum mechanical corrections

The solution of the system is known in case that the quantum mechanical potential is neglected. If we respect the q-potential, we get with

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{h}, \quad \varrho = \varrho_0 + \varrho', \quad p = p_0 + p', \quad (13)$$

the following equations

$$\operatorname{div} \mathbf{h} = 0, \quad \frac{\partial \mathbf{h}}{\partial t} = \operatorname{rot}(\mathbf{v} \times \mathbf{h}), \quad \frac{\partial \varrho'}{\partial t} + \operatorname{div}(\varrho_0 \mathbf{v}) = 0, \quad (14)$$

$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla \left(-\frac{\hbar^2}{2m^2} \frac{\Delta \sqrt{\varrho_0 + \varrho'}}{\sqrt{\varrho_0 + \varrho'}} \right) - \frac{1}{\varrho_0 + \varrho'} \nabla p' - \frac{1}{4\pi(\varrho_0 + \varrho')} \mathbf{H}_0 \times \operatorname{rot} \mathbf{h} \quad (15)$$

We see that in case of the incompressional fluid, the quantum corrections are zero. So the incompressional fluid is not quantum mechanical, but only classical.

Using the approximative relation

$$\nabla \left(\frac{\hbar^2}{2m^2} \frac{\Delta \sqrt{\varrho_0 + \varrho'}}{\sqrt{\varrho_0 + \varrho'}} \right) \approx \frac{\hbar^2}{4m^2} \frac{1}{\varrho_0} \operatorname{grad} \Delta \varrho' \quad (16)$$

and hydrodynamical relation

$$p' = u_0^2 \varrho', \quad (17)$$

where u_0^2 is the velocity of sound in the medium, we get

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{\hbar^2}{4m^2} \frac{1}{\varrho_0} \operatorname{grad} \Delta \varrho' - \frac{u_0^2}{\varrho_0} \operatorname{grad} \varrho' - \frac{1}{4\pi \varrho_0} (\mathbf{H}_0 \times \operatorname{rot} \mathbf{h}). \quad (18)$$

The solution of equations (14) and (18) can be realized in the form:

$$\mathbf{v} = \mathbf{v}_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)}, \quad \mathbf{h} = \mathbf{h}_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)}, \quad \varrho' = \varrho'_0 e^{i(\mathbf{k}\mathbf{r} - \omega t)}, \quad (19)$$

where \mathbf{k} is the wave vector and ω is a frequency of the wave. After insertion of eqs. (19) in equations (14) and (18), we get

$$\mathbf{k}\mathbf{h} = 0 \quad (20)$$

$$-\omega \mathbf{h} = \mathbf{k} \times (\mathbf{v} \times \mathbf{h}) \quad (21)$$

$$\omega \varrho' = \varrho_0 (\mathbf{k}\mathbf{v}) \quad (22)$$

$$-\omega \mathbf{v} + \frac{u_0^2}{\varrho_0} \varrho'_0 \mathbf{k} = -\frac{1}{4\pi \varrho_0} \mathbf{H}_0 \times (\mathbf{k} \times \mathbf{h}) - \frac{\hbar^2}{4m^2} \frac{\varrho'_0}{\varrho_0} k^2 \mathbf{k}. \quad (23)$$

We see from (20) that perturbation of the magnetic field \mathbf{h} is perpendicular to the wave vector \mathbf{k} . So, we can choose $\mathbf{k} \equiv (k_x, 0, 0)$ and $\mathbf{H}_0 \equiv (H_{0x}, H_{0y}, 0)$. Then, with $u = \omega/k$, we have the equations following from eq. (21) and (23) (where in eq. (23) the term ϱ'/ϱ_0 was excluded using eq. (22)) (Landau et al., 1982).

$$uh_z = -v_z H_{0x}, \quad uv_z = -\frac{H_{0x}}{4\pi\varrho_0} h_z, \quad (24)$$

$$uh_y = v_x H_{0y} - v_y H_{0x}, \quad uv_y = -\frac{H_{0x}}{4\pi\varrho_0} h_y, \quad (25)$$

$$\left(u - \frac{U_0^2}{u}\right) v_x = \frac{H_{0y}}{4\pi\varrho_0} h_y, \quad (26)$$

where

$$U_0^2 = u_0^2 + \frac{\hbar^2}{4m^2} k^2. \quad (27)$$

It follows from eq. (22) that

$$\varrho' = \frac{\varrho_0}{u} v_x \quad (28)$$

The necessary condition of solubility of eqs. (24) can be expressed as

$$u = \frac{|H_{0x}|}{\sqrt{4\pi\varrho_0}} \equiv u_A, \quad (29)$$

where A is the first letter of Alfvén.

If we will suppose that $H_{0x} > 0$ then we can remove the absolute value. The velocity v_z is also vibrating and it is easy to see that

$$v_z = -\frac{h_z}{\sqrt{4\pi\varrho_0}}. \quad (30)$$

Or, in general,

$$\mathbf{v} = -\frac{\mathbf{h}}{\sqrt{4\pi\varrho_0}} \quad (31)$$

The equation (30) can be written in the more general form as

$$\omega = \frac{1}{\sqrt{4\pi\varrho_0}} \mathbf{H}_0 \mathbf{k}, \quad (32)$$

from which it follows the group velocity in the form

$$\frac{\partial\omega}{\partial\mathbf{k}} = \frac{\mathbf{H}_0}{\sqrt{4\pi\varrho_0}} \quad (33)$$

and it does not depend on the direction of \mathbf{k} . So the group velocity of a wave is in the direction of the magnetic field \mathbf{H}_0 . These waves are so called Alfvén waves with the velocity

u_A . The analysis of the properties of the (MHD) waves is described in the Landau et al. textbook (Landau et al., 1982).

Let us remark that from eqs. (25) and (26) it follows the so called magnetosound waves as a result of the determinat of these equation which we put zero. Or, (Landau et al., 1982)

$$U_{\sigma M}^2 = \frac{1}{2} \left\{ \frac{H_0^2}{4\pi\rho_0} + U_0^2 \pm \left[\left(\frac{H_0^2}{4\pi\rho_0} + U_0^2 \right)^2 - \frac{H_{0x}^2 U_0^2}{\pi\rho_0} \right]^{1/2} \right\}. \quad (34)$$

So, we get two waves involving the quantum corrections. The first wave is quick wave with sign + and the second wave is slow wave with sign -.

3 The nonlinear (QMHD) equation

In case of the nonlinear Schrödinger equation with the logarithmic nonlinearity the basic equation is of the form (Parady, 1993, 2001; Bialynicky-Birula et al., 1976):

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V\Psi + b(\ln |\Psi|^2)\Psi, \quad (35)$$

where $b < 3 \times 10^{-15} eV$ (Gähler et al., 1981) is some constant.

The quantum equation of MGH with the nonlinear term is then

$$m \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \nabla \left(\frac{\hbar^2}{2m} \frac{\Delta \sqrt{n}}{\sqrt{n}} \right) + b(\ln |n|^2) - \frac{1}{n} \nabla p - \frac{1}{4\pi n} \mathbf{H} \times \text{rot } \mathbf{H}. \quad (36)$$

Or,

$$\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \nabla \left(\frac{\hbar^2}{2m^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) + \frac{b}{m} (\ln \left| \frac{\rho}{m} \right|^2) - \frac{1}{\rho} \nabla p - \frac{1}{4\pi \rho} \mathbf{H} \times \text{rot } \mathbf{H}. \quad (37)$$

It is evident that to finding the (QMHD) solutions will be more complicated than of the linear QMGH problems. Let us first remember the one-dimensional solutions of the one-dimensional nonlinear Schrödinger equation (Parady, 1993, 2001).

Let be $c, (\text{Im } c = 0), v, k, \omega$ some parameters and let us insert function

$$\Psi(x, t) = cG(x - vt)e^{ikx - i\omega t} \quad (38)$$

into the one-dimensional equation (35) with $V = 0$. Putting the imaginary part of the new equation to zero, we get

$$v = \frac{\hbar k}{m} \quad (39)$$

and for function G we get the following nonlinear equation (symbol $'$ denotes derivation with respect to $\xi = x - vt$):

$$G'' + AG + B(\ln G)G = 0, \quad (40)$$

where

$$A = \frac{2m}{\hbar}\omega - k^2 + \frac{2m}{\hbar^2}b \ln c^2 \quad (41)$$

$$B = \frac{4mb}{\hbar^2}. \quad (42)$$

After multiplication of eq. (40) by G' we get:

$$\frac{1}{2}[G'^2]' + \frac{A}{2}[G^2]' + B\left[\frac{G^2}{2} \ln G - \frac{G^2}{4}\right]' = 0, \quad (43)$$

or, after integration

$$G'^2 = -AG^2 - BG^2 \ln G + \frac{B}{2}G^2 + const. \quad (44)$$

If we choose the solution in such a way that $G(\infty) = 0$ and $G'(\infty) = 0$, we get $const. = 0$ and after elementary operations we get the following differential equation to be solved:

$$\frac{dG}{G\sqrt{a - B \ln G}} = d\xi, \quad (45)$$

where

$$a = \frac{B}{2} - A. \quad (46)$$

Equation (45) can be solved by the elementary integration and the result is

$$G = e^{\frac{a}{B}} e^{-\frac{B}{4}(\xi+d)^2}, \quad (47)$$

where d is some constant.

The corresponding soliton-wave function is evidently in the one-dimensional free particle case of the form:

$$\Psi(x, t) = ce^{\frac{a}{B}} e^{-\frac{B}{4}(x-vt+d)^2} e^{ikx - i\omega t}. \quad (48)$$

It is not necessary to change the standard probability interpretation of the wave function. It means that the normalization condition in our one-dimensional case is

$$\int_{-\infty}^{\infty} \Psi^* \Psi dx = 1. \quad (49)$$

Using the Gauss integral

$$\int_0^\infty e^{-\lambda^2 x^2} dx = \frac{\sqrt{\pi}}{2\lambda}, \quad (50)$$

we get with $\lambda = \left(\frac{B}{2}\right)^{\frac{1}{2}}$

$$c^2 e^{\frac{2a}{B}} = \left(\frac{B}{2\pi}\right)^{\frac{1}{2}} \quad (51)$$

and the density probability $\Psi^*\Psi = \delta_m(\xi)$ is of the form (with $d = 0$):

$$\delta_m(\xi) = \sqrt{\frac{m\alpha}{\pi}} e^{-\alpha m \xi^2} \quad ; \quad \alpha = \frac{2b}{\hbar^2}. \quad (52)$$

It may be easy to see that $\delta_m(\xi)$ is the delta-generating function and for $m \rightarrow \infty$ is just the Dirac δ -function.

It means that the motion of a particle with sufficiently big mass m is strongly localized and in other words it means that the motion of this particle is the classical one. Such behavior of a particle cannot be obtained in the standard quantum mechanics because the plane wave $\exp[ikx - i\omega t]$ corresponds to the free particle with no possibility of localization for $m \rightarrow \infty$.

Let us still remark that it is possible to show that coefficient c^2 is real and positive number (Parady, 1993, 2001).

We frequently read in the physical texts on the quantum mechanics that the classical limit of quantum mechanics is obtained only by the so called WKB method. However, the limit is only formal because in this case the probabilistic form of the solution is conserved while classical mechanics is strongly deterministic. In other words, statistical description of quantum mechanics is in no case reduced to the strong determinism of classical mechanics of one-particle system. So, only nonlinear quantum mechanics of the above form gives the correct classical limit expressed by the delta-function. More information on the problems which are solved by the nonlinear Schrödinger equation involving the collapse of the wave function and the Schrödinger cat paradox is described in author's articles (Parady, 1993, 1994, 2001). The extended version of the nonlinear quantum world is described in the preprint of Castro (2002).

4 Discussion

We have seen that the quantum corrections to the Alfvén waves follows from the hydrodynamical formulation of quantum mechanics involving the magnetic interaction and pressure. The Alfvén waves involve the Planck constant.

The Linear and nonlinear Alfvén waves with the quantum mechanical corrections being investigated in this article, is based on the previous articles by author (Parady,

2010a; 2010b).

By using QMHD, we have derived the three MHD modes called pure Alfvén, fast and slow magnetoacoustic waves with quantum corrections in the linear regime. A nonlinear QMHD for Alfvén waves has been also derived by making a possible generalization of the so-called nonlinear Schrödinger equation for the situation where the dissipation is described by the Navier-Stokes equation.

The theory can be generalized to the situation with dissipation. The dissipative terms can be evidently inserted in the equation (10) in order to get quantum Navier-Stokes magneto-hydrodynamical equation:

$$m \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla V_q - \frac{1}{n} \nabla p + \frac{\eta}{n} \Delta \mathbf{v} + \frac{1}{n} \left(\xi + \frac{\eta}{3} \right) \text{grad div } \mathbf{v} + \frac{1}{4\pi n} (\text{rot} \mathbf{H} \times \mathbf{H}), \quad (53)$$

where η and ξ are some classical constant which express the dissipation.

The Alfvén waves follow from the equation (53) in the approximative form in such a way they will involve the dissipation and the quantum corrections. We know that the Navier-Stokes equation are used in the determination of the turbulence. If we apply such methods to our system with the quantum corrections, we evidently get the so called quantum turbulence. To our knowledge such problem was not solved and it is not involved in the monographs on turbulence and monographs dealing with the theory of catastrophes. It is not excluded that the quantum turbulence plays the substantial role in the natural terrestrial catastrophes.

Let us still remark that Using the effective one fluid (QMHD) model, magnetosonic waves propagating obliquely to an external magnetic field are studied in an electron-ion Fermi plasma. In the linear approximation, the effect of quantum corrections on the fast and slow magnetosonic waves are discussed. It is shown that the system of QMHD equations admit the Kadomste-Petviashvilli soliton due to the balance between nonlinearity and dispersion caused by the obliqueness and quantum diffraction effects. It is observed that for both modes, the angle θ , the electron quantum diffraction, and statistic effects modify the shape of the solitary structure. It is also found that the results obtained for the quantum plasma differ significantly from the classical electron-ion plasmas (Mushtaq et al., 2009).

References

- Bialynicky-Birula I. and Mycielski, J. (1976). Nonlinear wave mechanics, Ann. Phys. (N.Y.) **100**, 62.
- Bohm D. and Vigier, J. (1954). Model of the causal interpretation of quantum theory in terms of a fluid with irregular fluctuations, Phys. Rev. **96**, No. 1, 208.

- Castro, C. J. Mahecha and Rodríguez B., (2002). Nonlinear QM as a fractal Brownian motion with complex diffusion constant, quant-ph/0202026.
- Gähler, R. Klein A. G. and Zeilinger, A. (1981). Neutron optical test of nonlinear wave mechanics, Phys. Rev. A **23** No. 4, 1611.
- Landau, L. D. and Lifshitz, E. M. *Electrodynamics of the continuous media*, Moscow, Nauka (1982). (in Russian).
- Madelung, E. (1926). Quantentheorie in hydrodynamischer Form, Z. Physik **40**, 322.
- Mushtaq, A. and Qamar, A. (2009). Parametric studies of nonlinear magnetosonic waves in two-dimensional quantum magnetoplasma, PHYSICS of PLASMAS **16**, 022301.
- Pardy, M. (1993). To the nonlinear quantum mechanics, Fac. Sci. Nat. Univ. Masaryk Brun. **23**, Physics, 114.
- Pardy, M. Possible Tests of Nonlinear Quantum Mechanics, in: *Waves and Particles in Light and Matter*, Ed. by Alwyn van der Merwe and Garuccio, A. Plenum Press New York (1994).
- Pardy, M. (2001). To the nonlinear quantum mechanics, quant-ph/0111105.
- Pardy, M. (2005). Volkov solution for two laser beams and ITER: hep-ph/0507141.
- Pardy, M. (2006). Volkov solution for two laser beams and ITER: Int. Journal of Theoretical Physics, 45, pages 647659.
- Pardy, M. (2010a). Some quantum effects in magnetic field, in: *Quantum Mechanics*, Nova Publishers, 2010, <https://www.novapublishers.com/>, Series: Physics, Research and Technology, ISBN: 978-1-61728-966-8, Editors: Jonathan P. Groffe, Nova Science Publishers.
- Pardy, M. (2010b). Quantum mechanical corrections to the Alfvén waves, arXiv:quant-ph/0510206v1 27 Oct 2005.
- Rosen, N. (1974). A classical picture of quantum mechanics, Nuovo Cimento **19** B No. 1, 90.
- Rosen, N. (1986). Quantum particles and classical particles, Foundation of Physics **16** No. 8, 687.
- Smirnov, V. P. (2010). Tokamak foundation in USSR/Russia 1950–1990, Nucl. Fusion **50**, 014003, (8pp).
- Wilhelm, E. (1970). Hydrodynamic model of quantum mechanics, Phys. Rev. D **1** No. 8, 2278.