Relations between e, π and golden ratios

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Abstract. We write out relations between the base of natural logarithms (e), the ratio of the circumference of a circle to its diameter (π), and the golden ratios (Φ_p) of the additive p-sequences. An additive p-sequence is a natural extension of the Fibonacci sequence in which every term is the sum of p *previous* terms given $p \geq 1$ initial values called *seeds*.

1 Introduction

Euler's identity (or Euler's equation) is given as

$$e^{i\pi} + 1 = 0, \tag{1}$$

where $e = 2.718\cdots$ is the base of natural logarithms, $i := \sqrt{-1}$ is the imaginary unit of complex numbers, and $\pi = 3.1415\cdots$ is the ratio of the circumference of a circle to its diameter. It is a special case of Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, for $\theta = \pi$. This expresses a deep mathematical beauty [1–3] as it involves three of the basic arithmetic operations: addition/subtraction, multiplication/division, and exponentiation/logarithm, and five fundamental mathematical constants: 0 (the additive identity), 1 (the multiplicative identity), e (Euler's number), i (the imaginary unit), and π (the fundamental circle constant).

As Euler's identity is an example of mathematical elegance, further generalizations of similar-type have been discovered.

- The n^{th} roots of unity ($n > 1$) add up to zero.

$$\sum_{k=0}^{n-1} e^{2i\pi\frac{k}{n}} = 0. \tag{2}$$

It yields Euler's identity (1) when $n = 2$.

- For quaternions [4], with the basis elements $\{i, j, k\}$ and real numbers a_n such that $a_1^2 + a_2^2 + a_3^2 = 1$,
$$e^{(a_1 i + a_2 j + a_3 k)\pi} + 1 = 0. \tag{3}$$
- For octonions, with the basis elements $\{i_1, i_2, \cdots, i_7\}$ and real numbers a_n such that $a_1^2 + a_2^2 + \cdots + a_7^2 = 1$,
$$e^{\left(\sum_{k=1}^{7} a_k i_k\right)\pi} + 1 = 0. \tag{4}$$

In this article, motivated by Euler's identity and its generalizations, we give relations between the base of natural logarithms (e), the ratio of the circumference of a circle to its diameter (π), and the golden ratios (Φ_p) of the additive p-sequences.

2 Additive p-sequences

An additive p-sequence [5] is a natural extension of the Fibonacci sequence [6–8] in which every term is the sum of p *previous* terms given $p \geq 1$ initial values called *seeds* $(s_0, s_1, \cdots, s_{p-1})$ such that $t_0 = s_0,\ t_1 = s_1, \cdots, t_{p-1} = s_{p-1}$, and

$$t_n(p) := t_{n-1}(p) + t_{n-2}(p) + \cdots + t_{n-p}(p) = \sum_{k=n-p}^{n-1} t_k(p). \tag{5}$$

This can be equivalently rewritten as

$$t_{n+p}(p) := t_{n+p-1}(p) + t_{n+p-2}(p) + \cdots + t_n(p). \tag{6}$$

Varying the values of seeds, it is possible to construct an infinite number of p-sequences.

By definition of $t_n(p)$, we have $t_{n+1}(p) > t_n(p)$ and $t_{n+1}(p) = 2t_n(p) - t_{n-p}(p) < 2t_n(p)$ [5]. Hence
$$1 < \Phi_p < 2. \tag{7}$$

The limiting ratio value $\left(\lim_{n\to\infty} \frac{t_{n+1}(p)}{t_n(p)}\right)$ of different p-sequences are different, say Φ_p, and tends toward 2 for p tending toward infinity. That is,

$$\Phi_p = \lim_{n\to\infty} \frac{t_{n+1}(p)}{t_n(p)} \xrightarrow{p\to\infty} 2. \tag{8}$$

Figure 1: Division of a line into 2 segments.

3 p-golden ratio

The golden ratio [9, 10] arises when we consider division of a line segment AB with a point P such that $\frac{BP}{AP} = \frac{AB}{BP}$, where $BP > AP$ (see Fig. 1). Given $AP = a$ and $BP = b$ are two positive numbers, the above problem translates as

$$\frac{b}{a} = \frac{a+b}{b}. \tag{9}$$

Taking $\frac{b}{a} = x$, the above equation can be rewritten as $x = 1 + \frac{1}{x}$. This reduces to the characteristic equation

$$X(x) = x^2 - x - 1 = 0, \tag{10}$$

whose positive solution is

$$\Phi = \frac{\sqrt{5}+1}{2} = 1.618\cdots. \tag{11}$$

The golden ratio allegedly appears everywhere: in geometry, math, science, art, architecture, nature, human body, music, painting. However, many hold skeptical views about this [11–15].

We wish to generalize the above case. That is, we consider division of a line segment AB into $p > 2$ segments (see Fig. 2) such that $AP_1(= a_1) < P_1P_2(= a_2) < \cdots < BP_{p-1}(= a_p)$ are p positive real numbers. We now demand that

$$\begin{aligned} &\frac{P_1P_2}{AP_1} = \frac{P_2P_3}{P_1P_2} = \cdots = \frac{AB}{P_{p-1}B} \\ \Leftrightarrow\ &\frac{a_2}{a_1} = \frac{a_3}{a_2} = \cdots = \frac{\sum_{k=1}^{p} a_k}{a_p}. \end{aligned} \tag{12}$$

If a unique positive value, say Φ_p, exists for the above ratio, we call it the p-golden ratio.

From Eq. (12) follows naturally the p-degree algebraic equation whose *positive solution* gives the value of Φ_p [5]:

$$X_p(x) \equiv x^p - \sum_{k=1}^{p-1} x^k - 1 = 0. \tag{13}$$

Note that $X_p(0) = -1$ for all p and $X_p(1) = -(p-1)$. Eq. (13) is the *characteristic equation* for Φ_p. Interestingly, Φ_p *coincides with the limiting ratio value* of the p-sequence [5].

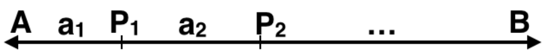

Figure 2: Division of a line into p segments.

4 Relations between e, π and Φ_p

In this section, firstly we present the relations between e, π and the 2-golden ratio. The Fibonacci sequence $\{0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots\}$ is a 2-sequence because it is generated by the sum of two previous terms. The positive real solution of the characteristic equation $x^2 - x - 1 = 0$ yields the 2-golden ratio,

$$\lim_{n\to\infty} \frac{t_{n+1}(2)}{t_n(2)} = \Phi_2 = \frac{\sqrt{5}+1}{2}. \tag{14}$$

Following relations hold between e, π and $\Phi_2 \equiv \Phi$.

$$\begin{align} e &\approx \Phi^2 + 0.1 = 2.7180339887, \tag{15}\\ e &\approx \Phi^2 + \frac{50000}{308253}\frac{1}{\Phi} = 2.7182818353, \tag{16}\\ \Phi &= 2\cos 36^\circ = e^{i\pi/5} + e^{-i\pi/5}, \tag{17}\\ \Phi(\Phi - 1) &= e^{i2\pi} = -e^{i\pi} = -ie^{i\pi/2}, \tag{18}\\ \Phi(e^{i\pi} + \Phi) &= 1. \tag{19} \end{align}$$

A transcendental expression cannot be equated with an algebraic expression. Eq. (15) and Eq. (16), however, give the polynomial approximations of e in terms of Φ. Because Φ_p is a solution of Eq. (13), we have

$$\begin{align} \Phi_p^p &= \Phi_p^{p-1} + \Phi_p^{p-2} + \cdots + \Phi_p + 1 = \sum_{k=0}^{p-1} \Phi_p^k, \tag{20}\\ \Phi_p^{p+1} &= \Phi_p^p + \Phi_p^{p-1} + \cdots + \Phi_p^2 + \Phi_p, \notag\\ &= 2\Phi_p^p - 1. \tag{21} \end{align}$$

Using these equations, we have the following relations between e, π and Φ_p ($p \geq 3$).

$$e^{i\pi} + \Phi_p^p - \sum_{k=1}^{p-1} \Phi_p^k = 0, \tag{22}$$

Eq. (22) has been obtained using the Euler's identity.

5 Conclusion

In summary, inspired by Euler's identity, we have provided several relations between the base of natural logarithms (e), the ratio of the circumference of a circle to its diameter (π), and the golden ratios (Φ_p) of the additive p-sequences.

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