PROOF OF π AND e IS IRRATIONAL NUMBER BY ITS TRANSCENDENCE

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Abstract. I will show how to prove π and e are irrational numbers with the fact that they are transcendental numbers.

Lemma 1. All transcendental numbers are irrational numbers.
Proof: Let x be a transcendental number. Assume x is a rational number.
$x = \frac{a}{b}$, where a and b are integers with $b \neq 0$.
However, $bx - a = 0$ and so x is an algebraic number, which contradicts the assumption. Therefore, x is an irrational number.

Lemma 2. Product of two algebraic numbers is also algebraic.
This can easily be proved by the resultant of the polynomial.

Theorem. (Lindemann–Weierstrass theorem) If $a_1, a_2, \dots, a_n$ are distinct algebraic numbers, then $e^{a_1}, e^{a_2}, \dots, e^{a_n}$ are linearly independent over the algebraic numbers. [1]

Corollary 1. e^{a} is a transcendental number if a is a nonzero algebraic number.
This can easily be proved by Lindemann–Weierstrass theorem.

Proposition. π is an irrational number.
Proof. First, we want to show π is a transcendental number.
Assume π is an algebraic number. Since i is an algebraic number, from lemma 2, πi is an algebraic number. From corollary 1, $e^{\pi i}$ is a transcendental number. However, by Euler's identity, $e^{\pi i} = - 1$ which clearly is an algebraic number. Therefore, the assumption leads to a contradiction. Therefore, π is a transcendental number. By lemma 1, π is an irrational number.

Proposition. e is an irrational number.
Proof: Put $a = 1$ into corollary 1. We can immediately get e is a transcendental number. By lemma 1, we can deduce that e is an irrational number.

REFERENCE

[1] Alan Baker. *Transcendental Number Theory*, Cambridge University Press, 1990