Personal observations on the equation
$x^4 + y^4 = z^4$
only case of UTF demonstrated by Pierre de Fermat and found in its documents .

(A study of Fermat's quartic equation without using "infinite descent")

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Abstract

In 1637 the Toulouse lawyer Pierre de Fermat enunciated a conjecture in number theory which denied the possibility of dividing the nth power of an integer into the sum of two nth integer powers when natural exponents greater than or equal to three are considered.

This is the famous "Fermat's last conjecture", now elevated to a theorem thanks to the proof given by Prof. A. Wiles in 1995.

In other words, the Diophantine equation $x^n + y^n = z^n$ with n $\geq$ 3 does not admit integer, primitive and non-trivial solutions. In my article, I set out my observations on a particular case of this theorem. The "admirable proof" which Fermat claimed to have was never found, except for the one in which n = 4 and which the French genius demonstrated with a new method coined by himself and known as the "method of infinite descent". The latter follows the same logic as the already known "ab absurdum reasoning" even if it is articulated with slightly more demanding algebraic procedures.

In this article I report my proof of the case n = 4 with the use of an elementary algebraic procedure which, avoiding Fermat's "infinite descent", brings the equation in question back to that of the well-known quadratic equation :

$$x^2 + 2y^2 = z^2$$

for which all the primitive, non-trivial solutions in the set of integers have already been parameterized.

The peculiarity of my method lies in transforming Fermat's quartic equation into a quadratic equation well studied by number theorists and whose solutions are already known. With this strategy it is possible to prove the thesis.

■ My observations on the Diophantine equation proved by P. de Fermat

Fermat's "big" theorem was amply proved by Professor Prof. A.Wiles in 1995 with the help of many professionals in the field. This gives an idea of how complex and tortuous the path followed was, often characterized by the use of new mathematical techniques, obviously unknown not only to Fermat, but to all eighteenth-century mathematicians. Suffice it to recall the "Kolyvagin-Flach method" used by the English scholar to associate its modular form with a particular elliptic equation (Frey's equation). It is therefore wholly improbable that the lawyer from Toulouse (Pierre de Fermat) was in possession of the "admirable proof" he speaks of when expounding his conjecture, given that he limited himself to making known only that in the particular case in which the exponents of the indeterminates are all equal to four.

P. de Fermat demonstrated that the equation $\boldsymbol{x^4 + y^4 = z^4}$does not admit primitive and non-trivial solutions in the integers, using, as already mentioned, a method coined by himself which he calls "infinite descent" . This is a special kind of proof "by contradiction", in which a given Diophantine equation is assumed to admit a "minimal integer primitive solution". If we arrive at the existence of a new primitive solution smaller than the hypothesized one, obviously we fall into an absurdity, for which we must consider the starting equation impossible in the integers . Revisiting UTF (Fermat's Last Theorem) in the special case n = 4 , I wondered if the equation $\boldsymbol{x^4 + y^4 = z^4}$ could be studied without using Fermat's "infinite descent", but by adopting an ordinary algebraic procedure, i.e. one based on elementary algebra and on consolidated results in the field of "number theory".

I thus found a profound connection between Fermat's equation for n = 4 and the well-known quadratic equation:

$$\boldsymbol{x^2 + 2y^2 = z^2} ,$$

of which all its primitive and non-trivial solutions in the ring of integers are known.

It is very easy to demonstrate, with suitable changes of variables and rearrangements of terms, that the quadratic equation mentioned above can be reduced to an even more famous equation already known in Fermat's time.

In fact, we learn from the historical documentation that he himself was the first to propose it to B.Frenicle de Bessy and in 1657 to many other mathematicians. In this case, Pell's equation as defined by Euler ,falls into the wider range of rational numbers, i.e. Q, but I will not use it here.
In my work I will first of all present a proof that leads to the parametrization of the primitive (non-trivial) integer solutions of the equation
$x^2 + 2y^2 = z^2$, after which I will focus on my observations about the connection between this and the UTF equation for n = 4, showing how knowledge of the solutions of the former is useful for arriving at proof of the irresolvability of Fermat's great theorem in the special case treated here, without using the method of infinite descent. Now follow three "lemmas" which prove what I said:

Lemma (1)

The quadratic equation:

$$(1)\ x^2 + 2y^2 = z^2$$

admits infinitely many integer, primitive and non-trivial solutions in Z + parametrizable as follows:

(x, y, z) = (a2-2b2; 2ab; a2 + 2b2) with a, b ∈Z^+: ab> 0 ∧ (a, b) = 1,

a> b√2

The proof of this lemma will be carried out using a "geometric method" known in the literature as the "Method of the bundle of Klein lines", valid for all conics, but generally for all second order curves.

If z ≠ 0, dividing both sides of equation (1) by z2 and working with the homogeneous coordinates X = x / z and Y = y / z, we arrive at the equation of a centered ellipse, having foci on abscissa axis, of unit semimajor axis and semiminor axis equal to 1 / √2:

(2) X2 + 2Y2 = 1

A rational point of the curve is certainly **P (-1; 0)**, therefore the equation of the proper bundle of Klein lines centered in P is given by:

(3) Y = m (X + 1) with m ∈Q

From solving the system between (2) and (3) we arrive at the equation:

(4) (2m2 + 1) X2 + 4m2 X + 2m2 -1 = 0

Whose solutions are X = -1 (not to be considered) e

X = (1-2m2) / (1 + 2m2) hence Y = 2m / (1 + 2m2)

Now I set m = b / a ∈Q with a, b∈Z ab ≠ 0 (a, b) = 1 and a > b√2
(x, y, z) = (a2 - 2b2 , 2ab, a2 + 2b2) with (x, y, z) = 1, because
(a, b) = 1.

Lemma (2)

The quartic equation: x4 + y4 = z4 does not admit primitive and non-trivial solutions in Z.

■Proof

I reason by contradiction, i.e. I establish that:

$\exists x_0 , y_0, z_0 \in Z+$: $(x_0 y_{0,} z_0 \neq 0) \wedge (x_0, y_0) = (x_0, z_0) = (y_{0,}, z_0) = 1$:

$$x_0^4 + y_0^4 = z_0^4$$

Therefore $(x_0^2, y_0^2, z_0^2) \in$ Tpp (primitive Pythagorean triple), from which it follows that:

.$z_0 \in D$ e $x_0 \not\equiv y_0 \ mod2$. I assume $x_0 \in$ D.

■ D = set of odd numbers;

■ P = set of even numbers.

Now :

$$x_0^4 + y_0^4 = z_0^4 \quad \rightarrow \quad (x_0^2 + y_0^2)^2 - 2x_0^2 y_0^2 = z_0^4 \quad \rightarrow$$

$$(x_0^2 + y_0^2)^2 = 2x_0^2 y_0^2 + z_0^4$$

and, operating the following substitution of the variables, we obtain:

$$\begin{cases} z_0^2 = X_0 \in \mathbf{D} \\ x_0 \, y_0 = \mathrm{Y}_0 \in \mathbf{P} \\ x_0^2 + y_0^2 = \mathrm{Z}_0 \in \mathbf{D} \end{cases}$$

Thus the above equation becomes:

$$X_0^2 + 2Y_0^2 = Z_0^2$$

of which ($X0$, $Y0$,$Z0$) constitutes a primitive solution in the ring of positive integers. It is clear that ($X0$, $Y0$) = 1, but it is easy to prove that the remaining pairs are also coprime: ($X0$, $Z0$) = ($Y0$, $Z0$) = **1** . If indeed it were ($X0$,$Z0$) =p ∈ D > p | 2$Y0$ from which it would immediately follow that p|$\boldsymbol{X0} \wedge \boldsymbol{p}| \boldsymbol{Y0}$, so it turns out that p = 1.

If also $(\boldsymbol{Y0}, \boldsymbol{Z0}) = \boldsymbol{q} \in \boldsymbol{Z}$> q | $\boldsymbol{X0}$, but this also implies that :

q | $\boldsymbol{Y0} \wedge \boldsymbol{q}$ |$\boldsymbol{Z0}$,then $(\boldsymbol{X0}, \boldsymbol{Y0}) > 1$. In this way I proved that :

$$(\boldsymbol{X0}, \boldsymbol{Y0}) = (\boldsymbol{X0}, \boldsymbol{Z0}) = (\boldsymbol{Y0}, \boldsymbol{Z0}) = 1 .$$

From lemma 1 it follows that:

$$\exists a,b \in \boldsymbol{Z} : ab>\boldsymbol{0} \wedge (a,b)=1 , a>b\sqrt{\boldsymbol{2}} :$$

$$\begin{cases} \boldsymbol{X0 = a^2 - 2b^2} \\ \boldsymbol{Y0 = 2ab} \\ \boldsymbol{Z0 = a^2 + 2b^2} \end{cases} \Rightarrow \begin{cases} \boldsymbol{z_0^2 = a^2 - 2b^2} \\ \boldsymbol{x_0 y_0 = 2ab} \\ \boldsymbol{x_0^2 + y_0^2 = a^2 + 2b^2} \end{cases}$$

If $\boldsymbol{Y0} = \boldsymbol{2ab}$ then $\boldsymbol{x0y0} = \boldsymbol{2ab}$ and $\boldsymbol{Z0} \in \boldsymbol{D}$ it is not difficult to prove (lemma 3) that $(\boldsymbol{x0}, \boldsymbol{2b}) = \boldsymbol{1}$ and $(\boldsymbol{a}, \boldsymbol{y0}) = \boldsymbol{1}$, remembering that $\boldsymbol{x0} \in \boldsymbol{D}$ it follows that we can immediately posit :

$$\boldsymbol{x0} = \boldsymbol{a} \wedge \boldsymbol{y0} = \boldsymbol{2b} \Rightarrow$$

$$\boldsymbol{x_0^2 + y_0^2 = a^2 + 2b^2} \Rightarrow$$

$$\boldsymbol{a^2 + 4b^2 = a^2 + 2b^2} \Rightarrow \boldsymbol{2b^2 = 0} \Rightarrow \boldsymbol{b=0}$$

This condition implies that the system admits only trivial solutions as we wanted to demonstrate.

Ultimately, Fermat's equation for n = 4 is impossible in the ring of integers, just as we wanted to prove without using "infinite descent". It is also interesting to note that a quadratic equation such as: X2 + 2Y2 = Z2 can be reduced to Fermat's biquadratic equation through the following substitutions:

$$\boldsymbol{X} = \boldsymbol{z2} , \boldsymbol{Y} = \boldsymbol{xy} , \boldsymbol{Z} = \boldsymbol{x2} + \boldsymbol{y2}$$

z4 + 2x2y2 = x4 + y4 + 2x2y2

(z2)2 + 2(xy)2 = (x2 + y2)2

x4 +y4 =z4,

which is just the UTF equation for n = 4.

Lemma 3 :

The following system defined on integers:

$$\begin{cases} x_0 y_0 = 2ab \\ x_0^2 + y_0^2 = a^2 + 2b^2 \\ (\mathrm{x0y0}\,) \;=\; (\mathrm{a,b}) \;=\; 1 \end{cases}$$

con $x_0, a \in \mathrm{D} \;\wedge\; y_0, b \in P$

admits only trivial solutions in Z^+ **.**

■ Proof

My assumption is that:

$(x_0\,, b\,) = d_1$ e $(y_0, a) = d_2 \;:\; d_1, d_2 \in D$..

Thus I can write that:

$$x_0 = d_1 X\,,\; y_0 = d_2 Y$$
$$a = d_2 A\;\;, b = d_1 B$$

from which it follows that:

$$(d_1, d_2) = (X,B)=(Y,A)=(A,B) =1$$

Also, if

$$x_0 y_0 = 2ab \;\rightarrow\; XY = 2AB$$

from which it follows that:

X=A and Y= 2B

In this way it can be written that:

$$x_0 = d_1 A\,,\; y_0 = 2 d_2 B$$
$$a = d_2 A\;\;, b = d_1 B \qquad \textbf{with}$$

$$d_1, d_2, A \in D \;\wedge\; B \in P$$

Thus, replacing everything in the relationship :

$$x_0^2 + y_0^2 = a^2 + 2b^2\,,$$

it is obtained that:

$$d_1^2 A^2 + 4 d_2^2 B^2 = d_2^2 A^2 + 2 d_1^2 B^2 \;\rightarrow$$

$$(d_1^2 - d_2^2)A^2 = (2d_1^2 - 4d_2^2)B^2$$

Now the two GCD d_1, d_2 are both odd and for them only the following two cases can occur:

1) $d_1 \neq d_2 \wedge (d_1, d_2) = 1$;
2) $d_1 = d_2 = 1$.

In the first case you can proceed as follows:

$$\begin{cases} d_1^2 - d_2^2 = t\,B^2 \\ 2d_1^2 - 4d_2^2 = tA^2 \end{cases} ,$$

but $2d_1^2 - 4d_2^2 = tA^2$ ➔

$d_1^2 - 2d_2^2 = \frac{t}{2}A^2$ ➔ **t/2 = s** $\in D$ ➔ $\begin{cases} d_1^2 - d_2^2 = 2s\,B^2 \\ d_1^2 - 2d_2^2 = sA^2 \end{cases}$ ➔

$$d_2^2 = s(2B2 - A2) \wedge d_1^2 = s(4B2 - A2)$$

but $(d_1, d_2) = 1$ ➔ **s=1** .

This leads to the following solution system:

$$\begin{cases} d_1^2 - d_2^2 = 2\,B^2 \\ d_1^2 - 2d_2^2 = A^2 \end{cases}$$

which can also be written as:

$$\begin{cases} d_1^2 = d_2^2 + 2\,B^2 \\ d_1^2 = 2d_2^2 + A^2 \end{cases}$$

As you can see the second equation is impossible since d_2∈D is not even as it should be!

Among other things, substituting $d_2^2 = d_1^2 - 2\,B^2$ in the second equation and performing simple calculations, we arrive at the following equation:

$$d_1^2 + A^2 = 4\,B^2$$

It is impossible due to the already known properties of pythagorean triples.

It can therefore be concluded that assuming the first case is valid,

$$d_1 \neq d_2 \wedge (d_1, d_2) = 1$$

It comes to an absurdity!!

The only remaining possibility is that:

$$d_1 = d_2 = 1$$

The direct implication of which is that:

$$x_0 = a \quad \wedge \quad y_0 = 2b$$

with the conclusion that proves the theorem.

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