The Generic Action Principles of Conformal Supergravity

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Abstract

All $N=4$ conformal supergravities in four space-time dimensions are constructed. These are the only $N=4$ supergravity theories whose actions are invariant under off-shell supersymmetry. They are encoded in terms of a holomorphic function that is homogeneous of zeroth degree in scalar fields that parametrize an $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset space. When this function equals a constant the Lagrangian is invariant under continuous $\mathrm{SU}(1,1)$ transformations. Based on the known non-linear transformation rules of the Weyl multiplet fields, the action of $N = 4$ conformal supergravity is constructed up to terms quadratic in the fermion fields. The bosonic sector corrects a recent result in the literature. The construction of these higher-derivative invariants also opens the door to various applications for non-conformal theories.

1 Introduction

Conformal supergravities in four dimensions are invariant under the local symmetries associated with the superconformal algebra $\mathfrak{su}(2,2|N)$. The transformation rules and corresponding invariant Lagrangians are known for $N = 1$ and 2 [1, 2]. For the $N = 4$ theory, the Weyl multiplet and its full non-linear transformations were determined in [3]. A unique feature of the latter theory is the presence of scalars fields which parametrize an $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset space. This U(1) factor extends the SU(4) R-symmetry to the U(4) that is generically present in the algebra [4]. Furthermore, it was shown that $N > 4$ theories cannot exist off-shell [5], as they would necessarily involve higher-spin fields and the supermultiplet would in general not contain the graviton. It is also worth pointing out that the $N \leq 4$ superconformal field representation and the transformation rules have been worked out in superspace [6]. Although the field representation and its off-shell transformation rules are known, the full non-linear action for $N = 4$ conformal supergravity remains to be constructed. Recently, a calculation was performed based on an on-shell $N = 4$ abelian gauge theory in a conformal supergravity background [7]. The integration of the abelian gauge multiplet led to the determination of the bosonic terms of the superconformal action [8]. These terms comprise the square of the Weyl tensor and are related to the conformal anomaly, as was discussed long ago in [9]. The resulting action is invariant under a continuous rigid $\mathrm{SU}(1,1)$ symmetry, which can be explained by the fact that the gauge theory action has $\mathrm{SU}(1,1)$ as an electric-magnetic duality group. In this paper we calculate the $\mathrm{SU}(1,1)$ invariant action of $N = 4$ conformal supergravity by exploiting the known transformation rules and imposing supersymmetry by iteration. This computation is of interest since it completes the result of [8] to quadratic order in the fermion fields. However, we also find that our results do not coincide. Actually, string theory indicates the existence of an extended class of actions in which the continuous $\mathrm{SU}(1,1)$ is broken. For instance, in IIA string compactifications on $\mathrm{K3} \times T^2$, the effective action contains terms quadratic in the Weyl tensor and its dual, multiplied by a modular function [10]. Further indications arise from the semiclassical approximation of the microscopic degeneracy formula for dyonic BPS black holes [11, 12, 13], which captures corrections to the macroscopic entropy originating from the same class of actions. This paper deals exclusively with the construction of the action invariant under the continuous $\mathrm{SU}(1,1)$. Conformal supergravity is the supersymmetric generalization of conformal gravity, whose Lagrangian is the square of the Weyl tensor. The combination of local supersymmetry and conformal symmetry necessarily implies the presence of additional local invariances, which include a special supersymmetry known as S-supersymmetry. The original supersymmetry is then called Q-supersymmetry. For $N=4$ the full non-linear transformation rules of the fields, which constitute the so-called Weyl supermultiplet, have been determined [3]. This is the largest possible conformal supergravity that can exist in four space-time dimensions [5], and so far a complete Lagrangian was not known. A unique feature is the presence of dimensionless scalar fields that parametrize an $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset space. The U(1) factor is realized as a local symmetry with a composite connection, which acts chirally on the fermions. Hence the so-called R-symmetry group is extended to $\mathrm{SU}(4)\times\mathrm{U}(1)$. As explained below there are good reasons to expect that a large variety of these theories will exist. This Letter reports important progress on this question as we derive the most general invariant Lagrangian, which turns out to depend on a single arbitrary holomorphic and homogeneous function of the coset fields. Here we will present its purely bosonic terms; full results will be reported elsewhere. When this function is constant these bosonic terms turn out to agree with a recent result derived by imposing supersymmetry on terms that are at most quadratic in the fermions. In the Lagrangian this function will, for instance, multiply the terms quadratic in the Weyl tensor. The possible existence of such a non-minimal coupling was suggested long ago in the literature. Meanwhile indirect evidence came from string theory, where the threshold corrections in the effective action of IIA string compactifications on K3 $\times\, T^2$ reveal the presence of terms proportional to the square of the Weyl tensor multiplied by a modular function [10]. The same terms emerge in the semiclassical approximation of microscopic degeneracy formulae for dyonic BPS black holes. Finally higher-derivative couplings derived for $N\!=\!4$ Poincaré supergravity do also exhibit non-trivial scalar interactions. The results of this Letter can provide more detailed information on such higher-derivative interactions. Likewise they can be utilized to study the subleading contributions to $N\!=\!4$ BPS black hole entropy in a fully supersymmetric description.

In this paper, we take a more pragmatic approach which relies on the construction of a generic supersymmetric action principle, also known as a *density formula*, directly at the component level. Such a density formula is built upon an abstract multiplet whose component fields appear linearly in the expression, along with some of the supergravity fields such as the vielbein. The multiplet in question is typically required to obey only very mild supersymmetry constraints (e.g. chirality), and it is in this sense that the action principle is generic. If one can build such a multiplet, for example by combining more fundamental constituents, the action principle can be applied. This approach is not distinct from superspace: whenever an invariant superspace exists, the corresponding component action always falls into this type, where the multiplet in question is identified with the superspace Lagrangian. The converse does not generally hold, and a density formula may exist in the absence of a corresponding superspace. As we will show, there exists such a density formula for four-dimensional $\mathcal{N} = 4$ theories with local superconformal symmetry. It is based on an abstract multiplet involving superconformally primary fields $C^{ij}{}_{kl}$, $\bar{C}^{ij}{}_{kl} = (C^{kl}{}_{ij})^*$, and $A^{ij}{}_{kl} = (A^{kl}{}_{ij})^*$, each in the $\mathbf{20'}$ of the SU(4) R-symmetry group, and whose supersymmetry transformations into the $\mathbf{60}$ and $\overline{\mathbf{60}}$ are constrained. That is,

$$\begin{aligned} \delta_\epsilon C^{ij}{}_{kl} &= \bar\epsilon^m\, \Xi^{ij}{}_{kl,m} + \bar\epsilon_m\, \Xi^{ij,m}{}_{kl}\,, \\ \delta_\epsilon A^{ij}{}_{kl} &= \bar\epsilon^m\, \Omega^{ij}{}_{kl,m} + \bar\epsilon_m\, \Omega^{ij,m}{}_{kl}\,, \end{aligned} \tag{1.1}$$

for fermions Ξ and Ω whose traceless parts are fixed as

$$[\Xi^{ij}{}_{kl,m}]_{\overline{\mathbf{60}}} = [2\Lambda_m A^{ij}{}_{kl}]_{\overline{\mathbf{60}}}\;, \qquad [\Xi^{ij,m}{}_{kl}]_{\mathbf{60}} = 0\;, \qquad [\Omega^{ij}{}_{kl,m}]_{\overline{\mathbf{60}}} = [\Lambda_m \bar{C}^{ij}{}_{kl}]_{\overline{\mathbf{60}}} \tag{1.2}$$

in terms of the dimension-1/2 fermion Λ_i of the Weyl multiplet. This implies that the fields $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$ must be intricately related. The density formula is

$$\begin{aligned} e^{-1}\mathcal{L} = {}& F + 2\,\bar\psi_{\mu i}(\Omega^{\mu i} + \gamma^\mu\Omega^i) + 2\,\bar\psi_\mu{}^i(\Omega^\mu{}_i + \gamma^\mu\Omega_i) \\ &+ \frac{1}{8}\bar\psi_{[\mu i}\gamma^\mu\psi_{\nu]}{}^j\;{}^{\nu i}{}_j - \frac{1}{8}\Big(i\,\bar\psi_{\mu i}\gamma^{\mu\nu}\psi^{ij}_{\nu j} + \frac{i}{2}{}^{\mu\nu\rho\sigma}\;\bar\psi_{\mu i}\psi_{\nu j\,\rho\sigma}{}^{ij} + \text{h.c.}\Big) \\ &+ \frac{i}{8}{}^{\mu\nu\rho\sigma}\;\Big(\bar\psi_{\mu i}\psi_{\nu j}\,\bar\psi_{\rho k}\gamma_\sigma\rho_{rs}{}^{k\;ijrs} + 2\,\bar\psi_{\mu i}\psi_{\nu j}\,\bar\psi_\rho{}^k\gamma_\sigma\kappa^{ij}{}_k + \text{h.c.}\Big) \\ &- \frac{i}{4}{}^{\mu\nu\rho\sigma}\;\Big(\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_{\rho k}\psi_{\sigma l}\;{}^{klrs}\,C^{ij}{}_{rs} + 2\,\bar\psi_{\mu i}\psi_{\nu j}\bar\psi_\rho{}^k\psi_\sigma{}^l\,A^{ij}{}_{kl} + \text{h.c.}\Big)\,. \end{aligned} \tag{1.3}$$

The fields $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$ of lowest Weyl weight multiply four gravitini ψ, while higher weight fields of the abstract multiplet appear with fewer gravitini: these include fermions $\rho_{ij}{}^k$, $\kappa^{ij}{}_k$, Ω_{ai} and Ω_i, as well as bosons ${}_a{}^i{}_j$, ${}_{ab}{}^{ij}$, ^{ij}, and F. They descend from the fields $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$ via supersymmetry in a manner that will be described in due course. Their superconformal transformations, which leave (1.3) invariant, turn out to be determined entirely by the basic supersymmetry constraints (1.2). Provided such fields can be constructed out of more fundamental constituents, invariant actions follow. The composite field F then contains all the bosonic terms of these actions. This is by no means the only possible density formula for $\mathcal{N} = 4$, but it turns out to be sufficient for our needs. Once one specifies the form of the basic fields $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$ in terms of the $\mathcal{N} = 4$ Weyl multiplet fields, it allows the direct construction of the class of superconformal Weyl squared actions which depend on a generic holomorphic function.

2 $N = 4$ conformal supergravity

$N = 4$ conformal supergravity [3] is built upon the gauging of the superconformal algebra $\mathfrak{su}(2,2|4)$. Its bosonic subalgebra contains the generators of the conformal group SU(2,2) and the generators of a chiral SU(4) R-symmetry. The fermionic generators consist of sixteen Q supercharges and sixteen S supercharges. In addition, the theory has a non-linearly realised rigid SU(1,1) symmetry and a local chiral U(1) symmetry. The latter extends the R-symmetry group to SU(4) × U(1). The field representation of the theory comprises the gauge fields associated to the various superconformal symmetries and the local U(1), as well as a set of matter fields. In this paper, we adopt the conventions of [3], unless stated otherwise.

The bosonic gauge fields associated to the SU(2,2|4) symmetries are the vierbein ${e_\mu}^a$, the spin connection ${\omega_\mu}^{ab}$, the dilatational gauge field b_μ, the conformal boost gauge field ${f_\mu}^a$ and the SU(4) gauge field ${V_\mu}^i{}_j$, while the fermionic ones are the Q- and S-supersymmetry gauge fields ${\psi_\mu}^i$ and ${\phi_\mu}^i$, respectively. Finally, the connection a_μ is associated with the local chiral U(1) symmetry. The complete set of gauge fields of $N = 4$ conformal supergravity is listed in table 1 along with their algebraic restrictions, their SU(4) representation, their weight w under local dilatations and their U(1) chiral weight c.

Table 1: Gauge fields of $N = 4$ conformal supergravity

	Field	Symmetries (Generators)	Name/Restrictions	SU(4)	w	c
8*Bosons	${e_\mu}^a$	Translations (P)	vierbein	1	-1	0
	${\omega_\mu}^{ab}$	Lorentz (M)	spin connection	1	0	0
	b_μ	Dilatation (D)	dilatational gauge field	1	0	0
	${V_\mu}^i{}_j$	SU(4) (V)	SU(4) gauge field	15	0	0
			${V_\mu}_i{}^j \equiv ({V_\mu}^i{}_j)^* = -{V_\mu}^j{}_i$			
			${V_\mu}^i{}_i = 0$			
	${f_\mu}^a$	Conformal boosts (K)	K-gauge field	1	1	0
	a_μ	U(1)	U(1) gauge field	1	0	0
3*Fermions	${\phi_\mu}^i$	S-supersymmetry (S)	S-gauge field	4	$\frac{1}{2}$	$-\frac{1}{2}$
			$\gamma_5\, {\phi_\mu}^i = -{\phi_\mu}^i$			
	${\psi_\mu}^i$	Q-supersymmetry (Q)	gravitino; $\gamma_5\, \psi^i_\mu = \psi^i_\mu$	4	$-\frac{1}{2}$	$-\frac{1}{2}$

The matter fields of the theory consist of three types of scalar fields $\phi_\alpha, E_{ij}, D^{ij}{}_{kl}$, an anti-symmetric tensor ${T_{ab}}^{ij}$ and two spin-1/2 fermions $\Lambda_i, \chi^{ij}{}_k$. We list them in table 2 with their various algebraic properties, and their representation assignments. The rigid SU(1,1) indices are denoted by $\alpha, \beta = 1, 2$.

An element of SU(1,1) can be written in terms of the doublet of complex scalars ϕ_α which satisfies

$$\phi^\alpha \phi_\alpha = 1\,, \tag{2.1}$$

where $\phi^\alpha \equiv \eta^{\alpha\beta}\phi^*_\beta$ with $\eta^{\alpha\beta} = \mathrm{diag}(+1, -1)$. Therefore, due to the presence of the local U(1), the scalars parametrise an SU(1,1)/U(1) coset.

Just as in ordinary gravity where the spin connection is a composite field, the gauge fields ${\omega_\mu}^{ab}$, ${f_\mu}^a$ and ${\phi_\mu}^i$ are expressed in terms of the other ones through a set of conventional constraints

Table 2: Matter fields of $N = 4$ conformal supergravity

	Field	Restrictions	SU(4)	w	c
7*Bosons	ϕ_α	$\phi^1 = \phi_1^*,\ \phi^2 = -\phi_2^*$	1	0	-1
	E_{ij}	$E_{ij} = E_{ji}$	10	1	-1
	$T_{ab}{}^{ij}$	$\frac{1}{2}\varepsilon_{ab}{}^{cd}T_{cd}{}^{ij} = -T_{ab}{}^{ij}$	6	1	-1
		$T_{ab}{}^{ij} = -T_{ab}{}^{ji}$			
	$D^{ij}{}_{kl}$	$D^{ij}{}_{kl} = \frac{1}{4}\varepsilon^{ijmn}\varepsilon_{klpq}D^{pq}{}_{mn}$	$20'$	2	0
		$D_{kl}{}^{ij} \equiv (D^{kl}{}_{ij})^* = D^{ij}{}_{kl}$			
		$D^{ij}{}_{kj} = 0$			
3*Fermions	Λ_i	$\gamma_5\Lambda_i = \Lambda_i$	4	$\frac{1}{2}$	$-\frac{3}{2}$
	$\chi^{ij}{}_k$	$\gamma_5\chi^{ij}{}_k = \chi^{ij}{}_k;\ \chi^{ij}{}_k = -\chi^{ji}{}_k$	20	$\frac{3}{2}$	$-\frac{1}{2}$
		$\chi^{ij}{}_j = 0$			

on the superconformal curvatures

$$\begin{aligned} R(P)_{\mu\nu}{}^a &= 0\,, \\ R(M)_{\mu\nu}{}^{ab}e^\nu{}_b &= 0\,, \\ \gamma^\mu R(Q)_{\mu\nu}{}^i &= 0\,. \end{aligned} \tag{2.2}$$

The U(1) gauge field a_μ is also composite and solves the supercovariant constraint

$$\phi^\alpha D_\mu \phi_\alpha = -\tfrac{1}{4}\bar\Lambda^i\gamma_\mu\Lambda_i\,. \tag{2.3}$$

The derivative D_μ is covariant with respect to all the gauge symmetries. By making use of the Bianchi identities for the curvatures, the constraints (2.2) lead to an additional set of identities which are summarised. The independent fields of tables 1 and 2 constitute the full Weyl supermultiplet of $N = 4$ conformal supergravity which contains $128 + 128$ off-shell degrees of freedom. The non-linear superconformal transformation rules of the fields were derived in [3]. The Q-supersymmetry transformations of the gauge fields read

$$\begin{aligned}
\delta_Q e_\mu{}^a =&\, \bar\epsilon^i\gamma^a\psi_{\mu i} + \text{h.c.}\,, \\
\delta_Q\psi_\mu{}^i =&\, 2\,\mathcal{D}_\mu\epsilon^i - \tfrac{1}{2}\gamma^{ab}T_{ab}{}^{ij}\gamma_\mu\epsilon_j + \varepsilon^{ijkl}\,\bar\psi_{\mu j}\epsilon_k\,\Lambda_l\,, \\
\delta_Q b_\mu =&\, \tfrac{1}{2}\bar\epsilon^i\phi_{\mu i} + \text{h.c.}\,, \\
\delta_Q V_\mu{}^i{}_j =&\, \bar\epsilon^i\phi_{\mu j} + \bar\epsilon^k\gamma_\mu\chi^i{}_{kj} - \tfrac{1}{2}\varepsilon_{jkmn}E^{ik}\,\bar\epsilon^m\psi_\mu{}^n - \tfrac{1}{6}E^{ik}\bar\epsilon_j\gamma_\mu\Lambda_k \\
&+ \tfrac{1}{4}\varepsilon^{iklm}\,T^{ab}{}_{lj}\,\bar\epsilon_k\gamma_{ab}\gamma_\mu\Lambda_m + \tfrac{1}{3}\bar\epsilon^i\gamma_\mu P\Lambda_j \\
&- \tfrac{1}{4}\varepsilon^{iklp}\varepsilon_{jmnp}\,\bar\epsilon^m\gamma_a\psi_{\mu k}\,\bar\Lambda_l\gamma^a\Lambda^n - (\text{h.c.; traceless})\,, \\
\delta_Q a_\mu =&\, \tfrac{1}{2}\mathrm{i}\bar\epsilon_i\gamma_\mu\bar P\Lambda^i + \tfrac{1}{4}\mathrm{i}E_{ij}\,\bar\Lambda^i\gamma_\mu\epsilon^j + \tfrac{1}{8}\mathrm{i}\varepsilon_{ijkl}\,T_{ab}{}^{kl}\,\bar\Lambda^i\gamma_\mu\gamma^{ab}\epsilon^j \\
&- \tfrac{1}{4}\mathrm{i}(\bar\Lambda^i\gamma_a\Lambda_j - \delta^i_j\,\bar\Lambda^k\gamma_a\Lambda_k)\,\bar\epsilon_i\gamma^a\psi_\mu{}^j + \text{h.c.}\,, \\
\delta_Q\omega_\mu{}^{ab} =&\, -\tfrac{1}{2}\bar\epsilon^i\gamma^{ab}\phi_{\mu i} + \bar\epsilon^i\gamma_\mu R(Q)^{ab}{}_i - 2\,T^{ab}{}_{ij}\,\bar\epsilon^i\psi_\mu{}^j + \text{h.c.}\,, \\
\delta_Q f_\mu{}^a =&\, -\tfrac{1}{8}e_{\mu b}\,\varepsilon^{abcd}\,\bar\epsilon_i R(S)_{cd}{}^i - \bar\epsilon_i\gamma_\mu D_b R(Q)^{ab\,i} - 2\,T_{\mu b}{}^{ij}\,\bar\epsilon_i R(Q)^{ab}{}_j \\
&+ \text{h.c.} + [\text{terms } \propto \psi_\mu]\,, \\
\delta_Q\phi_\mu{}^i =&\, -2f_\mu{}^a\gamma_a\epsilon^i + \tfrac{1}{4}T_{ab}{}^{ij}\,T^{cd}{}_{jk}\,\gamma_{cd}\gamma_\mu\gamma^{ab}\epsilon^k \\
&+ \tfrac{1}{6}\big[\gamma_\mu\gamma^{ab} - 3\,\gamma^{ab}\gamma_\mu\big]\big[R(V)_{ab}{}^i{}_j\,\epsilon^j + \tfrac{1}{2}\mathrm{i}F_{ab}\epsilon^i + \tfrac{1}{2}D_a T_{cd}{}^{ij}\gamma^{cd}\gamma_b\epsilon_j\big] \\
&+ [\text{terms } \propto \psi_\mu]\,,
\end{aligned} \tag{2.4}$$

while for the matter fields we have

$$
\begin{aligned}
\delta_Q\phi_\alpha &= -\,\bar\epsilon^i\Lambda_i\varepsilon_{\alpha\beta}\phi^\beta\,,\\
\delta_Q\bar P_a &= -\,\bar\epsilon^i D_a\Lambda_i - \tfrac14\bar\Lambda_i\gamma^{bc}T_{bc}{}^{ij}\gamma_a\epsilon_i - \tfrac12\bar\epsilon^i\Lambda_i\,\bar\Lambda^j\gamma_a\Lambda_j\,,\\
\delta_Q\Lambda_i &= -\,2\,\bar P\epsilon_i + E_{ij}\epsilon^j + \tfrac12\varepsilon_{ijkl}\,T_{bc}{}^{kl}\gamma^{bc}\,\epsilon^j\,,\\
\delta_Q E_{ij} &= 2\,\bar\epsilon_{(i}D\Lambda_{j)} - 2\,\bar\epsilon^k\chi^{mn}{}_{(i}\,\varepsilon_{j)kmn} - \bar\Lambda_i\Lambda_j\,\bar\epsilon_k\Lambda^k + 2\,\bar\Lambda_k\Lambda_{(i}\,\bar\epsilon_{j)}\Lambda^k\,,\\
\delta_Q T_{ab}{}^{ij} &= 2\,\bar\epsilon^{[i}R(Q)_{ab}{}^{j]} + \tfrac12\bar\epsilon^k\gamma_{ab}\chi^{ij}{}_k + \tfrac14\varepsilon^{ijkl}\,\bar\epsilon_k\gamma^c\gamma_{ab}D_c\Lambda_l - \tfrac16 E^{k[i}\,\bar\epsilon^{j]}\gamma_{ab}\Lambda_k + \tfrac13\bar\epsilon^{[i}\gamma_{ab}\bar P\Lambda^{j]}\,,\\
\delta_Q\chi^{ij}{}_k &= -\,\tfrac12\gamma^{ab}DT_{ab}{}^{ij}\epsilon_k - \gamma^{ab}R(V)_{ab}{}^{[i}{}_k\,\epsilon^{j]} - \tfrac12\varepsilon^{ijlm}\,DE_{kl}\,\epsilon_m + D^{ij}{}_{kl}\,\epsilon^l\\
&\quad - \tfrac16\varepsilon_{klmn}E^{l[i}\gamma^{ab}\big[T_{ab}{}^{j]n}\epsilon^m + T_{ab}{}^{mn}\epsilon^{j]}\big] + \tfrac12 E_{kl}\,E^{l[i}\,\epsilon^{j]} - \tfrac12\varepsilon^{ijlm}\,\bar P\gamma_{ab}T^{ab}{}_{kl}\,\epsilon_m\\
&\quad + \tfrac14\gamma^a\epsilon_n\big[2\,\varepsilon^{ijln}\bar\chi^m{}_{lk} - \varepsilon^{ijlm}\bar\chi^n{}_{lk}\big]\gamma_a\Lambda_m + \tfrac14\epsilon^{[i}\big[2\,\bar\Lambda^{j]}D\Lambda_k + \bar\Lambda_k D\Lambda^{j]}\big]\\
&\quad - \tfrac14\gamma^{ab}\epsilon^{[i}\big[2\,\bar\Lambda^{j]}\gamma_a D_b\Lambda_k - \bar\Lambda_k\gamma_a D_b\Lambda^{j]}\big] - \tfrac{5}{12}\varepsilon^{ijlm}\Lambda_m\,\bar\epsilon_l\big[E_{kn}\Lambda^n - 2\,P\Lambda_k\big]\\
&\quad + \tfrac{1}{12}\varepsilon^{ijlm}\Lambda_m\,\bar\epsilon_k\,\big[E_{ln}\Lambda^n - 2\,P\Lambda_l\big] - \tfrac12\gamma^{ab}T_{ab}{}^{ij}\gamma^c\epsilon_{[k}\,\bar\Lambda^l\gamma_c\Lambda_{l]}\\
&\quad - \tfrac12\gamma^{ab}T_{ab}{}^{l[i}\gamma^c\epsilon_{[k}\,\bar\Lambda^{j]}\gamma_c\Lambda_{l]} + \tfrac12\epsilon^{[i}\bar\Lambda^{j]}\Lambda^m\,\bar\Lambda_k\Lambda_m - (\text{traces})\,,\\
\delta_Q D^{ij}{}_{kl} &= -\,4\bar\epsilon^{[i}D\chi^{j]}{}_{kl} + \varepsilon_{klmn}\,\bar\epsilon^{[i}\big[-2\,E^{j]p}\chi^{mn}{}_p + \tfrac12\gamma^{ab}T_{ab}{}^{mn}\overset{\leftrightarrow}{D}\Lambda^{j]} + \tfrac13 E^{j]m}E^{np}\Lambda_p\\
&\qquad - \tfrac23\bar P\Lambda^m E^{j]n} + \tfrac12\gamma^{ab}T_{ab}{}^{mn}\Lambda_p\,\bar\Lambda^{j]}\Lambda^p\big]\\
&\quad + \bar\epsilon^{[i}\big[2\,\gamma^a\chi^m{}_{kl}\,\bar\Lambda^{j]}\gamma_a\Lambda_m + 2\,\bar P\,\gamma_{ab}T^{ab}{}_{kl}\,\Lambda^{j]} + \tfrac23\Lambda_{[k}E_{l]m}\,\bar\Lambda^{j]}\Lambda^m + \tfrac16\gamma^{ab}P\Lambda^{j]}\,\bar\Lambda_k\gamma_{ab}\Lambda_l\big]\\
&\qquad + \varepsilon^{ijmn}\,\bar\epsilon^p\,T^{ab}{}_{kl}\big[2\,T_{ab\,np}\Lambda_m + T_{ab\,mn}\Lambda_p\big] + (\text{h.c.; traceless})\,.
\end{aligned}
\tag{2.5}
$$

where ϵ^i is the Q-supersymmetry parameter and where $\mathcal{D}_\mu$ is covariant with respect to the all the bosonic symmetries except the conformal boosts. For instance, we have

$$
\begin{aligned}
\mathcal{D}_\mu\epsilon^i &= \big[\partial_\mu - \tfrac14\omega_\mu{}^{ab}\gamma_{ab} + \tfrac12(b_\mu + \mathrm{i}a_\mu)\big]\epsilon^i - V_\mu{}^i{}_j\,\epsilon^j\,,\\
\mathcal{D}_\mu\eta^i &= \big[\partial_\mu - \tfrac14\omega_\mu{}^{ab}\gamma_{ab} - \tfrac12(b_\mu - \mathrm{i}a_\mu)\big]\eta^i - V_\mu{}^i{}_j\,\eta^j\,,
\end{aligned}
\tag{2.6}
$$

The S-supersymmetry transformations of the fields are

$$
\begin{aligned}
\delta_S e_\mu{}^a &= 0\,,\\
\delta_S\psi_\mu{}^i &= -\,\gamma_\mu\eta^i\\
\delta_S b_\mu &= -\,\tfrac12\bar\psi_\mu{}^i\,\eta_i + \text{h.c.}\,,\\
\delta_S V_\mu{}^i{}_j &= -\,\big[\bar\psi_\mu{}^i\eta_j - \tfrac14\delta^i{}_j\,\bar\psi_\mu{}^k\eta_k\big] - \text{h.c.}\,,\\
\delta_S a_\mu &= 0\,,\\
\delta_S\omega_\mu{}^{ab} &= \tfrac12\bar\psi_\mu{}^i\gamma^{ab}\eta_i + \text{h.c.}\,,\\
\delta_S f_\mu{}^a &= \tfrac12\bar\eta_i\gamma^a\phi_\mu{}^i - \tfrac14\bar\eta_i R(Q)_\mu{}^{a\,i} + \tfrac{1}{12}\bar\eta_i\gamma^{bc}T_{bc}{}^{ij}\gamma^a\psi_{\mu j} + \text{h.c.}\,,\\
\delta_S\phi_\mu{}^i &= 2\,\mathcal{D}_\mu\eta^i - \tfrac16\gamma_\mu\gamma^{ab}T_{ab}{}^{ij}\eta_j + \tfrac12\varepsilon^{ijkl}\bar\eta_k\Lambda_l\psi_{\mu j}\,,\\
\delta_S\phi_\alpha &= 0\\
\delta_S\bar P_a &= -\,\tfrac12\bar\eta^i\gamma_a\Lambda_i\,,\\
\delta_S\Lambda_i &= 0\,,\\
\delta_S E_{ij} &= 2\,\bar\eta_{(i}\Lambda_{j)}\,,\\
\delta_S T_{ab}{}^{ij} &= -\,\tfrac14\varepsilon^{ijkl}\,\bar\eta_k\gamma_{ab}\Lambda_l\,,\\
\delta_S\chi^{ij}{}_k &= \tfrac12 T_{ab}{}^{ij}\,\gamma^{ab}\eta_k + \tfrac23\delta^{[i}_k\,T_{ab}{}^{j]l}\,\gamma^{ab}\eta_l - \tfrac12\varepsilon^{ijlm}E_{kl}\,\eta_m - \tfrac14\bar\Lambda_k\gamma^a\Lambda^{[i}\gamma_a\eta^{j]}\\
&\quad + \tfrac{1}{12}\delta^{[i}_k\big[\bar\Lambda_l\gamma^a\Lambda^l\gamma_a\eta^{j]} - \bar\Lambda_l\gamma^a\Lambda^{j]}\,\gamma_a\eta^l\big]\,,\\
\delta_S D^{ij}{}_{kl} &= 0\,.
\end{aligned}
\tag{2.7}
$$

As is clear from (2.4), (2.5) and (2.7), the coset space sector of the theory can be entirely described in terms of P_μ and $F_{\mu\nu}$. In what follows, we will make use of these $\mathrm{SU}(1,1)$ invariant quantities rather than the scalars ϕ_α. Note also that P_a has Weyl weight $w=1$ and is invariant under K-transformations. We finally present several identities which will be useful in the next sections. One can respectively derive

$$\begin{aligned}\varepsilon_{\beta\gamma} D_a\phi^\beta D_b\phi^\gamma &= 2\,\phi_\alpha D_{[a}\phi^\alpha \varepsilon_{\beta\gamma}\phi^\beta D_{b]}\phi^\gamma = \tfrac{1}{2}\bar\Lambda^i\gamma_{[a}\Lambda_i P_{b]}\,,\\ D_a\phi^\alpha D_b\phi_\alpha &= -\,P_a\bar P_b - \tfrac{1}{16}\bar\Lambda^i\gamma_a\Lambda_i\bar\Lambda^j\gamma_b\Lambda_j\,.\end{aligned}\tag{2.8}$$

It follows that

$$F_{ab} = 2\mathrm{i}\bar P_{[a}P_{b]} - \tfrac{1}{2}\mathrm{i}\big[\bar\Lambda^i\gamma_{[a}D_{b]}\Lambda_i - \text{h.c.}\big]\,,\tag{2.9}$$

$$D_{[a}\bar P_{b]} = \tfrac{1}{2}\bar\Lambda_i\gamma_{[a}\Lambda^i\bar P_{b]} + \tfrac{1}{4}\bar\Lambda_i R(Q)_{ab}{}^i\,,\tag{2.10}$$

which are the supersymmetric generalisations of the Maurer-Cartan equations associated with the coset space $\mathrm{SU}(1,1)/\mathrm{U}(1)$.

The gauge fields $\omega_\mu{}^{ab}$, $f_\mu{}^a$ and $\phi_\mu{}^a$ are composite. They are expressed in terms of the other fields through the set of constraints (2.2). The latter, when combined with the superconformal Bianchi identities, lead to the following useful relations

$$\begin{aligned}R(D)_{ab} &= 0\,,\\ R(M)_{abcd} &= R(M)_{cdab}\,,\\ \varepsilon^{aecd}R(M)_{cdeb} &= 0\,,\\ \tfrac{1}{4}\varepsilon^{abcd}\varepsilon^{efgh}R(M)_{cdgh} &= R(M)^{abef}\,,\\ \varepsilon^{cdef}D_bD_dR(M)_{efab} &= 0\,,\\ R(K)_{ab}{}^c &= D_eR(M)_{ab}{}^{ec}\,,\\ \varepsilon^{abcd}D_bR(V)_{cd}{}^i{}_j &= -\,\tfrac{1}{4}\varepsilon^{iklm}\bar\Lambda_m\gamma_b\gamma\cdot T_{jl}R(Q)^{ab\,k} - (\text{h.c.; traceless})\,,\\ D_aR(Q)^{ab\,i} &= -\,\tfrac{1}{4}\varepsilon^{abcd}\gamma_aR(S)_{cd}{}^i\\ R(Q)^{+\,i}_{ab} &= 0\,,\\ R(S)^{-\,i}_{ab} &= \not{D}R(Q)_{ab}{}^i\,,\\ \gamma^{ab}R(S)_{ab}{}^i &= 0\,,\\ \gamma^aR(S)^{+\,i}_{ab} &= 0\,,\\ \varepsilon^{abcd}D_bR(S)_{cd}{}^i &= -\,\tfrac{1}{3}\gamma^aT^{ij}\cdot R(S)_j - \tfrac{4}{3}T^{abij}D^dR(Q)_{db\,j} - \tfrac{1}{3}\gamma^aR(V)^i{}_j.R(Q)^j\\ &\quad -\tfrac{1}{6}\mathrm{i}\gamma^aF\cdot R(Q)^i + \tfrac{4}{3}D^gT_{gc}{}^{ij}R(Q)^{ac}{}_j - \tfrac{1}{4}\gamma\cdot T_{jk}\gamma^aT^{ij}.R(Q)^k\,.\end{aligned}\tag{2.1}$$

Note however that these relations are not independent. We recall that the (anti-)self dual part of a curvature is defined here as $R^\pm_{ab} = \frac{1}{2}(R_{ab}\pm\frac{1}{2}\varepsilon_{abcd}R^{cd})$. The transformations of $R_{ab}{}^c(K)$ and $R_{ab}{}^{i(-)}(S)$ can be easily derived from (2.1). Finally, for the purpose of section 4, we give the explicit expressions of the fermionic supercovariant curvatures

$$R(Q)_{\mu\nu}{}^i = 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \gamma_{[\mu}\phi_{\nu]}{}^i - \tfrac{1}{2}\gamma\cdot T^{ij}\gamma_{[\mu}\psi_{\nu]j} + \tfrac{1}{2}\varepsilon^{ijkl}\bar\psi_{\mu j}\psi_{\nu k}\Lambda_l\tag{2.2}$$

$$\begin{aligned}R(S)_{\mu\nu}{}^i &= 2\,\mathcal{D}_{[\mu}\phi_{\nu]}{}^i - 2f_{[\mu}{}^a\gamma_a\psi_{\nu]}{}^i - \tfrac{1}{6}\gamma_{[\mu}\gamma\cdot T^{ij}\phi_{\nu]j} - \tfrac{1}{2}\varepsilon^{ijkl}\bar\phi_{[\mu k}\Lambda_l\psi_{\nu]j} + \delta^{(cov)}_{Q|_{\psi_{[\nu}}}\phi_{\mu]}{}^i\\ &\quad + [\text{terms}\propto\psi^2]\end{aligned}\tag{2.3}$$

where the symbol $\delta^{(cov)}_{Q|_{\psi_b}}$ denotes the supercovariant part of a Q-variation with the parameter replaced by the gravitino.

The Q-supersymmetry and S-supersymmetry transformations of the supercovariant curvatures are

$$
\begin{aligned}
\delta_Q R(M)_{abcd} =& -\tfrac{1}{4}\bar\epsilon^i\gamma_{ab}R(S)^-_{cdi} - \tfrac{1}{4}\bar\epsilon^i\gamma_{cd}R(S)^-_{abi} + \tfrac{1}{4}\bar\epsilon^i\, \not{D}\gamma_{ab}R(Q)_{cdi} + \tfrac{1}{4}\bar\epsilon^i\, \not{D}\gamma_{cd}R(Q)_{abi} + \text{h.c.}\,,\\
\delta_Q R(Q)_{ab}{}^i =& -\tfrac{1}{2}R(M)_{abcd}\gamma^{cd}\epsilon^i + \tfrac{1}{4}\big[\gamma^{cd}\gamma_{ab} + \tfrac{1}{3}\gamma_{ab}\gamma^{cd}\big]\big[R(V)_{cd}{}^i{}_j\epsilon^j + \tfrac{1}{2}\mathrm{i}F_{cd}\epsilon^i + \not{D}T_{cd}{}^{ij}\epsilon_j\big]\,,\\
\delta_Q R(V)_{ab}{}^i{}_j =& \,\bar\epsilon^i R(S)_{abj} - 2\bar\epsilon^k\gamma_{[a}D_{b]}\chi^i{}_{kj} + 2\bar\epsilon_l\chi^i{}_{kj}T_{ab}{}^{kl} + \tfrac{1}{3}T_{abjl}\big[-2\,\bar\epsilon^l\bar P\Lambda^i - \bar\epsilon^l E^{ik}\Lambda_k\big]\\
&+ \tfrac{1}{8}\varepsilon^{iklm}\bar\epsilon^n\gamma_{[a}\gamma\cdot T_{kn}\gamma\cdot T_{lj}\gamma_{b]}\Lambda_m - \tfrac{1}{2}E^{ik}\varepsilon_{jkmn}\bar\epsilon^m R(Q)_{ab}{}^n\\
&- \tfrac{1}{4}\varepsilon^{iklp}\varepsilon_{jmnp}\bar\epsilon^m\gamma^c R(Q)_{abk}\bar\Lambda_l\gamma_c\Lambda^n + \tfrac{1}{3}\bar\epsilon_j\gamma_{[a}D_{b]}\big[E^{ik}\Lambda_k\big]\\
&+ \tfrac{1}{2}\varepsilon^{iklm}\bar\epsilon_k D_{[a}\big[\gamma\cdot T_{lj}\gamma_{b]}\Lambda_m\big] + \tfrac{2}{3}\bar\epsilon_j\gamma_{[a}D_{b]}\big[\bar P\Lambda^i\big] - (\text{h.c.; traceless})\,,\\
\delta_Q R(S)^{+i}_{ab} =& -2D^c R(M)^+{}_{abcd}\,g a^d\epsilon^i + \tfrac{1}{4}\big[\gamma^{cd}\gamma_{ab} + \tfrac{1}{3}\gamma_{ab}\gamma^{cd}\big]\big[\gamma^e\epsilon^j D_e R_{cd}{}^i{}_j + \tfrac{1}{2}\mathrm{i}\gamma^e\epsilon^j D_e F_{cd}\\
&+ \epsilon_j D_c D^e T_{ed}{}^{ij} + 2\epsilon_l T_{cdjk}T^{ij}.T^{kl} - 4\gamma^e\epsilon^k D^f T_{fe}{}^{ij}T_{cdjk} - 2\gamma^e\epsilon^k D^f T_{cdjk}T_{fe}{}^{ij}\big]\,,\\
\delta_S R(M)_{abcd} =& -\tfrac{3}{4}\bar\eta_i\gamma_{ab}R(Q)_{cd}{}^i - \tfrac{3}{4}\bar\eta_i\gamma_{cd}R(Q)_{ab}{}^i + \text{h.c.}\,,\\
\delta_S R(Q)_{ab}{}^i =& \,\tfrac{1}{2}\big[\gamma^{cd}\gamma_{ab} + \tfrac{1}{3}\gamma_{ab}\gamma^{cd}\big]T_{cd}{}^{ij}\eta_j\,,\\
\delta_S R(V)_{ab}{}^i{}_j =& \,\bar\eta^i R(Q)_{abj} + \varepsilon^{iklm}T_{abjl}\bar\eta_k\Lambda_m - \bar\eta^k\gamma_{ab}\chi^i{}_{kj} - \tfrac{1}{6}\bar\eta^i\gamma_{ab}\big[2\,\not{P}\Lambda_j - E_{jk}\Lambda^k\big]\\
&- (\text{h.c.; traceless})\,,\\
\delta_S R(S)^{+i}_{ab} =& -\tfrac{1}{2}R(M)_{abcd}\gamma^{cd}\eta^i + \tfrac{3}{4}\big[\gamma^{cd}\gamma_{ab} + \tfrac{1}{3}\gamma_{ab}\gamma^{cd}\big]\big[\eta^j R(V)_{cd}{}^i{}_j + \tfrac{1}{2}\mathrm{i}\eta^i F_{cd}\big]\,.
\end{aligned}
\tag{2.4}
$$

3 The quadratic action

In this section, we present the part of the action which is quadratic in the fields. It will be the starting point for the iterative procedure presented in section 4, which we will use to generate terms of higher-order in the fields. The action will be constructed such that all the derivatives and curvatures that appear are fully supercovariantized with respect to all the gauge transformations (bosonic as well as fermionic). Hence, we must insist that, throughout the paper, our counting of the fields always excludes the gauge fields which are implicitly contained within the supercovariant derivatives and curvatures.

The quadratic Lagrangian of $N=4$ conformal supergravity reads

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\mathrm{Q}} =& \,\tfrac{1}{2}R(M)^{abcd}R(M)^-_{abcd} + R(V)^{ab\,i}{}_j R(V)^-_{ab}{}^j{}_i\\
&- 4\,T_{ab}{}^{ij}D^a D_c T^{cb}{}_{ij} + \tfrac{1}{4}E_{ij}D^2E^{ij} + \frac{1}{8}D_{ij}{}^{kl}D_{kl}{}^{ij}\\
&- 2\,\bar P^a\big[D_a D^b P_b + D^2 P_a\big] - 2\,D^a P^b D_a\bar P_b - D^a P_a D^b\bar P_b\\
&+ \bar R(Q)_{ab}{}^i R(S)^{ab}{}_i - \bar\chi^{ij}{}_k\,\not{D}\chi^k{}_{ij} - \tfrac{1}{2}\bar\Lambda_i\left(D^2\,\not{D} + \not{D}D^2 - \not{D}^3\right)\Lambda^i\; + \text{h.c.}\,,
\end{aligned}
\tag{3.1}
$$

with $e = \det[e_\mu{}^a]$ and where the (anti)self-dual part of a generic second rank tensor R_{ab} is defined as $R^\pm_{ab} = \frac{1}{2}[R_{ab} \pm \frac{1}{2}\varepsilon_{abcd}R^{cd}]$. The expression (3.1) corresponds to the real part of the chiral invariant of the linearized theory given in [3]. The imaginary part of the chiral invariant is a total derivative.

The structures of the quadratic terms are uniquely fixed by requiring invariance under U(1), SU(4) and Lorentz symmetry, while the number of derivatives in each term is fixed by Weyl invariance. At the level of the action, the derivatives can be moved around using integration by parts at the expense of higher-order terms in the fermions. However, requiring K-invariance (i.e. under conformal boosts) fixes the position of the derivatives. Under these conditions, the quadratic terms for the fields $E^{ij}, T_{ab}{}^{ij}$ and Λ_i are uniquely determined. The case of the vectors P_μ is more subtle and will be discussed below.

The relative coefficients between the different quadratic terms are fixed by requiring Q-supersymmetry invariance at quadratic order in the fields. The K-invariance of the quadratic terms involving the vectors P_μ is not straightforward. Out of the four possible terms, all appearing in the Lagrangian (3.1), none is K-invariant. The two terms in which both derivatives act on the same field should not be treated as independent. Indeed, only their sum is relevant at quadratic order since their difference

$$D^2 P_a - D_a D^b P_b = D^b D_{[b} P_{a]} + [D^b, D_a] P_b \,, \tag{3.2}$$

is of higher-order in the fields due to (2.10). An arbitrary combination of the remaining three independent quadratic terms is generically not K-invariant. However, when considering the unique combination appearing in (3.1), one finds that it is K-invariant up to a term of higher-order in the fields

$$\begin{aligned}\delta_K\big[2\,\bar P^a\big(D_a D^b P_b + D^2 P_a\big)\\ + 2\, D_a P_b D^a \bar P^b + D_a P^a D_b \bar P^b + \text{h.c.}\big] = 4\,\Lambda_a^K \bar P_b D^{[b} P^{a]} + \text{h.c.}\,.\end{aligned} \tag{3.3}$$

Here Λ_a^K is the K-transformation parameter. We should emphasise that, at this point, requiring K-invariance of each of the supercovariant terms in the Lagrangian is not necessary. The advantage of imposing such a condition already at the level of the quadratic action is that terms with an explicit K-gauge field $f_\mu{}^a$ will not have to be introduced when deriving the interaction terms. This will be explained in section 4. Finally, it is important to emphasize that in this paper, we will exclusively consider the real part of the chiral invariant. Without this reality condition, the K-variation of the kinetic terms for P_μ is not of higher-order in the fields anymore and consequently, one is forced to introduce explicit K-gauge fields.

In view of their relevance for the present paper we first present some further details regarding the fields ϕ_α and ϕ^α, which we refer to as the holomorphic and the anti-holomorphic fields, respectively. The holomorphic fields carry U(1) charge equal to -1 and transform under Q-supersymmetry into the positive chirality spinors Λ_i, which themselves carry U(1) charge $-3/2$,

$$\phi_\alpha \to \mathrm{e}^{-\mathrm{i}\lambda(x)}\, \phi_\alpha \,, \qquad \delta\phi_\alpha = -\bar\epsilon^i \Lambda_i \, \varepsilon_{\alpha\beta}\, \phi^\beta \,. \tag{3.4}$$

The supercovariant constraint that determines the U(1) gauge field a_μ and the generalized supercovariant derivatives of the coset fields, P_a and $\bar P_a$, are defined by

$$\begin{aligned}\phi^\alpha D_a \phi_\alpha &= -\tfrac14 \bar\Lambda^i \gamma_a \Lambda_i \,,\\ P_a = \phi^\alpha\, \varepsilon_{\alpha\beta}\, D_a \phi^\beta \,, &\quad \bar P_a = -\phi_\alpha\, \varepsilon^{\alpha\beta}\, D_a \phi_\beta \,,\end{aligned} \tag{3.5}$$

where D_a denotes the fully superconformal covariant derivative. Note that P_a and $\bar P_a$ carry Weyl weight +1 and U(1) weights +2 and -2, respectively. From these definitions one may derive the supercovariant extension of the Maurer-Cartan equations associated with the $\mathrm{SU}(1,1)/\mathrm{U}(1)$ coset space,

$$\begin{aligned}F(a)_{ab} &= -2\mathrm{i}\, P_{[a}\, \bar P_{b]} - \tfrac12 \mathrm{i}\big(\bar\Lambda^i \gamma_{[a} D_{b]} \Lambda_i - \text{h.c.}\big)\,,\\ D_{[a} P_{b]} &= -\tfrac12 \bar\Lambda_i \gamma_{[a} \Lambda^i\, P_{b]} + \tfrac14 \bar\Lambda^i\, R(Q)_{ab\,i}\,,\end{aligned} \tag{3.6}$$

where $F(a)_{ab}$ and $R(Q)_{ab\,i}$ denote the supercovariant U(1) and Q-supersymmetry curvatures, respectively. Note that the expressions (3.4) and (3.5), when combined with those for the anti-holomorphic fields, reflect the structure of the three left-invariant vector fields associated with the group $\mathrm{SU}(1,1)$,

$$\begin{aligned}\mathcal{D}^0 &= \phi^\alpha \frac{\partial}{\partial\phi^\alpha} - \phi_\alpha \frac{\partial}{\partial\phi_\alpha}\,,\\ \mathcal{D}^\dagger &= \phi_\alpha\, \varepsilon^{\alpha\beta} \frac{\partial}{\partial\phi^\beta}\,, \qquad \mathcal{D} = -\phi^\alpha\, \varepsilon_{\alpha\beta} \frac{\partial}{\partial\phi_\beta}\,,\end{aligned} \tag{3.7}$$

which satisfy the commutation relations $\left[\mathcal{D}^0, \mathcal{D}\right] = 2\,\mathcal{D}$ and $\left[\mathcal{D}, \mathcal{D}^\dagger\right] = \mathcal{D}^0$. Using these definitions the supersymmetry variation and the supercovariant derivative of arbitrary functions $\mathcal{H}(\phi_\alpha, \phi^\beta)$ can be written as

$$\begin{aligned} \delta\mathcal{H} &= -\left[\bar{\epsilon}^i \Lambda_i\, \mathcal{D} + \bar{\epsilon}_i \Lambda^i\, \mathcal{D}^\dagger\right] \mathcal{H}\,, \\ D_a\mathcal{H} &= \left[\bar{P}_a\, \mathcal{D} + P_a\, \mathcal{D}^\dagger + \tfrac{1}{4}\bar{\Lambda}^i \gamma_a \Lambda_i\, \mathcal{D}^0\right] \mathcal{H}\,. \end{aligned} \tag{3.8}$$

The class of Lagrangians presented below involves a function $\mathcal{H}(\phi_\alpha)$ that is homogeneous of zeroth degree in the holomorphic variables, so that $\mathcal{D}^\dagger\mathcal{H}(\phi_\alpha) = 0$ and $\mathcal{D}^0\mathcal{H}(\phi_\alpha) = 0$. Using the above commutation relations, it then follows that $\mathcal{D}^\dagger\,\mathcal{D}^n\mathcal{H}(\phi_\alpha) \propto \mathcal{D}^{n-1}\mathcal{H}(\phi_\alpha)$ for $n > 1$, and vanishes for $n = 1$ so that $\mathcal{D}\mathcal{H}(\phi_\alpha)$ is holomorphic while $\mathcal{D}^2\mathcal{H}$ is not. Let us now turn to the derivation of this result. It makes use of the fact that any supersymmetric component Lagrangian can be written as the Hodge dual of a four-form built in terms of the vierbein, gravitini, and possibly other connections, multiplied by supercovariant coefficient functions that we will treat as composite fields. This approach is known as the superform method. Since for $N=4$ supergravity chiral superspace does not exist, we aim to construct such a density formula directly, assuming that only the vierbein and gravitini may appear explicitly within the four-form. Schematically we will thus consider a four-form decomposed into five types of forms, namely ψ^4, $e\,\psi^3$, $e^2\,\psi^2$, $e^3\,\psi$ and finally e^4. The Weyl weight of these forms ranges from $w = -2$ for the first one to $w = -4$ for the last one. This last form will be multiplied by a composite coefficient function with $w = 4$ that contains all the purely bosonic terms of the Lagrangian specified in (5.3) as well as fermionic terms.

4 Building up higher-order terms

In this section, we present the iterative procedure used to construct the supersymmetric completion of the quadratic Lagrangian (3.1). The non-linearity of the supersymmetry transformations rules will require us to add successive layers of terms of higher-order in the fields to the Lagrangian. The higher-order terms will be chosen such that their supersymmetry variations precisely cancel against the variations of the pre-existing lower-order terms. Ultimately, this program terminates when all the necessary terms have been added such that the Lagrangian is fully invariant under supersymmetry. Requiring Q-supersymmetry invariance turns out to be enough to ensure invariance under all the symmetries of $N = 4$ conformal supergravity. This is due to the specific superalgebra obeyed by the different generators [3]. Indeed, the commutator of two infinitesimal Q-supersymmetry transformations yields the full set of superconformal transformations including the U(1) transformation.

4.1 Structure of the full Lagrangian

This supersymmetrization procedure is unambiguous, yet lengthy, and provided sufficient computational efforts are invested it is guaranteed to give the full off-shell superconformal invariant. In practice however, the computation rapidly becomes unmanageable due to the rich field content and the non-linearity of the transformation rules. Therefore it becomes essential to systematise the work by making use of certain structure patterns appearing in the computation. Hence, we argue that the full Lagrangian can be written in the following form

$$\mathcal{L} = \mathcal{L}_0 + \psi\mathcal{L}_\psi + \phi\mathcal{L}_\phi + \psi^2\mathcal{L}_{\psi^2} + \psi\phi\mathcal{L}_{\psi\phi} + \phi^2\mathcal{L}_{\phi^2} + \psi^3\mathcal{L}_{\psi^3} + \psi^2\phi\mathcal{L}_{\psi^2\phi} + \psi^4\mathcal{L}_{\psi^4}\,, \tag{4.1}$$

where here, ψ and ϕ schematically denote the gravitino and the S-gauge field, respectively. The quantities $\mathcal{L}_0, \mathcal{L}_\psi, \mathcal{L}_\phi, \mathcal{L}_{\psi^2}, \mathcal{L}_{\phi^2}, \mathcal{L}_{\psi\phi},\ \mathcal{L}_{\psi^3},\ \mathcal{L}_{\psi^2\phi},\ \mathcal{L}_{\psi^4}$ only depend on supercovariant fields, i.e. matter fields, supercovariant curvatures and their supercovariant derivatives. Note that the terms of lowest-order in the fields in $\mathcal{L}_0$ correspond to the quadratic Lagrangian (3.1). Consequently, the other supercovariant quantities in (4.1) are at least of quadratic order in the fields. The expression (4.1) only contains terms up to four explicit gauge fields (ψ or ϕ). This can be understood as follows. Under an infinitesimal Q-supersymmetry variation (Q-variation), a gravitino transforms into the gradient of the Q-supersymmetry parameter. In order for this variation to be subsequently canceled, it first has to be integrated by parts such that when the derivative hits any of the other explicit gauge fields, it yields a curvature (Q or S). This requires the explicit gauge fields to appear fully anti-symmetrized in their vector indices and therefore rules out the possibility of terms with more than four explicit gauge fields. The same reasoning holds for an infinitesimal S-supersymmetry variation acting on ϕ. However, for our current analysis the terms with more than two explicit gauge fields are not required since we are only looking to construct the Lagrangian up to quadratic order in the fermion fields. We will therefore not attempt to derive them explicitly.

4.2 Constructing the interaction terms

In this subsection, we outline the iterative procedure used to construct the various supercovariant quantities appearing in the schematic expression (4.1) of the full $N = 4$ conformal supergravity Lagrangian. To this purpose, let us first write a part of (4.1) with explicit indices

$$\mathcal{L} = \mathcal{L}_0 + \left[\frac{1}{2}\bar{\psi}_a{}^i\mathcal{L}_{\psi}{}^a{}_i + \frac{1}{2}\bar{\phi}_a{}^i\mathcal{L}_{\phi}{}^a{}_i + \frac{1}{4}\bar{\psi}_b{}^i\mathcal{L}_{\psi^2}{}^{ab}{}_{ij}\psi_a{}^j + \frac{1}{4}\bar{\psi}_{bi}\mathcal{L}_{\psi^2}{}^{ab\,i}{}_j\psi_a{}^j \right.$$
$$\left. + \frac{1}{2}\bar{\psi}_b{}^i\mathcal{L}_{\psi\phi}{}^{ab}{}_{ij}\phi_a{}^j + \frac{1}{2}\bar{\psi}_{bi}\mathcal{L}_{\psi\phi}{}^{ab\,i}{}_j\phi_a{}^j + \text{h.c.}\right] . \tag{4.2}$$

Since we are only interested in the Lagrangian up to quadratic order in the fermion fields, we have truncated the full Lagrangian to the above expression. For the same reason, $\mathcal{L}_0$ is restricted to terms up to quadratic order in the fermions, while $\mathcal{L}_{\psi}{}^a{}_i, \mathcal{L}_{\phi}{}^a{}_i$ and $\mathcal{L}_{\psi^2}{}^{ab}{}_{ij}, \mathcal{L}_{\psi^2}{}^{ab\,i}{}_j, \mathcal{L}_{\psi\phi}{}^{ab}{}_{ij}, \mathcal{L}_{\psi\phi}{}^{ab\,i}{}_j$ are only linear in the fermions and purely bosonic, respectively. Note also that, as discussed in section 4.1, the last four quantities are antisymmetric in their vector indices. In what follows, we will work at specific orders in the supercovariant fields. To this purpose, we define $\mathcal{L}_0^{(n)}, \mathcal{L}_{\psi}^{(n)a}{}_i, \mathcal{L}_{\phi}^{(n)a}{}_i, \mathcal{L}_{\psi^2}^{(n)ab}{}_{ij}, \mathcal{L}_{\psi^2}^{(n)ab\,i}{}_j$ and $\mathcal{L}_{\psi\phi}^{(n)ab}{}_{ij}, \mathcal{L}_{\psi\phi}^{(n)ab\,i}{}_j$ which contain the terms of order n in the supercovariant fields of the quantities appearing in (4.1). In order to explain how the Q-variations at a specific order cancel against each other, we compute below the Q-variations of the various terms appearing in the Lagrangian at order n. To this purpose, we introduce the symbols $\delta_{K|_{f_a}}, \delta_{Q|_{\psi_a}}$ and $\delta_{S|_{\phi_a}}$ which denote gauge transformations where the parameters are replaced by the associated gauge fields. Additionally, we define $\delta_Q^{(cov)}$ as the supercovariant part of a Q-variation. In what follows, we insist that all the variations which are of cubic order, or more than cubic order, in the fermions (gauge and matter fields) will be suppressed.

$$\delta_Q\mathcal{L}_0^{(n)} \sim [\delta_Q e]e^{-1}\mathcal{L}_0^{(n)} + e\,\delta_Q[e^{-1}\mathcal{L}_0^{(n)}]\,, \tag{4.3}$$

$$\tfrac{1}{2}\delta_Q[\bar{\phi}_a{}^i\mathcal{L}_{\phi}^{(n)a}{}_i + \text{h.c.}] \sim f_a{}^b\bar{\epsilon}^i\gamma_b\mathcal{L}_{\phi}^{(n)a}{}_i + \tfrac{1}{2}[\delta_Q^{(cov)}\bar{\phi}_a{}^i]\mathcal{L}_{\phi}^{(n)a}{}_i + \tfrac{e}{2}\bar{\phi}_a{}^i\delta_Q[e^{-1}\mathcal{L}_{\phi}^{(n)a}{}_i] + \text{h.c.}\,, \tag{4.4}$$

$$\begin{aligned}\tfrac{1}{2}\delta_Q[\bar{\psi}_a{}^i\mathcal{L}_{\psi}^{(n)a}{}_i + \text{h.c.}] &\sim \mathcal{D}_a\bar{\epsilon}^i\mathcal{L}_{\psi}^{(n)a}{}_i - \tfrac{1}{4}\bar{\epsilon}_j\gamma_a\gamma\cdot T^{ij}\mathcal{L}_{\psi}^{(n)a}{}_i + \tfrac{e}{2}\bar{\psi}_a{}^i\delta_Q[e^{-1}\mathcal{L}_{\psi}^{(n)a}{}_i] + \text{h.c.}\\ &\sim -e\,\bar{\epsilon}^iD_a[e^{-1}\mathcal{L}_{\psi}^{(n)a}{}_i] - \bar{\epsilon}^i\delta_{K|_{f_a}}\mathcal{L}_{\psi}^{(n)a}{}_i - \tfrac{e}{2}\bar{\epsilon}^i\delta_{Q|_{\psi_a}}[e^{-1}\mathcal{L}_{\psi}^{(n)a}{}_i]\\ &\quad - \tfrac{e}{2}\bar{\epsilon}^i\delta_{S|_{\phi_a}}[e^{-1}\mathcal{L}_{\psi}^{(n)a}{}_i] - \tfrac{1}{4}\bar{\epsilon}_j\gamma_a\gamma\cdot T^{ij}\mathcal{L}_{\psi}^{(n)a}{}_i + \tfrac{e}{2}\bar{\psi}_a{}^i\delta_Q[e^{-1}\mathcal{L}_{\psi}^{(n)a}{}_i]\\ &\quad + \text{h.c.}\,,\end{aligned} \tag{4.5}$$

where in (4.5), we dropped a total derivative. Note that the term involving the field ${T_{ab}}^{ij}$ comes from the covariant part of the variation of $\psi_a{}^i$. It will appear similarly in the subsequent variations. We continue with

$$
\begin{aligned}
&\tfrac{1}{4}\delta_Q[\bar{\psi}_b{}^i \mathcal{L}^{(n)ab}_{\psi^2}{}_{ij}\psi_a{}^j + \bar{\psi}_{bi}\mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j\psi_a{}^j + \text{h.c.}]\\
&\qquad\qquad \sim\ [\mathcal{D}_b\bar{\epsilon}^i]\mathcal{L}^{(n)ab}_{\psi^2}{}_{ij}\psi_a{}^j + [\mathcal{D}_b\bar{\epsilon}^i]\mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j\psi_a{}^j\\
&\qquad\qquad\quad - \tfrac{1}{4}[\bar{\epsilon}_k\gamma_b\gamma\cdot T^{ik}\mathcal{L}^{(n)ab}_{\psi^2}{}_{ij}\psi_a{}^j + \bar{\epsilon}^k\gamma_b\gamma\cdot T_{ik}\mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j\psi_a{}^j] + \text{h.c.}\\
&\qquad\qquad \sim\ -e\,\bar{\epsilon}^i D_b[e^{-1}\mathcal{L}^{(n)ab}_{\psi^2}{}_{ij}]\psi_a{}^j - e\,\bar{\epsilon}_i D_b[e^{-1}\mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j]\psi_a{}^j\\
&\qquad\qquad\quad - \bar{\epsilon}^i[\delta_{K|_{f_b}}\mathcal{L}^{(n)ab}_{\psi^2}{}_{ij}]\psi_a{}^j - \bar{\epsilon}_i[\delta_{K|_{f_b}}\mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j]\psi_a{}^j\\
&\qquad\qquad\quad - \tfrac{1}{2}\left[\bar{\epsilon}^i\mathcal{L}^{(n)ab}_{\psi^2}{}_{ij} + \bar{\epsilon}_i\mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j\right]\left[\gamma_b\phi_a{}^j + R(Q)_{ba}{}^j + \tfrac{1}{2}\gamma\cdot T^{jk}\gamma_b\psi_{ak}\right]\\
&\qquad\qquad\quad - \tfrac{1}{4}[\bar{\epsilon}_k\gamma_b\gamma\cdot T^{ik}\mathcal{L}^{(n)ab}_{\psi^2}{}_{ij}\psi_a{}^j + \bar{\epsilon}^k\gamma_b\gamma\cdot T_{ik}\mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j\psi_a{}^j] + \text{h.c.}\,, \qquad (4.6)
\end{aligned}
$$

where we have again dropped a total derivative. In the sixth line, we have used that $\mathcal{L}^{(n)ab}_{\psi^2}{}_{ij}, \mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j$ are antisymmetric in their vector indices and we have rewritten the curl of the gravitino making use of the explicit expression of $R(Q)_{ab}{}^i$ given in (2.2). Finally, we have

$$
\begin{aligned}
&\tfrac{1}{2}\delta_Q[\bar{\psi}_b{}^i \mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}\phi_a{}^j + \bar{\psi}_{bi}\mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j\phi_a{}^j + \text{h.c.}]\\
&\quad \sim\ [\mathcal{D}_b\bar{\epsilon}^i]\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}\phi_a{}^j + [\mathcal{D}_b\bar{\epsilon}_i]\mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j\phi_a{}^j\\
&\quad\quad - \tfrac{1}{4}[\bar{\epsilon}_k\gamma_b\gamma\cdot T^{ik}\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}\phi_a{}^j + \bar{\epsilon}^k\gamma_b\gamma\cdot T_{ik}\mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j\phi_a{}^j]\\
&\quad\quad - \bar{\psi}_b{}^i\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}\gamma_c\,\epsilon^j f_b^c - \bar{\psi}_{bi}\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}\gamma_c\,\epsilon^j f_b^c + \tfrac{1}{2}[\bar{\psi}_b{}^i\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij} + \bar{\psi}_{bi}\mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j]\delta_Q^{(cov)}\phi_a^j + \text{h.c.}\\
&\quad \sim -e\,\bar{\epsilon}^i D_b[e^{-1}\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}]\phi_a{}^j - e\,\bar{\epsilon}_i D_b[e^{-1}\mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j]\phi_a{}^j\\
&\quad\quad - \tfrac{1}{4}[\bar{\epsilon}_k\gamma_b\gamma\cdot T^{ik}\mathcal{L}^{(n)ab}_{\psi^2}{}_{ij}\phi_a{}^j + \bar{\epsilon}^k\gamma_b\gamma\cdot T_{ik}\mathcal{L}^{(n)ab\,i}_{\psi^2}{}_j\phi_a{}^j]\\
&\quad\quad - \bar{\psi}_b{}^i\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}\gamma_c\,\epsilon^j f_b^c - \bar{\psi}_{bi}\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}\gamma_c\,\epsilon^j f_b^c + \tfrac{1}{2}[\bar{\psi}_b{}^i\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij} + \bar{\psi}_{bi}\mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j]\delta_Q^{(cov)}\phi_a^j\\
&\quad\quad - \tfrac{1}{2}\left[\bar{\epsilon}^i\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij} + \bar{\epsilon}_i\mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j\right]\left[R(S)_{ba}^j - 2\gamma_c\psi_a^j f_b^c + \tfrac{1}{6}\gamma_b\gamma\cdot T^{jk}\phi_{ak} - \delta^{(cov)}_{Q|_{\psi_a}}\phi_b^j\right] + \text{h.c.} \qquad (4.7)
\end{aligned}
$$

where after dropping a total derivative, we used in the last line that $\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}, \mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j$ are antisymmetric in their vector indices. This allowed us to rewrite the curl of the S-gauge field through the expression of $R(S)_{ab}{}^i$ given in (2.3). Note that we have also used $\delta_K\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij} = \delta_K\mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j = 0$. This is because $n \geq 2$, and in our case, $\mathcal{L}^{(n)ab}_{\psi\phi}{}_{ij}, \mathcal{L}^{(n)ab\,i}_{\psi\phi}{}_j$ are bosonic quantities with Weyl weight.

We now present in detail how the different variations appearing in (4.3)–(4.7) cancel each other out up to order n in the supercovariant fields. The purely supercovariant variations must cancel as

$$
\begin{aligned}
&\tfrac{1}{2}\sum_{k=2}^{n}\delta_Q\left(e^{-1}\mathcal{L}^{(k)}_0\right) - \tfrac{1}{4}\bar{\epsilon}_j\gamma_a\gamma.T^{ij}\sum_{k=2}^{n-1}\left(e^{-1}\mathcal{L}^{(k)a}_{\psi}{}_i\right) + \tfrac{1}{2}\left[\delta_Q^{(cov)}\bar{\phi}_a^i\right]\sum_{k=1}^{n-1}\left(e^{-1}\mathcal{L}^{(k)a}_{\phi}{}_i\right)\\
&- \tfrac{1}{2}\bar{\epsilon}^i\sum_{k=2}^{n-1}\left(e^{-1}\mathcal{L}^{(k)ab}_{\psi^2}{}_{ij}\right)R(Q)_{ba}{}^j - \tfrac{1}{2}\bar{\epsilon}_i\sum_{k=2}^{n-1}\left(e^{-1}\mathcal{L}^{(k)abi}_{\psi^2}{}_j\right)R(Q)_{ba}{}^j\\
&- \tfrac{1}{2}\bar{\epsilon}^i\sum_{k=2}^{n-1}\left(e^{-1}\mathcal{L}^{(k)ab}_{\psi\phi}{}_{ij}\right)R(S)_{ba}{}^j - \tfrac{1}{2}\bar{\epsilon}_i\sum_{k=2}^{n-1}\left(e^{-1}\mathcal{L}^{(k)abi}_{\psi\phi}{}_j\right)R(S)_{ba}{}^j + \text{h.c.}\\
&\qquad\qquad\qquad = \left[\bar{\epsilon}^i D_a\sum_{k=2}^{n}\left(e^{-1}\mathcal{L}^{(k)a}_{\psi}{}_i\right) + \text{h.c.}\right] + \mathcal{O}(n+1)\,, \qquad (4.8)
\end{aligned}
$$

where $\mathcal{O}(n+1)$ denote variations whose number of supercovariant fields is equal to or greater than $n+1$. We carry on with the variations containing an explicit K-gauge field. They have to satisfy

$$f_a{}^b \bar{\epsilon}^i \gamma_b \sum_{j=2}^{n} \mathcal{L}_{\phi}^{(k)a}{}_i - \bar{\epsilon}^i \delta_{K|f_a} \sum_{k=2}^{n} \mathcal{L}_{\psi}^{(k)a}{}_i + \text{h.c.} = \mathcal{O}(n+1)\,. \tag{4.9}$$

The variations containing an explicit gravitino must satisfy

$$\begin{aligned}
&\tfrac{1}{2}[\delta_Q e] \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_0^{(k)}\right) - \tfrac{e}{2}\bar{\epsilon}^i \delta_{Q|\psi_a} \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi}^{(k)a}{}_i\right) + \tfrac{e}{2}\bar{\psi}_a{}^i \delta_Q \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi}^{(k)a}{}_i\right) \\
&- \tfrac{1}{4}\bar{\epsilon}_i \gamma_b \gamma.T^{ji} \sum_{k=2}^{n-1} \left(\mathcal{L}_{\psi^2}^{(k)ab}{}_{jl}\right) \psi_a{}^l - \tfrac{1}{4}\bar{\epsilon}^i \gamma_b \gamma.T_{ji} \sum_{k=2}^{n-1} \left(\mathcal{L}_{\psi^2}^{(k)abj}{}_l\right) \psi_a{}^l \\
&- \tfrac{1}{4}\bar{\epsilon}^i \sum_{k=2}^{n-1} \left(\mathcal{L}_{\psi^2}^{(k)ab}{}_{ij}\right) \gamma.T^{jl}\gamma_b \psi_{al} - \tfrac{1}{4}\bar{\epsilon}_i \sum_{k=2}^{n-1} \left(\mathcal{L}_{\psi^2}^{(k)abi}{}_j\right) \gamma.T^{jl}\gamma_b \psi_{al} \\
&+ \tfrac{1}{2}\bar{\psi}_{bi} \sum_{k=2}^{n-1} \left(\mathcal{L}_{\psi\phi}^{(k)abi}{}_j\right) \delta_Q^{(cov)} \phi_a{}^j + \tfrac{1}{2}\bar{\psi}_b^i \sum_{k=2}^{n-1} \left(\mathcal{L}_{\psi\phi}^{(k)ab}{}_{ij}\right) \delta_Q^{(cov)} \phi_a^j \\
&+ \tfrac{1}{2}\bar{\epsilon}_i \sum_{k=2}^{n-1} \left(\mathcal{L}_{\psi\phi}^{(k)abi}{}_j\right) \delta_{Q|\psi_a}^{(cov)} \phi_b^j + \tfrac{1}{2}\bar{\epsilon}^i \sum_{k=2}^{n-1} \left(e\mathcal{L}_{\psi\phi}^{(k)ab}{}_{ij}\right) \delta_{Q|\psi_a}^{(cov)} \phi_b^j + \text{h.c.} \\
&\qquad = e\left[\bar{\epsilon}^i D_b \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi^2}^{(k)ab}{}_{ij}\right) \psi_a{}^j + \bar{\epsilon}_i D_b \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi^2}^{(k)abi}{}_j\right) \psi_a{}^j + \text{h.c.}\right] + \mathcal{O}(n+1)\,.
\end{aligned} \tag{4.10}$$

We continue with the variations containing a bare S-gauge field

$$\begin{aligned}
&- \tfrac{1}{2}\bar{\epsilon}^i \delta_{S|\phi_a} \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi}^{(k)a}{}_i\right) + \tfrac{1}{2}\bar{\phi}_a{}^i \delta_Q \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\phi}^{(k)a}{}_i\right) \\
&- \tfrac{1}{2}\bar{\epsilon}^i \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi^2}^{(k)ab}{}_{ij}\right) \gamma_b \phi_a^j - \tfrac{1}{2}\bar{\epsilon}_i \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi^2}^{(k)abi}{}_j\right) \gamma_b \phi_a^j \\
&- \tfrac{1}{12}\bar{\epsilon}^i \sum_{k=2}^{n-1} \left(e^{-1}\mathcal{L}_{\psi\phi}^{(k)ab}{}_{ij}\right) \gamma_b \gamma.T^{jl} \phi_{al} - \tfrac{1}{12}\bar{\epsilon}_i \sum_{k=2}^{n-1} \left(e^{-1}\mathcal{L}_{\psi\phi}^{(k)abi}{}_j\right) \gamma_b \gamma.T^{jl} \phi_{al} \\
&- \tfrac{1}{4}\bar{\epsilon}_j \gamma_b \gamma.T^{ij} \sum_{k=2}^{n-1} \left(e^{-1}\mathcal{L}_{\psi\phi}^{(k)ab}{}_{il}\right) \phi_a{}^l - \tfrac{1}{4}\bar{\epsilon}^j \gamma_b \gamma.T_{ij} \sum_{k=2}^{n-1} \left(e^{-1}\mathcal{L}_{\psi\phi}^{(k)abi}{}_l\right) \phi_a{}^l + \text{h.c.} \\
&\qquad = \left[\bar{\epsilon}^i D_b \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi\phi}^{(k)ab}{}_{ij}\right) \phi_a{}^j + \bar{\epsilon}_i D_b \sum_{k=2}^{n} \left(e^{-1}\mathcal{L}_{\psi\phi}^{(k)ab\,i}{}_j\right) \phi_a{}^j + \text{h.c.}\right] + \mathcal{O}(n+1)\,.
\end{aligned} \tag{4.11}$$

The Lagrangian (4.2) is build iteratively using the equations (4.8)-(4.11). The first step of the iterative procedure starts at the lowest-order, i.e. at $n = 2$. At this point, the left-hand side of equation (4.8) obviously only contains the first term and the expression of $\mathcal{L}_0^{(2)}$ is already know as it corresponds to the quadratic Lagrangian given in (3.1). This allows us to derive $\mathcal{L}_{\psi}^{(2)a}{}_i$. Subsequently, $\mathcal{L}_{\phi}^{(2)a}{}_i$ and $\mathcal{L}_{\psi^2}^{(2)ab}{}_{ij}, \mathcal{L}_{\psi^2}^{(2)ab\,i}{}_j$ are determined by imposing (4.9) and (4.10), respectively. This, in turn, allows to compute $\mathcal{L}_{\psi\phi}^{(2)ab}{}_{ij}$ and $\mathcal{L}_{\psi\phi}^{(2)ab\,i}{}_j$ from (4.11). At the $(n-1)$th iteration step, we consider the cancellation of the supersymmetry variations of order n in the supercovariant fields. We start with equation (4.8), where every term on the left-hand side is known from previous iterations, except for $\mathcal{L}_0^{(n)}$. At this stage, one has to determine $\mathcal{L}_0^{(n)}$ so that the whole left-hand side cancels at order n up to a total supercovariant derivative. The quantity on which the derivative acts upon is then $\mathcal{L}_{\psi}^{(n)a}{}_i$. This will then lead to $\mathcal{L}_{\phi}^{(n)a}{}_i$, $\mathcal{L}_{\psi^2}^{(n)ab}{}_{ij}$, $\mathcal{L}_{\psi^2}^{(n)abi}{}_j$, $\mathcal{L}_{\psi\phi}^{(n)ab}{}_{ij}$ and $\mathcal{L}_{\psi\phi}^{(n)abi}{}_j$ by solving the equations (4.9), (4.10) and (4.11). It is important to mention that at every step of the iteration, the equations (4.8)–(4.11) should be solved one after the other as each equation requires an input obtained by solving the previous one. In this way, we build all the terms of the Lagrangian (4.2) up to quadratic order in the fermion fields.

5 Building all $\mathcal{N}=4$ conformal supergravity actions

In this section, we provide the foundation for the construction of all $\mathcal{N}=4$ conformal supergravity actions. Making use of the density formula built in the previous section, we only need to specify the lowest Weight covariant fields $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$. Our goal will be to build candidates for these composites, using only the fields of the Weyl multiplet as our constituents. Once such composite fields are specified, the supersymmetry transformations can be used to build all of the other composite fields appearing in the density formula. Invariance under local superconformal transformations is then guaranteed. This approach is however not *a priori* guaranteed to lead to all possible conformal supergravity actions. To establish that the class we construct is actually exhaustive, we show that its supercurrent (the multiplet containing the energy-momentum tensor) corresponds to the most general supercurrent of conformal supergravity.

5.1 Ansatz for $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$

Let us attempt to construct $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$ out of the constituent fields of the $\mathcal{N}=4$ Weyl multiplet. We will need basic "building blocks" $X^{ij}{}_{kl}$ corresponding to S-invariant combinations of Weyl weight two in the $\mathbf{20'}$. It turns out that there are essentially four possible combinations which we denote X_n, with $n=1,\cdots,4$ indicating the degree of homogeneity in the covariant Weyl multiplet fields. X_1 is real, and we simply denote it as $D^{ij}{}_{kl}$ from now on. X_2, X_3, and X_4 are complex. We then choose the following ansatz for $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$:

$$
\begin{aligned}
C^{ij}{}_{kl} &= C_1^{(-2)} D^{ij}{}_{kl} & A^{ij}{}_{kl} &= A_1^{(0)}\, D^{ij}{}_{kl} \\
&\quad + C_2^{(0)} X_2{}^{ij}{}_{kl} + C_2^{(-4)} \bar{X}_2{}^{ij}{}_{kl} & &\quad + A_2^{(+2)} X_2{}^{ij}{}_{kl} + \bar{A}_2^{(-2)} \bar{X}_2{}^{ij}{}_{kl} \\
&\quad + C_3^{(+2)} X_3{}^{ij}{}_{kl} + C_3^{(-6)} \bar{X}_3{}^{ij}{}_{kl} & &\quad + A_3^{(+4)} X_3{}^{ij}{}_{kl} + \bar{A}_3^{(-4)} \bar{X}_3{}^{ij}{}_{kl} \\
&\quad + C_4^{(+4)} X_4{}^{ij}{}_{kl} + C_4^{(-8)} \bar{X}_4{}^{ij}{}_{kl}\ , & &\quad + A_4^{(+6)} X_4{}^{ij}{}_{kl} + \bar{A}_4^{(-6)} \bar{X}_4{}^{ij}{}_{kl}\ ,
\end{aligned}
\tag{5.1}
$$

with $\bar{C}^{ij}{}_{kl}$ given by complex conjugation. The factors $C_1^{(-2)}$, $A_1^{(0)}$, $A_2^{(+2)}$, etc., are functions of the coset scalars $\phi_{\boldsymbol{\alpha}}\,,\phi^{\boldsymbol{\alpha}}$, and their superscript correspond to their U(1) charge. Their complex conjugates are denoted by $\bar{C}_1^{(+2)}$, $\bar{A}_1^{(0)}$, $\bar{A}_2^{(-2)}$, etc.

The supersymmetry constraints (1.2) on $C^{ij}{}_{kl}$ and $A^{ij}{}_{kl}$ should then become constraints on these functions of the coset scalars. This is indeed the case since the four combinations X_n turn out to transform into each other under supersymmetry when we restrict to the largest SU(4) representations:

$$
\begin{aligned}
&[\nabla_{\alpha m} D^{ij}{}_{kl}]_{\mathbf{60}} = 0\ , \\
&[\nabla_{\alpha m} X_2{}^{ij}{}_{kl} + \Lambda_{\alpha m} D^{ij}{}_{kl}]_{\mathbf{60}} = 0\ , \quad [\bar\nabla^m X_2{}^{ij}{}_{kl} - 2\bar\Lambda^m X_3{}^{ij}{}_{kl}]_{\overline{\mathbf{60}}} = 0\ , \\
&[\nabla_{\alpha m} X_3{}^{ij}{}_{kl} + \Lambda_{\alpha m} X_2{}^{ij}{}_{kl}]_{\mathbf{60}} = 0\ , \quad [\bar\nabla^m X_3{}^{ij}{}_{kl} - 6\bar\Lambda^m X_4{}^{ij}{}_{kl}]_{\overline{\mathbf{60}}} = 0\ , \\
&[\nabla_{\alpha m} X_4{}^{ij}{}_{kl} + \Lambda_{\alpha m} X_3{}^{ij}{}_{kl}]_{\mathbf{60}} = 0\ , \quad [\bar\nabla^m X_4{}^{ij}{}_{kl}]_{\overline{\mathbf{60}}} = 0\ , \\
&[\Lambda_{\alpha m} X_4{}^{ij}{}_{kl}]_{\mathbf{60}} = 0\ .
\end{aligned}
\tag{5.2}
$$

The final condition arises because five Λ_i's cannot be placed into the **60**. From these results, one can derive a set of differential equations on the coset functions and search for a solution. To explain these conditions, we first make a brief detour to discuss the structure of the coset space geometry.

5.2 Presentation of results

From the expressions of the lowest dimension composite (5.1) in terms of the Weyl multiplet fields and the supersymmetry transformations rules, we generate the full $\mathcal{N} = 4$ conformal supergravity Lagrangian. The purely bosonic part $\mathcal{L}_B$ of the Lagrangian was already presented in the current literature and can be written as

$$
\begin{aligned}
e^{-1}\mathcal{L}_B = \mathcal{H}\Big[\, & \tfrac{1}{2} R(M)^{abcd}\, R(M)^-_{abcd} + R(V)^{ab\,i}{}_j\, R(V)^{-\,j}_{ab}{}_i + \tfrac{1}{8} D^{ij}{}_{kl}\, D^{kl}{}_{ij} + \tfrac{1}{4} E_{ij}\, D^2 E^{ij} \\
& - 4\, T_{ab}{}^{ij} D^a D_c T^{cb}{}_{ij} - \bar{P}^a D_a D_b P^b + P^2 \bar{P}^2 + \tfrac{1}{3}(P^a \bar{P}_a)^2 - \tfrac{1}{6} P^a \bar{P}_a\, E_{ij}\, E^{ij} \\
& - 8\, P_a\, \bar{P}^c\, T^{ab}{}_{ij}\, T_{bc}{}^{ij} - \tfrac{1}{16} E_{ij}\, E^{jk}\, E_{kl}\, E^{li} + \tfrac{1}{48} (E_{ij}\, E^{ij})^2 + T^{ab}{}_{ij}\, T_{ab\,kl}\, T^{cd\,ij}\, T_{cd}{}^{kl} \\
& - T^{ab}{}_{ij}\, T_{cd}{}^{jk}\, T_{ab\,kl}\, T^{cd\,li} - \tfrac{1}{2} E^{ij}\, T^{ab\,kl}\, R(V)_{ab}{}^m{}_i\, \varepsilon_{jklm} + \tfrac{1}{2} E_{ij}\, T^{ab}{}_{kl}\, R(V)_{ab}{}^i{}_m\, \varepsilon^{jklm} \\
& - \tfrac{1}{16} E_{ij} E_{kl}\, T^{ab}{}_{mn}\, T_{ab\,pq}\, \varepsilon^{ikmn}\, \varepsilon^{jlpq} - \tfrac{1}{16} E^{ij} E^{kl}\, T^{ab\,mn}\, T_{ab}{}^{pq}\, \varepsilon_{ikmn}\, \varepsilon_{jlpq} \\
& - 2\, T^{ab\,ij} \left(P_{[a} D_{c]} T_b{}^{c\,kl} + \tfrac{1}{6} P^c D_c T_{ab}{}^{kl} + \tfrac{1}{3} T_{ab}{}^{kl} D_c P^c \right) \varepsilon_{ijkl} \\
& - 2\, T^{ab}{}_{ij} \left(\bar{P}_{[a} D_{c]} T_b{}^c{}_{kl} - \tfrac{1}{2} \bar{P}^c D_c T_{ab\,kl} \right) \varepsilon^{ijkl} \Big] \\
+ \mathcal{D}\mathcal{H}\Big[\, & \tfrac{1}{4} T_{ab}{}^{ij}\, T_{cd}{}^{kl} R(M)^{abcd}\, \varepsilon_{ijkl} + E_{ij}\, T^{ab\,ik}\, R(V)_{ab}{}^j{}_k + T^{ab\,ij}\, T_a{}^{c\,kl}\, R(V)_{bc}{}^m{}_k\, \varepsilon_{ijlm} \\
& - \tfrac{1}{24} E_{ij}\, E^{ij}\, T^{ab\,kl}\, T_{ab}{}^{mn}\, \varepsilon_{klmn} - \tfrac{1}{6} E^{ij}\, T_{ab}{}^{kl}\, T^{ac\,mn}\, T^b{}_c{}^{pq}\, \varepsilon_{iklm}\, \varepsilon_{jpqn} \\
& - \tfrac{1}{8} D^{ij}{}_{kl} \left(T^{ab\,mn}\, T_{ab}{}^{kl}\, \varepsilon_{ijmn} - \tfrac{1}{2} E_{im}\, E_{jn}\, \varepsilon^{klmn} \right) \Big] \\
+ \mathcal{D}^2\mathcal{H}\Big[\, & \tfrac{1}{32} T^{ab\,ij}\, T^{cd\,pq}\, T_{ab}{}^{mn}\, T_{cd}{}^{kl}\, \varepsilon_{ijkl}\, \varepsilon_{mnpq} - \tfrac{1}{64} T^{ab\,ij}\, T^{cd\,pq}\, T_{ab}{}^{kl}\, T_{cd}{}^{mn}\, \varepsilon_{ijkl}\, \varepsilon_{mnpq} \\
& + \tfrac{1}{6} E_{ij}\, T_{ab}{}^{ik}\, T^{ac\,jl}\, T^b{}_c{}^{mn}\, \varepsilon_{klmn} + \tfrac{1}{384} E_{ij}\, E_{kl}\, E_{mn}\, E_{pq}\, \varepsilon^{ikmp}\, \varepsilon^{jlnq} \\
& - \tfrac{1}{8} E_{ij}\, E_{kl}\, T_{ab}{}^{ik}\, T^{ab\,jl} \Big] \\
+ 2\, \mathcal{H}\, & e_a{}^\mu f_\mu{}^c\, \eta_{cb} \Big[P^a\, \bar{P}^b - P^d\, \bar{P}_d\, \eta^{ab} \Big] + \text{h.c.}
\end{aligned}
\tag{5.3}
$$

The coset derivatives $\mathcal{D} \equiv \mathcal{D}^{++}$ and $\mathcal{D}^\dagger \equiv \mathcal{D}^{--}$ are defined and here we use the notations where the U(1) charges are suppressed. D_a is the fully supercovariant derivative (including the gravitino connection) and it coincides with the projection to components of the superspace derivative ∇_a. All covariant fields of the Weyl multiplet play a role in the action, including E^{ij} and $D^{ij}{}_{kl}$, as well as the SU(4) curvature $R(V)_{ab}{}^i{}_j$ and the Lorentz curvature $R(M)_{abcd}$. P_a is the supercovariant vielbein on the coset space.

From the point of view of the density formula, this Lagrangian corresponds to the bosonic part of the composite field F. The conformal supergravity Lagrangian obtained from our action principle a priori depends on the real and complex functions of the coset scalars and $\mathcal{I}$, respectively. It was further argued that the dependence on these functions can be removed by extracting a total derivative. The elimination of $\mathcal{I}$ in this way however typically generates terms that depend on $\mathcal{H}$. These terms can modify the structure of the density formula. In particular, at the purely bosonic level this total derivative introduces a dependence on the bare K-gauge field. This explains the last term in (5.3) whose presence also ensures the invariance of the kinetic term for the coset scalars under conformal boosts. The expression (5.3) is then fully invariant all the bosonic symmetries. A very stringent check of our result can be performed by setting the function $\mathcal{H}$ to a constant. The Lagrangian is then invariant under rigid SU(1,1) transformations and the bosonic terms (5.3) reduce precisely to the result. In this case, the bare K-gauge field can also be eliminated by extracting a total derivative and writing a kinetic term for the coset scalars which is invariant under conformal boosts up to fermionic terms. For any other holomorphic function, the rigid SU(1,1) invariance is broken.

Let us now present all the supercovariant terms which are quadratic in the fermion fields. They are still all contained in the field F and for legibility we will decompose them according to the number of coset derivatives acting on the holomorphic function $\mathcal{H}$. Once again, all the terms depending on the function $\mathcal{I}$ can be eliminated by splitting off a total derivative. The terms which do not depend on derivatives of $\mathcal{H}$ read

$$
\begin{aligned}
\mathcal{H}\Big[&\bar R(Q)^i_{ab}\,R(S)^{ab}_i - \tfrac34\bar\chi^{ij}{}_k D\chi_{ij}{}^k - \tfrac14\bar\chi_{ij}{}^k D\chi^{ij}{}_k - \tfrac38\bar\Lambda^i\left(DD^2+D^2D-D^3\right)\Lambda_i\\
&-\tfrac18\bar\Lambda_i\left(DD^2+D^2D-D^3\right)\Lambda^i - \tfrac18\bar\chi^{lm}{}_k\gamma\cdot DT^{ij}\Lambda^k\varepsilon_{ijlm} - \tfrac18\bar\chi^k{}_{lm}\gamma\cdot DT_{ij}\Lambda_k\,\varepsilon^{ijlm}\\
&+\tfrac18\bar\chi^{lm}{}_k\gamma\cdot T^{ij}D\Lambda^k\varepsilon_{ijlm} + \tfrac18\bar\chi^k{}_{lm}\gamma\cdot T_{ij}D\Lambda_k\varepsilon^{ijlm} - \tfrac14E^{ij}\bar\chi^{kl}{}_i\chi^{mn}{}_j\varepsilon_{klmn}\\
&-\tfrac14E_{ij}\,\bar\chi^i{}_{kl}\chi^j{}_{mn}\,\varepsilon^{klmn} - \tfrac12\bar R(Q)^{ab\,i}DT_{ab}{}^{kl}\Lambda^j\varepsilon_{ijkl} - \tfrac12\bar R(Q)^{ab}_i DT_{ab\,kl}\,\Lambda_j\,\varepsilon^{ijkl}\\
&-D_c\bar R(Q)^{ab\,i}\gamma_c\Lambda^jT_{ab}{}^{kl}\varepsilon_{ijkl} - \tfrac12D_c\bar R(Q)^{ab}_i\gamma_c\Lambda_jT_{ab\,kl}\,\varepsilon^{ijkl} + 2\,D_a\bar\Lambda_iR(Q)^{ab\,i}P_b\\
&+\tfrac12T_{ij}\cdot T_{kl}\,\bar\Lambda_m\chi^{kl}{}_n\,\varepsilon^{ijmn} + \tfrac12T^{ij}\cdot T^{kl}\bar\Lambda^m\chi_{kl}{}^n\varepsilon_{ijmn} + \tfrac12P^a\bar\Lambda_k\gamma\cdot T^{ij}\gamma_a\chi^k{}_{ij}\\
&+\tfrac12\bar P^a\bar\Lambda^k\gamma\cdot T_{ij}\gamma_a\chi^{ij}{}_k - \tfrac1{12}E^{ij}E^{kl}\bar\Lambda_i\chi^{mn}{}_k\,\varepsilon_{jlmn} - \tfrac1{12}E_{ij}E_{kl}\bar\Lambda^i\chi^k{}_{mn}\,\varepsilon^{jlmn}\\
&-\tfrac16E^{ij}\bar P^a\bar\Lambda^k\gamma_a\chi^{lm}{}_i\,\varepsilon_{jklm} - \tfrac16E_{ij}P^a\bar\Lambda_m\gamma_a\chi^i{}_{kl}\,\varepsilon^{jklm} - \tfrac16E^{ij}D_aE_{jk}\,\bar\Lambda_i\gamma^a\Lambda^k\\
&-\tfrac1{48}E^{ij}D_aE_{ij}\,\bar\Lambda_k\gamma^a\Lambda^k - \tfrac1{48}E_{ij}D_aE^{ij}\bar\Lambda_k\gamma^a\Lambda^k - \tfrac1{16}E^{ij}E_{ij}\bar\Lambda_k\gamma^aD_a\Lambda^k\\
&+\tfrac1{48}E^{ij}E_{ij}\,D_a\bar\Lambda_k\gamma^a\Lambda^k + \tfrac1{12}E^{ik}E_{ij}\,\bar\Lambda_k\gamma^aD_a\Lambda^j - \tfrac14E_{ik}E^{ij}D_a\bar\Lambda_j\gamma^a\Lambda^k\\
&+\tfrac5{12}D_aE^{ij}P^a\bar\Lambda_i\Lambda_j + \tfrac5{12}D_aE_{ij}\bar P^a\bar\Lambda^i\Lambda^j - \tfrac13E^{ij}P^a\bar\Lambda_i\gamma_{ab}D_b\Lambda_j + \tfrac13E^{ij}D_aP^a\bar\Lambda_i\Lambda_j\\
&+\tfrac13E_{ij}\,D_a\bar P^a\bar\Lambda^i\Lambda^j - \tfrac14\bar\Lambda^i\gamma^a\Lambda_iD_b\bar P_aP^b - \tfrac5{12}\bar\Lambda^i\gamma^a\Lambda_iD_bP_a\bar P^b + \tfrac23\bar\Lambda^i\gamma^aD_b\Lambda_i\bar P_aP^b\\
&-\tfrac23D_a\bar\Lambda^i\gamma^b\Lambda_i\bar P^aP^b + \tfrac1{12}\bar\Lambda^i\gamma^b\Lambda_iD_a\bar P^aP^b - \tfrac1{12}\bar\Lambda^i\gamma^b\Lambda_iD_aP^a\bar P^b - \tfrac23D_a\bar\Lambda^i\gamma^a\Lambda_i\bar P^bP_b\\
&+\tfrac43\bar\Lambda_i\gamma_aD_b\Lambda^i\bar P_cP_d\varepsilon^{abcd} - 2T_{ab\,ij}T^{ac\,ik}\bar\Lambda_k\gamma^bD_c\Lambda^j - 2T_{ab\,ij}D_cT^{ac\,ik}\bar\Lambda_k\gamma^b\Lambda^j\\
&-2\,T_{abij}T^{acij}D_c\bar\Lambda_k\gamma^b\Lambda^k - 2\,D_cT_{abij}T^{acij}\bar\Lambda_k\gamma^b\Lambda^k + \tfrac13P_aT^{ab}{}_{ij}\bar\Lambda_kD_b\Lambda_l\varepsilon^{ijkl}\\
&+\tfrac13\bar P_aT^{ab\,ij}\bar\Lambda^kD_b\Lambda^l\varepsilon_{ijkl} - \tfrac12P_cD^bT_{ab\,ij}\bar\Lambda_k\gamma^{ac}\Lambda_l\varepsilon^{ijkl} - \tfrac12\bar P^cD_bT^{ab\,ij}\bar\Lambda^k\gamma_{ac}\Lambda^l\varepsilon_{ijkl}\\
&-\tfrac13D^cP_bT^{ab}{}_{ij}\bar\Lambda_k\gamma_{ac}\Lambda_l\varepsilon^{ijkl} - \tfrac13D^c\bar P_bT^{ab\,ij}\bar\Lambda^k\gamma_{ac}\Lambda^l\varepsilon_{ijkl} + \tfrac16E^{ij}D_bT^{ab\,lm}\bar\Lambda_i\gamma_a\Lambda^k\varepsilon_{jklm}\\
&-\tfrac16E_{ij}D_bT^{ab}{}_{lm}\bar\Lambda_k\gamma_a\Lambda^i\varepsilon^{jklm} - \tfrac13E_{ij}T^{ab}{}_{kl}D_b\bar\Lambda_m\gamma_a\Lambda^i\varepsilon^{jklm} + \tfrac13E^{ij}T^{ab\,kl}\bar\Lambda_i\gamma_aD_b\Lambda^m\varepsilon_{jklm}\\
&-\tfrac12D_bE^{km}T^{ab\,ij}\bar\Lambda_k\gamma_a\Lambda^l\varepsilon_{ijlm} + D_bE_{km}T^{ab}{}_{ij}\bar\Lambda_l\gamma_a\Lambda^k\varepsilon^{ijlm} - \tfrac1{24}E^{ij}T_{ab\,kl}T^{ab}{}_{mn}\bar\Lambda_i\Lambda_j\varepsilon^{klmn}\\
&-\tfrac1{24}E_{ij}T_{ab}{}^{kl}T^{ab\,mn}\bar\Lambda^i\Lambda^j\varepsilon_{klmn} - \tfrac13E_{ij}P^aT_{ab}{}^{ik}\bar\Lambda_k\gamma^b\Lambda^j + \tfrac13E^{ij}P^aT_{ab\,ik}\bar\Lambda_j\gamma^b\Lambda^k\\
&-\tfrac16P^cP_cT_{ab}{}^{ij}\bar\Lambda_i\gamma^{ab}\Lambda_j - \tfrac16\bar P^c\bar P_cT_{abij}\bar\Lambda^i\gamma^{ab}\Lambda^j - \tfrac1{24}T_{ab}{}^{ij}T^{ab\,kl}\bar\Lambda_m\gamma_c\Lambda^mP^c\varepsilon_{ijkl}\\
&+\tfrac1{24}T_{abij}T^{ab}{}_{kl}\bar\Lambda_m\gamma_c\Lambda^m\bar P^c\varepsilon^{ijkl} - \tfrac12\bar\Lambda_i\gamma^a\Lambda^jD^bR(V)_{ab}{}^i{}_j\Big] + \text{h.c.}\,.
\end{aligned}
\tag{5.4}
$$

This result can once more be checked by setting the function $\mathcal{H}$ to a constant. The remaining terms which are quadratic in fermions depend on derivatives of $\mathcal{H}$. Those with a single derivative can be written as

$$\begin{aligned}
\mathcal{DH}\Big[& \tfrac{1}{4}\bar\chi^{ij}{}_k\gamma_a\chi^k{}_{ij}\bar P^a - \tfrac{1}{2}\bar\chi^{ik}{}_l\chi^{jl}{}_k E_{ij} - \tfrac{1}{4}\bar\chi^{kl}{}_m\gamma^{ab}\chi^{mn}{}_k T_{ab}{}^{ij}\varepsilon_{ijln} - \tfrac{1}{2}E_{ij}\bar R(Q)^i_{ab}R(Q)^{ab\,j} \\
& - \tfrac{3}{4}T_{ab}{}^{ij}\bar R(Q)^{ab\,k}\chi^{lm}{}_k\varepsilon_{ijlm} - \tfrac{1}{2}D^{ij}{}_{kl}\bar\Lambda_i\chi^{kl}{}_j - \tfrac{1}{2}R(V)_{ab}{}^j{}_k\bar\Lambda_i\gamma^{ab}\chi^{ik}{}_j - \tfrac{1}{4}E_{ij}E^{ik}\bar\Lambda_l\chi^{jl}{}_k \\
& - \tfrac{1}{4}E_{ij}\bar\Lambda_k D\chi^i{}_{lm}\varepsilon^{jklm} + \tfrac{1}{24}E^{ij}T_{ab}{}^{kl}\bar\Lambda_i\gamma^{ab}\chi^{mn}{}_j\varepsilon_{klmn} + \tfrac{1}{8}E^{ij}T_{ab}{}^{kl}\bar\Lambda_m\gamma^{ab}\chi^{mn}{}_i\varepsilon_{jkln} \\
& + \tfrac{1}{4}\bar\Lambda_i\gamma\cdot T^{jk}D\chi^i{}_{jk} - \tfrac{1}{12}\bar\chi^{lm}{}_k\gamma\cdot T^{ij}\bar P\Lambda^k\varepsilon_{ijlm} + \tfrac{1}{8}\bar\chi^k{}_{lm}\gamma\cdot T_{ij}\bar P\Lambda_k\varepsilon^{ijlm} - \tfrac{3}{8}\bar\Lambda^i\gamma_a D^2\Lambda_i\bar P^a \\
& + \tfrac{1}{3}E^{ij}T_{ab}{}^{kl}\bar\Lambda_i R(Q)^{ab\,m}\varepsilon_{jklm} + \tfrac{1}{6}T^{ab}{}_{ij}\bar P^c\bar\Lambda_k\gamma_c R(Q)_{abl}\varepsilon^{ijkl} - \tfrac{3}{4}T_{ab}{}^{ij}\bar P^c\bar R(Q)^{ab\,k}\gamma_c\Lambda^l\varepsilon_{ijkl} \\
& + \tfrac{1}{3}\bar\Lambda_i R(Q)^i_{ab}\bar P^a P^b - \tfrac{7}{6}\bar\Lambda_i R(Q)^{ab\,j}R(V)_{ab}{}^i{}_j - \tfrac{3}{2}\bar\Lambda_i T^{ij}\cdot R(S)_j + \tfrac{1}{4}\bar\Lambda_i\gamma^{cd}R(Q)^{ab\,i}R(M)_{abcd} \\
& + \tfrac{1}{8}\bar\Lambda^i\gamma_d\Lambda_j R(V)_{ab}{}^j{}_i\bar P_c\varepsilon^{abcd} + \tfrac{1}{6}\bar\Lambda_k\Lambda_l T^{ab}{}_{ij}R(V)_{ab}{}^k{}_m\varepsilon^{ijlm} + \tfrac{8}{3}\bar\Lambda^i\gamma^b\Lambda_i P^a\bar P_a\bar P_b + \tfrac{7}{24}D_a\bar\Lambda^i\gamma_b D_c\Lambda_i\bar P_d\varepsilon^{abcd} \\
& - \tfrac{7}{12}\bar\Lambda^i\gamma^b D_a D_b\Lambda_i\bar P^a - \tfrac{1}{4}\bar\Lambda^i D\Lambda_i D_a\bar P^a - \tfrac{1}{2}\bar\Lambda^i\gamma^a D_b\Lambda_i D_a\bar P^b + \tfrac{1}{24}\bar\Lambda_i\gamma^a D^2\Lambda^i\bar P_a \\
& + \tfrac{1}{24}D^b\bar\Lambda_i\gamma^a D_b\Lambda^i\bar P_a - \tfrac{1}{12}\bar\Lambda_i\gamma^b D_a D_b\Lambda^i\bar P^a - \tfrac{1}{12}\bar\Lambda_i D\Lambda^i D_a\bar P^a - \tfrac{1}{6}\bar\Lambda_i\gamma^a D_b\Lambda^i D_a\bar P^b \\
& - \tfrac{1}{8}D_a\bar\Lambda_i D\Lambda^i\bar P^a + \tfrac{1}{8}D_a\bar\Lambda^i D\Lambda_i\bar P^a - \tfrac{1}{6}\bar\Lambda^i\gamma^a\Lambda_i D_a D_b\bar P^b - \tfrac{17}{12}\bar\Lambda^i\gamma^a\Lambda_i P_a\bar P^b\bar P_b \\
& + \tfrac{5}{6}E_{ij}T_{ab\,kl}\bar\Lambda_m\gamma^b\Lambda^i\bar P^a\varepsilon^{jklm} + \tfrac{1}{18}E^{ij}T_{ab}{}^{kl}\bar\Lambda_i\gamma^b\Lambda^m\bar P^a\varepsilon_{jklm} + \tfrac{1}{6}E_{ij}T_{ab}{}^{ik}D^b\bar\Lambda_k\gamma^a\Lambda^j \\
& + \tfrac{1}{6}D^b E_{jk}T_{ab}{}^{ij}\bar\Lambda_i\gamma^a\Lambda^k + \tfrac{1}{3}E_{ij}T_{ab}{}^{ik}\bar\Lambda_k\gamma^a D^b\Lambda^j - \tfrac{1}{3}E_{ij}D^b T_{ab}{}^{ik}\bar\Lambda_k\gamma^a\Lambda^j \\
& - T_{abij}\bar\Lambda_k\gamma^{ac}D^b D_c\Lambda_l\varepsilon^{ijkl} + \tfrac{1}{6}D_a D^c T_{bc\,kl}\bar\Lambda_i\gamma^{ab}\Lambda_j\varepsilon^{ijkl} - \tfrac{1}{6}T^{ab}{}_{ij}D_a\bar\Lambda_k D_b\Lambda_l\varepsilon^{ijkl} \\
& - \tfrac{1}{6}D_a T^{ab}{}_{ij}\bar\Lambda_k D_b\Lambda_l\varepsilon^{ijkl} + \tfrac{1}{12}D_a T_{bc\,ij}\bar\Lambda_k\gamma^{ab}D^c\Lambda_l\varepsilon^{ijkl} - \tfrac{1}{12}D^c T_{ac\,ij}\bar\Lambda_k\gamma^{ab}D_b\Lambda_l\varepsilon^{ijkl} \\
& - \tfrac{1}{3}T_{ab\,ij}D^b\bar\Lambda_k\gamma^{ac}D_c\Lambda_l\varepsilon^{ijkl} + \tfrac{2}{3}P_a T^{ab\,ij}\bar\Lambda_i D_b\Lambda_j - \tfrac{1}{2}D^c T_{bc}{}^{ij}P_a\bar\Lambda_i\gamma^{ab}\Lambda_j \\
& - \tfrac{1}{24}P^a D_a T_{bc}{}^{ij}\bar\Lambda_i\gamma^{bc}\Lambda_j + \tfrac{1}{3}P_a T_{bc}{}^{ij}\bar\Lambda_i\gamma^{ab}D^c\Lambda_j + \tfrac{5}{12}P^a T_{bc}{}^{ij}\bar\Lambda_i\gamma^{bc}D_a\Lambda_j \\
& - \tfrac{5}{9}T_{ac}{}^{ij}\bar P^a\bar P_b\bar\Lambda^k\gamma^{bc}\Lambda^l\varepsilon_{ijkl} + \tfrac{7}{12}T_{abij}\bar P^b P_c\bar\Lambda_k\gamma^{ac}\Lambda_l\varepsilon^{ijkl} - \tfrac{1}{3}E_{ij}E^{ik}\bar\Lambda_k\gamma_a\Lambda^j\bar P^a \\
& + \tfrac{1}{48}E_{ij}E^{ij}\bar\Lambda_k\gamma^a\Lambda^k\bar P_a - \tfrac{2}{3}T_{ac}{}^{kl}T^{abij}\bar\Lambda_k\gamma_b D^c\Lambda^m\varepsilon_{ijlm} + \tfrac{1}{8}T^{kl}\cdot T^{ij}\bar\Lambda_m D\Lambda^m\varepsilon_{ijkl} \\
& + \tfrac{1}{3}T_{ac}{}^{kl}T^{abij}D^c\bar\Lambda_k\gamma_b\Lambda^m\varepsilon_{ijlm} - \tfrac{1}{24}T^{kl}\cdot T^{ij}\bar\Lambda^m D\Lambda_m\varepsilon_{ijkl} - \tfrac{1}{3}T_{ac}{}^{ij}D_b T^{abkl}\bar\Lambda_k\gamma^c\Lambda^m\varepsilon_{ijlm} \\
& + \tfrac{1}{6}T_{ac}{}^{ij}D_b T^{abkl}\bar\Lambda_m\gamma^c\Lambda^m\varepsilon_{ijkl} + \tfrac{1}{3}T_{ab}{}^{ij}D_c T^{abkl}\bar\Lambda_k\gamma^c\Lambda^m\varepsilon_{ijlm} + \tfrac{1}{3}T_{ab}{}^{ij}D_c T^{abkl}\bar\Lambda_m\gamma^c\Lambda^m\varepsilon_{ijkl} \\
& + \tfrac{1}{8}D^2E^{ij}\bar\Lambda_i\Lambda_j + \tfrac{1}{2}\bar P^a\bar P_a E_{ij}\bar\Lambda^i\Lambda^j - \tfrac{1}{6}\bar P^a P_a E^{ij}\bar\Lambda_i\Lambda_j - \tfrac{1}{24}\bar\Lambda_i\gamma\cdot R(V)^k{}_j\Lambda_k E^{ij} \\
& + \tfrac{1}{24}E_{ij}E^{ij}E^{kl}\bar\Lambda_k\Lambda_l - \tfrac{1}{16}E_{ij}E^{ik}E^{jl}\bar\Lambda_k\Lambda_l - \tfrac{7}{288}E^{ij}E^{kl}T_{ab}{}^{mn}\bar\Lambda_i\gamma^{ab}\Lambda_k\varepsilon_{jlmn} \\
& + \tfrac{1}{16}T_{ab\,ij}T^{ab}{}_{kl}T_{cd}{}^{ij}\bar\Lambda_m\gamma^{cd}\Lambda_n\varepsilon^{klmn} + \tfrac{1}{16}T_{ab\,ij}T^{ab}{}_{kl}T_{cd}{}^{kl}\bar\Lambda_m\gamma^{cd}\Lambda_n\varepsilon^{ijmn} \\
& - \tfrac{1}{8}E_{ij}T^{ab}{}_{kl}T_{ab\,mn}\bar\Lambda_p\Lambda_q\varepsilon^{iklp}\varepsilon^{jmnq}\Big] + \text{h.c.}\,,
\end{aligned} \tag{5.5}$$

while those with two and three derivatives of the function read

$$\begin{aligned}
\mathcal{D}^2\mathcal{H}\Big[& \tfrac{1}{4}E_{ij}T_{ab}{}^{ik}\bar\Lambda_l\gamma^{ab}\chi^{jl}{}_k - \tfrac{5}{8}E_{ij}T_{ab}{}^{ik}\bar\Lambda_k R(Q)^{ab\,j} - \tfrac{1}{8}E_{ij}E_{kl}\bar\Lambda_m\chi^{ik}{}_n\varepsilon^{jlmn} \\
& + \tfrac{1}{4}\bar\Lambda_k\chi^{mn}{}_l T^{ij}\cdot T^{kl}\varepsilon_{ijmn} + \tfrac{1}{4}\bar\Lambda_m\gamma^b{}_c\chi^{mn}{}_i T_{ab}{}^{ij}T^{ac\,kl}\varepsilon_{jkln} + \tfrac{5}{64}\bar\Lambda_m\gamma^{cd}R(Q)^m_{ab}T^{ab\,ij}T_{cd}{}^{kl}\varepsilon_{ijkl} \\
& + \tfrac{5}{8}\bar\Lambda_i R(Q)^m_{bc}T^{ab\,ij}T_a{}^{c\,kl}\varepsilon_{jklm} - \tfrac{3}{4}\bar\Lambda^i\gamma^a D_b\Lambda_i\bar P_a\bar P^b - \tfrac{5}{24}\bar\Lambda^i D\Lambda_i\bar P^a\bar P_a - \tfrac{5}{12}\bar\Lambda^i\gamma_a\Lambda_i\bar P^a D_b\bar P^b \\
& - \tfrac{1}{2}\bar\Lambda^i\gamma^b\Lambda_i\bar P^a D_a\bar P_b + \tfrac{1}{12}\bar\Lambda_i\gamma^a D_b\Lambda^i\bar P_a\bar P^b - \tfrac{1}{8}\bar\Lambda_i D\Lambda^i\bar P^a\bar P_a - \tfrac{1}{6}E_{ij}\bar P^b\bar\Lambda_k\gamma^a\Lambda^i T_{ab}{}^{jk} \\
& - \tfrac{1}{6}T_{ab\,ij}\bar P^a\bar\Lambda_k D^b\Lambda_l\varepsilon^{ijkl} + \tfrac{1}{24}\bar P_a D^c T_{bcij}\bar\Lambda_k\gamma^{ab}\Lambda_l\varepsilon^{ijkl} - \tfrac{7}{12}T_{ab\,ij}\bar P_c\bar\Lambda_k\gamma^{ac}D^b\Lambda_l\varepsilon^{ijkl} \\
& - \tfrac{1}{8}T_{ab\,ij}D_c\bar P^b\bar\Lambda_k\gamma^{ac}\Lambda_l\varepsilon^{ijkl} + \tfrac{1}{6}T_{bc}{}^{ij}P^a\bar P_a\bar\Lambda_i\gamma^{bc}\Lambda_j + \tfrac{1}{48}T_{ab}{}^{ij}T^{abkl}\bar P_c\bar\Lambda_m\gamma^c\Lambda^m\varepsilon_{ijkl} \\
& - \tfrac{1}{2}\bar\Lambda_i\gamma_b{}^c\Lambda_k R(V)_{ac}{}^k{}_j T^{ab\,ij} + \tfrac{1}{2}\bar\Lambda_i\Lambda_k R(V)_{ab}{}^k{}_j T^{ab\,ij} + \tfrac{1}{16}\bar\Lambda_k\gamma^{ab}\Lambda_l R(V)_{ab}{}^i{}_m E_{ij}\varepsilon^{jklm}
\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{8}\bar\Lambda_i\gamma^{cd}\Lambda_j R(M)_{abcd}T^{ab\,ij}-\tfrac{1}{16}\bar\Lambda_i\gamma^{ab}\Lambda_j T_{ab}{}^{kl}D^{ij}{}_{kl}+\tfrac{1}{16}\bar\Lambda_j\Lambda_n E_{im}D^{ij}{}_{kl}\varepsilon^{klmn}\\
&-\tfrac{1}{48}E_{ij}E^{ij}T_{ab}{}^{kl}\bar\Lambda_k\gamma^{ab}\Lambda_l+\tfrac{1}{24}E_{ij}E^{ik}T_{ab}{}^{jl}\bar\Lambda_k\gamma^{ab}\Lambda_l-\tfrac{1}{48}E^{ij}T^{ab\,kl}T_{ab}{}^{mn}\bar\Lambda_i\Lambda_j\varepsilon_{klmn}\\
&-\tfrac{1}{12}E^{ij}T^{ab\,kl}T_{ac}{}^{mn}\bar\Lambda_i\gamma_b{}^c\Lambda_k\varepsilon_{jlmn}\Big]\\
+\mathcal{D}^3\mathcal{H}\Big[&\tfrac{1}{8}T_{ab\,ij}\bar P^a\bar P_c\bar\Lambda_k\gamma^{bc}\Lambda_l\varepsilon^{ijkl}-\tfrac{1}{64}E_{ij}E_{kl}T_{ab}{}^{ik}\bar\Lambda_m\gamma^{ab}\Lambda_n\varepsilon^{jlmn}-\tfrac{1}{12}T^{ab\,ij}T_{ac}{}^{kl}T_b{}^{c\,mn}\bar\Lambda_i\Lambda_m\varepsilon_{jkln}\\
&+\tfrac{1}{64}T^{ab\,ij}T_{ab}{}^{kl}T_{cd}{}^{mn}\bar\Lambda_i\gamma^{cd}\Lambda_j\varepsilon_{klmn}-\tfrac{1}{32}T_{ab}{}^{ij}T^{ac\,kl}T_{cd}{}^{mn}\bar\Lambda_i\gamma^{bd}\Lambda_j\varepsilon_{klmn}\\
&+\tfrac{1}{32}T_{ab}{}^{ij}T^{ac\,kl}T_{cd}{}^{mn}\bar\Lambda_k\gamma^{bd}\Lambda_l\varepsilon_{ijmn}+\tfrac{1}{192}E_{ij}E_{kl}E_{mn}\bar\Lambda_p\Lambda_q\varepsilon^{ikmp}\varepsilon^{jlnq}-\tfrac{3}{8}\bar\Lambda^i\gamma_b\Lambda_i\bar P_a\bar P^a\bar P^b\Big]\\
&+\text{h.c.}
\end{aligned}
\tag{5.6}
$$

We emphasize that (5.4),(5.5) and (7.7) correspond to all the terms quadratic in the supercovariant fermion fields. The remaining fermionic terms at this order then necessarily involve bare gravitini and/or S-supersymmetry gauge fields. From the perspective of the density formula, these bare gravitini terms are associated with the composite fields Ω^i, $\Omega_a{}^i$, ^{ij}, ${}^{ij}{}_{ab}$ and ${}_a{}^i{}_j$. However, there will also be contributions coming from the various total derivatives we have extracted in order to write the result as above.

6 Terms with explicit fermionic gauge fields

In this section, we present all the terms at quadratic order in the fermion fields which contain explicit fermionic gauge fields. Therefore, we give the expression for the supercovariant quantities

$$
\mathcal{L}_{\psi}{}^a{}_i\,,\ \mathcal{L}_{\phi}{}^a{}_i\,,\ \mathcal{L}_{\psi^2}{}^{ab}{}_{ij}\,,\ \mathcal{L}_{\psi^2}{}^{ab\,i}{}_j\,,\ \mathcal{L}_{\psi\phi}{}^{ab}{}_{ij}\,,\ \mathcal{L}_{\psi\phi}{}^{ab\,i}{}_j\,,
\tag{6.1}
$$

which, as described in (4.2), appear in the Lagrangian coupled to bare fermionic gauge fields. For the purpose of this paper, we can restrict ourselves to the terms in $\mathcal{L}_{\psi}{}^a{}_i$ and $\mathcal{L}_{\phi}{}^a{}_i$ which are linear in fermions. Likewise, it is enough to only consider the bosonic terms in $\mathcal{L}_{\psi^2}{}^{ab}{}_{ij}, \mathcal{L}_{\psi^2}{}^{ab\,i}{}_j, \mathcal{L}_{\psi\phi}{}^{ab}{}_{ij}, \mathcal{L}_{\psi\phi}{}^{ab\,i}{}_j$.

Let us first consider $\mathcal{L}_{\psi}{}^a{}_i$ which is contracted with a gravitino in the Lagrangian. For the reader's convenience, we split this quantity into

$$
\mathcal{L}_{\psi}{}^a{}_i=\mathcal{L}_{\psi}^{(2)}{}^a{}_i+\mathcal{L}_{\psi}^{(3)}{}^a{}_i+\mathcal{L}_{\psi}^{(4)}{}^a{}_i+\dots\,,
\tag{6.2}
$$

where $\mathcal{L}_{\psi}^{(2)}{}^a{}_i, \mathcal{L}_{\psi}^{(3)}{}^a{}_i$ and $\mathcal{L}_{\psi}^{(4)}{}^a{}_i$ are quadratic, cubic and quartic in the fields, respectively. Due to Weyl weight restrictions, the dots denote terms which are of higher-order in fermions. The quadratic part reads

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\psi}^{(2)}{}^a{}_i=&\,\gamma^a\chi^l_{jk}D^{jk}{}_{li}+\tfrac{1}{2}\gamma^a\gamma\cdot T_{jk}\,\not{D}\chi^{jk}{}_i+2\gamma^a R(Q)_{cdj}R(V)^{cdj}{}_i+\tfrac{1}{2}\gamma^a\gamma_{eb}R(Q)_{cdi}R(M)^{ebcd}\\
&-2\gamma^a R(S)_{cd}{}^j T^{cd}{}_{ij}-\epsilon_{ijkl}\gamma^a\gamma_{bd}\Lambda^j D^b D_c T^{cdkl}+\tfrac{1}{2}\epsilon_{ijkl}\gamma^a\,\not{D}\chi^{jk}{}_m E^{lm}+\tfrac{1}{2}\gamma^a\Lambda^j D^2 E_{ij}\\
&+\gamma^a\big[\left(D_d\not{D}+\not{D}D_d+\gamma_d D^2\right)\Lambda_i\big]P^d+\tfrac{1}{2}\gamma^a\gamma^d\Lambda_i\big[D_dD_bP^b+D^2P_d\big]+\gamma^a\,\not{D}\Lambda_i D_dP^d\\
&+2\gamma^a\gamma^d D^b\Lambda_i D_{(d}P_{b)}+\gamma^a\gamma\cdot R(V)^j{}_k\chi^k{}_{ij}\,,
\end{aligned}
\tag{6.3}
$$

while the cubic part is

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\psi}^{(3)}{}^a{}_i=&-\tfrac{1}{2}\gamma^a\chi^k{}_{ji}E_{kl}E^{lj}-\gamma_b R(Q)_i\cdot T_{jk}T^{ab\,jk}-6\,\gamma_b R(Q)_k\cdot T_{ij}T^{ab\,jk}+4\,\gamma^b\chi^l_{jk}T_{db\,li}T^{ad\,jk}\\
&-\varepsilon_{jklm}\gamma^c\gamma_d\chi_i^{jk}P_cT^{ad\,lm}+\varepsilon_{jklm}\gamma\cdot T^{lm}\chi^{jk}{}_iP^a-2\,\varepsilon_{ijkl}\gamma^c\gamma^a R(Q)^l\cdot T^{jk}P_c\\
&+3\,\varepsilon_{ijkl}R(Q)^l\cdot T^{jk}P^a+\tfrac{1}{2}\varepsilon^{jklm}\gamma_b\chi^n{}_{jk}E_{ni}T^{ab}{}_{lm}-\varepsilon^{jklm}\gamma_b\chi^n{}_{ij}E_{kn}T^{ab}{}_{lm}
\end{aligned}
$$

$$
\begin{aligned}
&- \varepsilon_{jklm}\gamma_b\chi^j_{ni}E^{kn}T^{ab\,lm} - \tfrac{1}{12}\gamma^a\gamma\cdot R(V)^j{}_i\Lambda^k E_{jk} - \tfrac{1}{6}\gamma.\gamma^a R(V)^j{}_i\Lambda^k E_{jk} \\
&+ \tfrac{1}{4}\gamma^a\gamma\cdot R(V)^j{}_k\Lambda^k E_{ji} + \tfrac{1}{4}\varepsilon^{jklm}\gamma^a R(Q)_j\cdot T_{kl}E_{mi} + 2R(Q)^{ab\,j}P_b E_{ij} \\
&+ \gamma_{cd}\Lambda_i R(M)^{abcd}P_b + \gamma^c R(Q)^{ab}{}_i\bar P_c P_b - \tfrac{19}{3}\varepsilon^{jklm}\Lambda_m D^b T_{bc\,li}T^{ac}{}_{jk} \\
&- 6\,\varepsilon^{jklm}D^b\Lambda_m T_{bc\,li}T^{ac}{}_{jk} + 3\,\varepsilon^{jklm}\Lambda_m D^b T^{ac}{}_{li}T_{bc\,jk} + 2\,\varepsilon^{jklm}\Lambda_m T^{ac}{}_{li}D^b T_{bc\,jk} \\
&- 2\,\varepsilon^{jklm}\Lambda_m D^a\big[T_{li}\cdot T_{jk}\big] - 2\,\varepsilon^{jklm}\gamma_{dc}D_b\Lambda_m T^{bd}{}_{li}T^{ac}{}_{jk} + \tfrac{5}{3}\varepsilon^{jklm}\gamma_{dc}\Lambda_m D_b T^{bd}{}_{li}T^{ac}{}_{jk} \\
&- 2\,\varepsilon^{jklm}\gamma_{dc}\Lambda_m T^{ad}{}_{li}D_b T^{bc}{}_{jk} + \varepsilon^{jklm}\gamma_{dc}\Lambda_m D_b T^{ad}{}_{li}T^{bc}{}_{jk} + \tfrac{1}{2}\varepsilon_{ijkl}\gamma_b\Lambda^l T_{cd}{}^{jk}R(M)^{abcd} \\
&+ 2\,\Lambda_k D_c E^{jk}T^{ca}{}_{ji} + 2\,\Lambda_k E^{jk}D_c T^{ca}{}_{ji} + \tfrac{1}{3}\gamma^{ba}\Lambda_k D^c E^{jk}T_{cb\,ji} + \tfrac{1}{3}\gamma^{ba}D^c\Lambda_k E^{jk}T_{cb\,ji} \\
&- \tfrac{1}{3}\gamma^{ba}\Lambda_k E^{jk}D^c T_{cb\,ji} + \tfrac{16}{3}\gamma^c\Lambda^j\bar P^b D^a T_{bcji} + \tfrac{16}{3}\gamma^c\Lambda^j D^a\bar P^b T_{bcji} + \tfrac{4}{3}\gamma^c D^a\Lambda^j\bar P^b T_{bcji} \\
&+ \tfrac{2}{3}\gamma_c D_b\Lambda^j\bar P^b T^{ac}{}_{ji} + \tfrac{10}{3}\gamma^c\Lambda^j\bar P^a D^b T_{bcji} - \tfrac{10}{3}\gamma_c\Lambda^j\bar P^b D_b T^{ac}{}_{ji} + \tfrac{2}{3}\gamma^c D^b\Lambda^j\bar P^a T_{bcji} \\
&+ \tfrac{2}{3}\gamma_c\Lambda^j D^b\bar P_b T^{ac}{}_{ji} + \tfrac{2}{3}\gamma^c\Lambda^j D^b\bar P^a T_{bcji} + \tfrac{4}{3}\gamma^b\Lambda^j\bar P_b D_c T^{ca}{}_{ji} - \tfrac{20}{3}\gamma^a\Lambda^j\bar P^b D^c T_{cbji} \\
&+ \tfrac{28}{3}\gamma^b\Lambda^j\bar P_c D_b T^{ac}{}_{ji} - \tfrac{8}{3}\gamma^a D^c\Lambda^j\bar P^b T_{cbji} + \tfrac{16}{3}\gamma^b\Lambda^j D_b\bar P_c T^{ac}{}_{ji} + \tfrac{4}{3}\gamma^b D_b\Lambda^j\bar P_c T^{ac}{}_{ji} \\
&+ \tfrac{1}{4}\big[\tfrac{1}{3}\gamma_{cd}\gamma^{ab} + \gamma^{ab}\gamma_{cd}\big]\Lambda_j R(V)^{cdj}{}_i P_b - \tfrac{1}{6}\gamma_{cd}\gamma^{ab}\Lambda_j R(V)^{cdj}{}_i P_b - \tfrac{1}{2}\gamma_{cd}\Lambda_j R(V)^{cdj}{}_i P^a \\
&- \tfrac{1}{3}\Lambda_j R(V)^{ba\,j}{}_i P_b - \tfrac{1}{3}\gamma_{cd}\Lambda_j R(V)^{ca\,j}{}_i P^d - 4\,\epsilon_{ijkl}\gamma^c\Lambda^j T_{cb}{}^{km}R(V)^{ab\,l}{}_m \\
&- \tfrac{1}{2}\epsilon_{ijkl}\gamma^a\Lambda^m T^{ef\,jk}R(V)_{ef}{}^l{}_m + \tfrac{2}{3}\epsilon_{jklm}\gamma_b\Lambda^m T_c{}^{[a\,kl}R(V)^{b]c\,j}{}_i \\
&+ \tfrac{5}{6}\epsilon_{jklm}\gamma^a\Lambda^m T^{cd\,kl}R(V)_{cd}{}^j{}_i\,.
\end{aligned}
\tag{6.4}
$$

The quartic part takes the following form

$$
\begin{aligned}
e^{-1}\mathcal{L}^{(4)\,a}_{\psi\ \ i} = {}& \tfrac{1}{12}\gamma^a\Lambda^j E_{ij}E_{kl}E^{kl} - \tfrac{1}{6}\gamma^a\Lambda^k E_{lk}E_{ij}E^{lj} + \tfrac{1}{6}\Lambda_i E_{jk}E^{jk}P^a + \tfrac{1}{6}\gamma^{ab}\Lambda_i E_{jk}E^{jk}P_b \\
&+ \tfrac{2}{3}\Lambda_k E_{ij}E^{kj}P^a - \tfrac{1}{3}\gamma^{ab}\Lambda_k E_{ij}E^{kj}P_b - \tfrac{2}{3}\gamma^a\Lambda^j E_{ij}\bar P_b P^b - \tfrac{4}{3}\gamma^b\Lambda^j E_{ij}P^a\bar P_b \\
&+ \tfrac{10}{3}\gamma^b\Lambda^j E_{ij}P_b\bar P^a - \tfrac{1}{3}\varepsilon^{abcd}\gamma_d\Lambda^j P_b\bar P_c E_{ji} + \tfrac{7}{3}\gamma^{bc}\Lambda_i P^a P_b\bar P_c - \tfrac{2}{3}\gamma^{ac}\Lambda_i P_b P_c\bar P^b \\
&- \tfrac{2}{3}\gamma^{ac}\Lambda_i P_b P^b\bar P_c - \tfrac{5}{3}\Lambda_i P_b P^b\bar P^a + \tfrac{1}{3}\Lambda_i P_b P^a\bar P^b - \tfrac{7}{3}\varepsilon_{ijkl}\gamma^a\Lambda^j T_{bc}{}^{kl}P^b\bar P^c \\
&+ \tfrac{3}{2}\varepsilon_{ijkl}\gamma_b\Lambda^j T^{ackl}P^b\bar P_c + \tfrac{23}{6}\varepsilon_{ijkl}\gamma_b\Lambda^j T^{ackl}P_c\bar P^b - \tfrac{5}{6}\varepsilon_{ijkl}\gamma^b\Lambda^j T_{bc}{}^{kl}P^a\bar P^c \\
&- \tfrac{1}{2}\varepsilon_{ijkl}\gamma^b\Lambda^j T_{bc}{}^{kl}P^c\bar P^a + 4\,\varepsilon_{ijkl}\gamma_b\Lambda^j T^{bakl}\bar P_c P^c - \tfrac{1}{6}\varepsilon_{ijkl}\gamma^b\Lambda^m T_{ba}{}^{jk}E_{mn}E^{nl} \\
&+ \tfrac{2}{3}\varepsilon^{jklm}\gamma_b\Lambda^n E_{mn}E_{il}T^{ba}{}_{jk} - \tfrac{1}{2}\varepsilon_{jklm}\gamma_b\Lambda^l E_{ni}E^{nm}T^{bajk} + 2\,\gamma^c\Lambda^l E_{li}T^{ab}{}_{jk}T_{cb}{}^{jk} \\
&+ 2\,\varepsilon_{ijkl}\varepsilon_{mnpq}\gamma^c\Lambda^m E^{ln}T^{abpq}T_{cb}{}^{jk} + \tfrac{1}{2}\varepsilon_{ijkl}\varepsilon_{mnpq}\gamma^a\Lambda^n E^{ml}T^{pq}\cdot T^{jk} \\
&- \tfrac{23}{3}\gamma^c\Lambda^l E_{lj}T^{ab}{}_{ik}T_{cb}{}^{jk} + 2\,\gamma^c\Lambda^j E_{ik}T^{ab}{}_{lj}T_{cb}{}^{lk} + \tfrac{4}{3}\varepsilon^{jklm}\Lambda_l E_{im}T^{ab}{}_{jk}P_b \\
&- \tfrac{2}{3}\varepsilon_{ijkl}\gamma_{cb}\Lambda_m E^{ml}T^{abjk}P^c - 2\,\varepsilon^{jklm}\gamma_{cb}\Lambda_m E_{il}T^{ab}{}_{jk}P^c + \tfrac{1}{3}\varepsilon_{ijkl}\gamma^{bc}\Lambda_m E^{ml}T_{bc}{}^{jk}P^a \\
&+ \tfrac{46}{3}\Lambda_k T^{abkj}T_{bcji}P^c + \tfrac{28}{3}\Lambda_i T^{abjk}T_{bcjk}P^c + \tfrac{34}{3}\gamma^{cd}\Lambda_k T^{abkj}T_{bcji}P_d \\
&+ \tfrac{4}{3}\gamma^{cd}\Lambda_i T^{abjk}T_{bcjk}P_d + \tfrac{14}{3}\gamma\cdot T^{kj}\Lambda_k T^{ab}{}_{ji}P_b - \tfrac{2}{3}\gamma\cdot T^{jk}\Lambda_i T^{ab}{}_{jk}P_b \\
&- 2\,\varepsilon_{jklm}\gamma_b\Lambda^l T^{jk}\cdot T^{mn}T^{ab}{}_{in} + \varepsilon_{ijkl}\gamma_b\Lambda^l T^{jk}\cdot T^{mn}T^{ab}{}_{mn} \\
&+ 8\,\varepsilon_{jklm}\gamma_d\Lambda^l T^{ab}{}_{in}T_{bc}{}^{jk}T^{cdmn}\,.
\end{aligned}
\tag{6.5}
$$

We now present the expression of $\mathcal{L}_{\phi\,i}^{\ a}$ which appears contracted with an S-gauge field in the Lagrangian. The terms linear in fermions can only be of cubic order in the fields

$$
e^{-1}\mathcal{L}_{\phi\,i}^{\ a} = \tfrac{1}{2}\gamma_b\Lambda_j E^{lj}T^{ab}{}_{li} + \tfrac{19}{3}\Lambda^j\bar P_b T^{ba}{}_{ji} - \tfrac{1}{6}\gamma^{bc}\Lambda^j\bar P^a T_{bc\,ji}
$$

$$+\tfrac{1}{3}\gamma^{ab}\Lambda^j\bar{P}^cT_{bcji}+\tfrac{2}{3}\varepsilon^{jklm}\gamma^b\Lambda_mT_{bcli}T^{ac}{}_{jk}+2\,\varepsilon^{jklm}\gamma^a\Lambda_mT_{li}\cdot T_{jk}\,. \tag{6.6}$$

We move on to $\mathcal{L}_{\psi^2}{}^{ab}{}_{ij}$ and $\mathcal{L}_{\psi^2}{}^{ab\,i}{}_j$ which enter the Lagrangian contracted with two gravitini. For clarity, we split them into

$$\mathcal{L}_{\psi^2}{}^{ab}{}_{ij}=\,\mathcal{L}^{(2)\,ab}_{\psi^2}{}_{ij}+\mathcal{L}^{(3)\,ab}_{\psi^2}{}_{ij}+\dots\,, \tag{6.7}$$
$$\mathcal{L}_{\psi^2}{}^{ab\,i}{}_j=\,\mathcal{L}^{(2)\,ab\,i}_{\psi^2}{}_j+\mathcal{L}^{(3)\,ab\,i}_{\psi^2}{}_j+\dots\,, \tag{6.8}$$

where $\mathcal{L}^{(2)\,ab}_{\psi^2}{}_{ij},\mathcal{L}^{(2)\,ab\,i}_{\psi^2}{}_j$ and $\mathcal{L}^{(3)\,ab}_{\psi^2}{}_{ij},\mathcal{L}^{(3)\,ab\,i}_{\psi^2}{}_j$ contain terms quadratic and cubic in the bosonic fields, respectively. The higher-order terms, denoted by the dots, are fermionic. The expressions of the quadratic parts are

$$\begin{aligned}
\mathcal{L}^{(2)\,ab}_{\psi^2}{}_{ij}=&\tfrac{1}{2}\varepsilon_{iklm}\gamma^{ab}E^{nm}D_{jn}{}^{kl}-\tfrac{1}{2}\varepsilon_{ijkl}\gamma^{[a}\gamma_{cd}\gamma^{b]}R(V)^{cdl}{}_mE^{mk}-2\gamma^{ab}P^cD_cE_{ij}-\gamma^{ab}D^cP_cE_{ij}\\
&-2\,T^{ab}{}_{kl}D_{ij}{}^{kl}+\tfrac{1}{2}\gamma^{[a}\gamma_{ef}\gamma_{cd}\gamma^{b]}R(V)^{ef\,k}{}_jT^{cd}{}_{ik}-\tfrac{1}{2}\gamma^{[a}\gamma_{cd}\gamma_{ef}\gamma^{b]}R(V)^{ef\,k}{}_iT^{cd}{}_{kj}\\
&+4\,R(M)^{abcd}T_{cdij}+4\,\varepsilon_{ijkl}\varepsilon^{abcd}P_cD^eT_{ed}{}^{kl}-4\,\varepsilon_{ijkl}P^cD_cT^{bakl}\\
&-2\,\varepsilon_{ijkl}D_cP^cT^{bakl}\,,
\end{aligned} \tag{6.9}$$
$$\mathcal{L}^{(2)\,ab\,i}_{\psi^2}{}_j=\,-4\,\varepsilon^{abcd}\delta^i_j\gamma_dD^e\left[P_{[c}\bar{P}_{e]}\right]\,. \tag{6.10}$$

while the cubic parts read

$$\begin{aligned}
\mathcal{L}^{(3)\,ab}_{\psi^2}{}_{ij}=&\tfrac{1}{3}T^{ab}{}_{l[i}E_{j]k}E^{kl}-\tfrac{2}{3}T^{ab}{}_{ij}E_{kl}E^{kl}+\tfrac{1}{2}\varepsilon_{ijkl}\varepsilon_{mnpq}E^{ml}E^{pk}T^{abqn}+8\,T^{bakl}T_{lj}\cdot T_{ik}\\
&+16\,\gamma\cdot T^{kl}T_{lj}{}^{[a}{}_cT^{b]c}{}_{ik}+8\,\bar{P}_cP^{[a}T^{b]c}{}_{ij}-16\,P_c\bar{P}^{[a}T^{b]c}{}_{ij}+4\,\varepsilon^{klmn}E_{n[i}T_{j]mc}{}^{[a}T^{b]c}{}_{kl}\\
&+\tfrac{1}{6}\varepsilon_{klm(i}T_{j)n}{}^{ba}\gamma\cdot T^{lm}E^{kn}\,,
\end{aligned} \tag{6.11}$$
$$\begin{aligned}
\mathcal{L}^{(3)\,ab\,i}_{\psi^2}{}_j=&\,-2\,\gamma_cE^{ki}\bar{P}^{[a}T^{b]c}{}_{kj}-\gamma^cE^{ki}\bar{P}_cT^{ab}{}_{kj}-2\,\gamma_cE_{kj}P^{[a}T^{b]cki}-\gamma^cE_{kj}P_cT^{abki}\\
&+\tfrac{16}{3}\varepsilon^{ilkm}\gamma^cT^{ab}{}_{jl}\bar{P}^dT_{cdkm}+8\,\varepsilon^{iklm}\gamma^{[b}T^{a]c}{}_{lm}\bar{P}^dT_{cdjk}-4\,\varepsilon^{iklm}\gamma^c\bar{P}^{[b}T^{a]d}{}_{lm}T_{dcjk}\\
&-2\,\varepsilon^{iklm}\gamma^{[a}\bar{P}^{b]}T_{jk}.T_{lm}-\tfrac{16}{3}\varepsilon_{jklm}\gamma_cT^{abik}P_dT^{dclm}+8\,\varepsilon_{jklm}\gamma^{[b}T^{a]clm}P^dT_{cd}{}^{ik}\\
&-4\,\varepsilon_{jklm}\gamma^cP^{[b}T^{a]dlm}T_{dc}{}^{ik}-2\,\varepsilon_{jklm}\gamma^{[a}P^{b]}T^{ik}\cdot T^{lm}\,.
\end{aligned} \tag{6.12}$$

Finally we present the results for $\mathcal{L}_{\psi\phi}{}^{ab}{}_{ij}$ and $\mathcal{L}_{\psi\phi}{}^{ab\,i}{}_j$ which are coupled to a gravitino and an S-gauge field. The only bosonic terms are clearly at most quadratic

$$\mathcal{L}_{\psi\phi}{}^{ab}{}_{ij}=2\,\varepsilon_{ijkl}\varepsilon^{abcd}\gamma_dP^eT_{ec}{}^{kl}\,, \tag{6.13}$$
$$\mathcal{L}_{\psi\phi}{}^{ab}{}_{ij}=\,-4\,\delta^i{}_jP^{[a}\bar{P}^{b]}\,. \tag{6.14}$$

7 The general results

In this section, we present all the supercovariant terms of the $N=4$ conformal supergravity Lagrangian up to quadratic order in the fermion fields, obtained through the iterative procedure presented in section 4. We argued the Lagrangian takes the form (4.1). Within this scheme, the purely supercovariant terms at all order in the fields, bosonic or fermionic, are cast within the quantity denoted by $\mathcal{L}_0$. Let us now split $\mathcal{L}_0$ into

$$\mathcal{L}_0=\mathcal{L}_{\rm Q}+\mathcal{L}_{\rm B}+\mathcal{L}_{\rm F}+\dots\,, \tag{7.1}$$

where $\mathcal{L}_{\rm Q}$, $\mathcal{L}_{\rm B}$ and $\mathcal{L}_{\rm F}$ are respectively the quadratic Lagrangian (3.1), all the purely bosonic supercovariant interaction terms and the supercovariant interaction terms quadratic in the fermion fields. Here, the dots denote terms which are quartic, sextic and octic in the fermion fields and which, therefore, are outside of the scope of this paper.

We first recall the quadratic Lagrangian

$$
\begin{aligned}
e^{-1}\mathcal{L}_0 = {}& \tfrac{1}{2} R(M)^{abcd}\, R(M)^-_{abcd} + R(V)^{ab\,i}{}_j\, R(V)^-_{ab}{}^j{}_i \\
& - 4\, T_{ab}{}^{ij}\, D^a D_c T^{cb}{}_{ij} + \tfrac{1}{4} E_{ij}\, D^2 E^{ij} + \tfrac{1}{8} D_{ij}{}^{kl}\, D_{kl}{}^{ij} \\
& - 2\, \bar P^a \big[D_a D^b P_b + D^2 P_a \big] - 2\, D^a P^b D_a \bar P_b - D^a P_a D^b \bar P_b \\
& + \bar R(Q)_{ab}{}^i R(S)^{ab}{}_i - \bar\chi^{ij}{}_k\, \not{D} \chi^k{}_{ij} - \tfrac{1}{2} \bar\Lambda_i \left(D^2\, \not{D} + \not{D} D^2 - \not{D}^3 \right) \Lambda^i \; + \text{h.c.}\,,
\end{aligned} \tag{7.2}
$$

which was discussed in section 3 and served as the basis for the iterative procedure.

The bosonic interaction terms at all order in the fields are

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm B} = {}& \tfrac{1}{3} P^a\, \bar P_a\, P^b\, \bar P_b + P^a\, P_a\, \bar P^b\, \bar P_b \\
& - \tfrac{1}{16} E_{ij}\, E^{jk}\, E_{kl}\, E^{li} + \tfrac{1}{48} \big[E_{ij}\, E^{ij} \big]^2 \\
& - \tfrac{1}{6} E_{ij}\, E^{ij}\, P^a\, \bar P_a - 8\, T^{ab\,ij}\, T_{bc\,ij}\, P_a\, \bar P^c \\
& + T^{ab\,ij}\, T_{ab}{}^{kl}\, T_{cd\,ij}\, T^{cd}{}_{kl} - T^{ab\,ik}\, T_{ab}{}^{jl}\, T_{cd\,ij}\, T^{cd}{}_{kl} \\
& + \varepsilon^{ijkl}\, T^{ab}{}_{ij}\, E_{km}\, R(V)_{ab}{}^m{}_l \\
& - \varepsilon^{ijkl}\, \bar P^c \, [4\, D_a T^{ab}{}_{ij}\, T_{bc\,kl} - D_c T^{ab}{}_{ij}\, T_{ab\,kl}] \\
& - \tfrac{1}{8} \varepsilon_{ijkl}\, \varepsilon_{mnpq}\, T^{ab\,ij}\, T_{ab}{}^{mn}\, E^{kp}\, E^{lq} + \text{h.c.}\,,
\end{aligned} \tag{7.3}
$$

which involve cubic and quartic terms in the fields. Quintic terms are forbidden due to the Weyl weights of the bosons.

The interaction terms which are quadratic in the fermion fields read

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm F} = {}& \tfrac{1}{4}\varepsilon_{ijkl}\bar\chi^{ij}{}_m \gamma\cdot T^{kl}\, \not{D}\Lambda^m - \tfrac{1}{4}\varepsilon_{ijkl}\bar\chi^{ij}{}_m \gamma\cdot T^{kl}\overleftarrow{\not{D}}\,\Lambda^m \\
& - \tfrac{1}{2}\varepsilon_{ijkl}\bar\chi^{ij}{}_m \chi^{kl}{}_n E^{mn} - \tfrac{3}{4}\varepsilon^{ijkl}\bar R(Q)_i\cdot \not{D} T_{jk}\Lambda_l - \tfrac{3}{2}\varepsilon^{ijkl}\bar R(Q)_i \overleftarrow{\not{D}}\cdot T_{jk}\Lambda_l \\
& + 2\, D^a\bar\Lambda^i R(Q)_{abi}\bar P^b + \tfrac{1}{2}\bar\Lambda_j\gamma^b\Lambda^i D^a R(V)_{ab}{}^j{}_i + \varepsilon^{ijkl}\bar\chi^{mn}{}_l\Lambda_i T_{mn}\cdot T_{jk} \\
& + \bar\chi^{ij}{}_k\gamma^a\gamma\cdot T_{ij}\Lambda^k\bar P_a + \tfrac{1}{6}\varepsilon_{ijkl}\bar\chi^{ij}{}_m\Lambda_n E^{mk}E^{ln} - \tfrac{1}{3}\varepsilon_{ijkl}\bar\chi^{ij}{}_m\gamma^a\Lambda^k E^{ml}\bar P_a \\
& - \tfrac{1}{12}\bar\Lambda_i\,\not{D}\Lambda^i E_{jk}E^{jk} + \tfrac{1}{3}\bar\Lambda_i\,\not{D}\Lambda^j E_{jk}E^{ki} - \tfrac{1}{6}\bar\Lambda_i\gamma^a\Lambda^j D_a E_{jk}E^{ki} + \tfrac{5}{6}\bar\Lambda_i\Lambda_j D_a E^{ij}P^a \\
& + \tfrac{2}{3}\bar\Lambda_i\Lambda_j E^{ij} D_a P^a + \tfrac{1}{3}\bar\Lambda_i\gamma^{ab}D_a\Lambda_j E^{ij}P_b + \tfrac{4}{3}\bar\Lambda_i\gamma_a D_b\Lambda^i\bar P^b P^a - \tfrac{1}{6}\bar\Lambda_i\gamma_a\Lambda^i D_b\bar P^a P^b \\
& - \tfrac{1}{6}\bar\Lambda_i\gamma_a\Lambda^i D_b\bar P^b P^a + \tfrac{2}{3}\bar\Lambda_i\,\not{D}\Lambda^i P_b\bar P^b + \tfrac{4}{3}\varepsilon^{abcd}\bar\Lambda_i\gamma_a D_b\Lambda^i\bar P_c P_d - 2\, D_a\bar\Lambda_i\gamma^c\Lambda^i T^{ab}{}_{jk}T_{cb}{}^{jk} \\
& - 2\,\bar\Lambda_i\gamma^c\Lambda^i D_a T^{ab}{}_{jk}T_{cb}{}^{jk} + 2\, D_a\bar\Lambda_i\gamma^c\Lambda^j T^{ab}{}_{jk}T_{cb}{}^{ik} + 2\,\bar\Lambda_i\gamma^c\Lambda^j D_a T^{ab}{}_{jk}T_{cb}{}^{ik} \\
& - \tfrac{2}{3}\varepsilon^{ijkl}\bar\Lambda_i D^a\Lambda_j T_{abkl}P^b + \varepsilon^{ijkl}\bar\Lambda_i\gamma^{ab}\Lambda_j D_a T_{bckl}P^c + \tfrac{2}{3}\varepsilon^{ijkl}\bar\Lambda_i\gamma^{ab}\Lambda_j T_{bckl}D_a P^c \\
& - \tfrac{2}{3}\varepsilon^{ijkl}D^a\bar\Lambda_i\gamma^b\Lambda^m E_{mj}T_{abkl} + \varepsilon_{ijkl}\bar\Lambda_m\gamma^b\Lambda^i D^a E^{jm}T_{ab}{}^{kl} + \tfrac{1}{3}\varepsilon_{ijkl}\bar\Lambda_m\gamma^b\Lambda^i E^{jm}D^a T_{ab}{}^{kl} \\
& - \tfrac{1}{8}\varepsilon^{klmn}\bar\Lambda_i\Lambda_j T_{kl}\cdot T_{mn}E^{ij} + \tfrac{1}{6}\varepsilon^{klmn}\bar\Lambda_k\Lambda_i T_{jl}\cdot T_{mn}E^{ij} + \tfrac{2}{3}\bar\Lambda_i\gamma^a\Lambda^j E_{jk}T_{ab}{}^{ki}P^b \\
& - \tfrac{1}{3}\bar\Lambda_i\gamma^{ab}\Lambda_j T_{ab}{}^{ij}P^c P_c - \tfrac{1}{12}\varepsilon_{jklm}\bar\Lambda_i\gamma^a\Lambda^i T^{jk}\cdot T^{lm}P_a + \text{h.c.}\,.
\end{aligned} \tag{7.4}
$$

They involve cubic, quartic and quintic terms in the fields. Note that there are no terms of sextic, septic or octic order in the fields as, due to the restrictions on the Weyl weights, these would be of higher-order in the fermion fields. Finally, (7.2), (7.3) and (7.4) are $\mathrm{SU}(1,1)$ invariant and their sum is K-invariant.

As we already mentioned in section 1, the bosonic part of the $N = 4$ conformal supergravity Lagrangian has been derived in [8]. Because it was obtained in a different set of conventions, we have converted their result in the conventions of the present paper to facilitate the comparison with our results. In particular, this requires to covariantize the curvatures and derivatives with respect to the conformal boosts and to switch to a different parametrisation of the coset space. Up to a Gauss-Bonnet term, the Lagrangian in [8] is then equivalent to

$$
\begin{aligned}
e^{-1}\mathcal{L} = {} & \tfrac{1}{2} R(M)^{abcd}\, R(M)^{-}_{abcd} + R(V)^{ab\,i}{}_j\, R(V)^{-}_{ab}{}^j{}_i \\
& - 4\, T_{ab}{}^{ij}\, D^a D_c T^{cb}{}_{ij} + \tfrac{1}{4} E_{ij}\, D^2 E^{ij} + \tfrac{1}{8} D_{ij}{}^{kl}\, D_{kl}{}^{ij} \\
& - 2\, \bar{P}^a \big[D_a D^b P_b + D^2 P_a \big] - 2\, D^a P^b D_a \bar{P}_b - D^a P_a D^b \bar{P}_b \\
& + \tfrac{4}{3} P^a\, \bar{P}_a\, P^b\, \bar{P}_b + P^a\, P_a\, \bar{P}^b\, \bar{P}_b \\
& - \tfrac{1}{24} E_{ij}\, E^{jk}\, E_{kl}\, E^{li} \\
& + \tfrac{1}{12} E_{ij} E^{ij}\, P^a \bar{P}_a - 4\, T^{ab\,ij}\, T_{bc\,ij}\, P_a\, \bar{P}^c \\
& + \tfrac{5}{12} T_{ab}{}^{ij}\, T^{ab\,kl}\, T^{cd}{}_{ij}\, T^{cd\,kl} + \tfrac{1}{6} T^{ab\,ik}\, T_{ab}{}^{jl}\, T_{cd\,ij}\, T^{cd}{}_{kl} + \text{h.c.}\,. \qquad (7.5)
\end{aligned}
$$

We now compare the above expression with (7.3) and the bosonic part of (7.2). Clearly, the quadratic Lagrangians agree as the first three lines of (7.5) coincide with the bosonic part of (7.2). We note, however, a number of differences when comparing interaction terms. The most obvious one is perhaps the presence of terms cubic in the fields in our results while none appear in (7.5). Further differences concern the quartic terms in the fields. Indeed, the last term of the second line and the last line in (7.3) are not present in (7.5). Moreover, none of the coefficients of the remaining terms match.

The complete form of supergravity Lagrangian coupled to bare fermionic gauge fields is

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\psi}{}^a{}_i = {} & \gamma^a \chi^l{}_{jk} D^{jk}{}_{li} + \tfrac{1}{2}\gamma^a\gamma\cdot T_{jk}\, \not{D}\chi^{jk}{}_i + 2\gamma^a R(Q)_{cdj} R(V)^{cdj}{}_i + \tfrac{1}{2}\gamma^a\gamma_{eb} R(Q)_{cdi} R(M)^{ebcd} \\
& - 2\gamma^a R(S)_{cd}{}^j T^{cd}{}_{ij} - \epsilon_{ijkl}\gamma^a\gamma_{bd}\Lambda^j D^b D_c T^{cdkl} + \tfrac{1}{2}\epsilon_{ijkl}\gamma^a\, \not{D}\chi^{jk}{}_m E^{lm} + \tfrac{1}{2}\gamma^a\Lambda^j D^2 E_{ij} \\
& + \gamma^a\big[\big(D_d\, \not{D} + \not{D} D_d + \gamma_d D^2\big)\,\Lambda_i\big] P^d + \tfrac{1}{2}\gamma^a\gamma^d\Lambda_i\big[D_d D_b P^b + D^2 P_d\big] + \gamma^a\, \not{D}\Lambda_i D_d P^d \\
& - \tfrac{1}{2}\gamma^a\chi^k{}_{ji} E_{kl} E^{lj} - \gamma_b R(Q)_i\cdot T_{jk} T^{ab\,jk} - 6\,\gamma_b R(Q)_k\cdot T_{ij} T^{ab\,jk} + 4\,\gamma^b\chi^l{}_{jk} T_{db\,li} T^{ad\,jk} \\
& - \varepsilon_{jklm}\gamma^c\gamma_d\chi_i^{jk} P_c T^{ad\,lm} + \varepsilon_{jklm}\gamma\cdot T^{lm}\chi^{jk}{}_i P^a - 2\,\varepsilon_{ijkl}\gamma^c\gamma^a R(Q)^l\cdot T^{jk} P_c \\
& + 3\,\varepsilon_{ijkl} R(Q)^l\cdot T^{jk} P^a + \tfrac{1}{2}\varepsilon^{jklm}\gamma_b\chi^n{}_{jk} E_{ni} T^{ab}{}_{lm} - \varepsilon^{jklm}\gamma_b\chi^n{}_{ij} E_{kn} T^{ab}{}_{lm} \\
& - \varepsilon_{jklm}\gamma_b\chi^j{}_{ni} E^{kn} T^{ab\,lm} - \tfrac{1}{12}\gamma^a\gamma\cdot R(V)^j{}_i\Lambda^k E_{jk} - \tfrac{1}{6}\gamma.\gamma^a R(V)^j{}_i\Lambda^k E_{jk} \\
& + \tfrac{1}{4}\gamma^a\gamma\cdot R(V)^j{}_k\Lambda^k E_{ji} + \tfrac{1}{4}\varepsilon^{jklm}\gamma^a R(Q)_j\cdot T_{kl} E_{mi} + 2R(Q)^{ab\,j} P_b E_{ij} \\
& + \gamma_{cd}\Lambda_i R(M)^{abcd} P_b + \gamma^c R(Q)^{ab}{}_i \bar{P}_c P_b - \tfrac{19}{3}\varepsilon^{jklm}\Lambda_m D^b T_{bc\,li} T^{ac}{}_{jk} \\
& - 6\,\varepsilon^{jklm} D^b\Lambda_m T_{bc\,li} T^{ac}{}_{jk} + 3\,\varepsilon^{jklm}\Lambda_m D^b T^{ac}{}_{li} T_{bc\,jk} + 2\,\varepsilon^{jklm}\Lambda_m T^{ac}{}_{li} D^b T_{bc\,jk} \\
& - 2\,\varepsilon^{jklm}\Lambda_m D^a\big[T_{li}\cdot T_{jk}\big] - 2\,\varepsilon^{jklm}\gamma_{dc} D_b\Lambda_m T^{bd}{}_{li} T^{ac}{}_{jk} + \tfrac{5}{3}\varepsilon^{jklm}\gamma_{dc}\Lambda_m D_b T^{bd}{}_{li} T^{ac}{}_{jk} \\
& - 2\,\varepsilon^{jklm}\gamma_{dc}\Lambda_m T^{ad}{}_{li} D_b T^{bc}{}_{jk} + \varepsilon^{jklm}\gamma_{dc}\Lambda_m D_b T^{ad}{}_{li} T^{bc}{}_{jk} + \tfrac{1}{2}\varepsilon_{ijkl}\gamma_b\Lambda^l T_{cd}{}^{jk} R(M)^{abcd} \\
& + 2\,\Lambda_k D_c E^{jk} T^{ca}{}_{ji} + 2\,\Lambda_k E^{jk} D_c T^{ca}{}_{ji} + \tfrac{1}{3}\gamma^{ba}\Lambda_k D^c E^{jk} T_{cb\,ji} + \tfrac{1}{3}\gamma^{ba} D^c\Lambda_k E^{jk} T_{cb\,ji} \\
& - \tfrac{1}{3}\gamma^{ba}\Lambda_k E^{jk} D^c T_{cb\,ji} + \tfrac{16}{3}\gamma^c\Lambda^j \bar{P}^b D^a T_{bcji} + \tfrac{16}{3}\gamma^c\Lambda^j D^a \bar{P}^b T_{bcji} + \tfrac{4}{3}\gamma^c D^a\Lambda^j \bar{P}^b T_{bcji} \\
& + \tfrac{2}{3}\gamma_c D_b\Lambda^j \bar{P}^b T^{ac}{}_{ji} + \tfrac{10}{3}\gamma^c\Lambda^j \bar{P}^a D^b T_{bcji} - \tfrac{10}{3}\gamma_c\Lambda^j \bar{P}^b D_b T^{ac}{}_{ji} + \tfrac{2}{3}\gamma^c D^b\Lambda^j \bar{P}^a T_{bcji} \\
& + \tfrac{2}{3}\gamma_c\Lambda^j D^b \bar{P}_b T^{ac}{}_{ji} + \tfrac{2}{3}\gamma^c\Lambda^j D^b \bar{P}^a T_{bcji} + \tfrac{4}{3}\gamma^b\Lambda^j \bar{P}_b D_c T^{ca}{}_{ji} - \tfrac{20}{3}\gamma^a\Lambda^j \bar{P}^b D^c T_{cbji}
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{28}{3}\gamma^b\Lambda^j\bar{P}_cD_bT^{ac}{}_{ji}-\tfrac{8}{3}\gamma^aD^c\Lambda^j\bar{P}^bT_{cbji}+\tfrac{16}{3}\gamma^b\Lambda^jD_b\bar{P}_cT^{ac}{}_{ji}+\tfrac{4}{3}\gamma^bD_b\Lambda^j\bar{P}_cT^{ac}{}_{ji}\\
&+\tfrac{1}{4}\big[\tfrac{1}{3}\gamma_{cd}\gamma^{ab}+\gamma^{ab}\gamma_{cd}\big]\Lambda_jR(V)^{cdj}{}_iP_b-\tfrac{1}{6}\gamma_{cd}\gamma^{ab}\Lambda_jR(V)^{cdj}{}_iP_b-\tfrac{1}{2}\gamma_{cd}\Lambda_jR(V)^{cdj}{}_iP^a\\
&-\tfrac{1}{3}\Lambda_jR(V)^{ba\,j}{}_iP_b-\tfrac{1}{3}\gamma_{cd}\Lambda_jR(V)^{ca\,j}{}_iP^d-4\,\epsilon_{ijkl}\gamma^c\Lambda^jT_{cb}{}^{km}R(V)^{ab\,l}{}_m\\
&+\tfrac{1}{12}\gamma^a\Lambda^jE_{ij}E_{kl}E^{kl}-\tfrac{1}{6}\gamma^a\Lambda^kE_{lk}E_{ij}E^{lj}+\tfrac{1}{6}\Lambda_iE_{jk}E^{jk}P^a+\tfrac{1}{6}\gamma^{ab}\Lambda_iE_{jk}E^{jk}P_b\\
&+\tfrac{2}{3}\Lambda_kE_{ij}E^{kj}P^a-\tfrac{1}{3}\gamma^{ab}\Lambda_kE_{ij}E^{kj}P_b-\tfrac{2}{3}\gamma^a\Lambda^jE_{ij}\bar{P}_bP^b-\tfrac{4}{3}\gamma^b\Lambda^jE_{ij}P^a\bar{P}_b\\
&+\tfrac{10}{3}\gamma^b\Lambda^jE_{ij}P_b\bar{P}^a-\tfrac{1}{3}\varepsilon^{abcd}\gamma_d\Lambda^jP_b\bar{P}_cE_{ji}+\tfrac{7}{3}\gamma^{bc}\Lambda_iP^aP_b\bar{P}_c-\tfrac{2}{3}\gamma^{ac}\Lambda_iP_bP_c\bar{P}^b\\
&-\tfrac{2}{3}\gamma^{ac}\Lambda_iP_bP^b\bar{P}_c-\tfrac{5}{3}\Lambda_iP_bP^b\bar{P}^a+\tfrac{1}{3}\Lambda_iP_bP^a\bar{P}^b-\tfrac{7}{3}\varepsilon_{ijkl}\gamma^a\Lambda^jT_{bc}{}^{kl}P^b\bar{P}^c\\
&+\tfrac{3}{2}\varepsilon_{ijkl}\gamma_b\Lambda^jT^{ackl}P^b\bar{P}_c+\tfrac{23}{6}\varepsilon_{ijkl}\gamma_b\Lambda^jT^{ackl}P_c\bar{P}^b-\tfrac{5}{6}\varepsilon_{ijkl}\gamma^b\Lambda^jT_{bc}{}^{kl}P^a\bar{P}^c\\
&-\tfrac{1}{2}\varepsilon_{ijkl}\gamma^b\Lambda^jT_{bc}{}^{kl}P^c\bar{P}^a+4\,\varepsilon_{ijkl}\gamma_b\Lambda^jT^{bakl}\bar{P}_cP^c-\tfrac{1}{6}\varepsilon_{ijkl}\gamma^b\Lambda^mT_{ba}{}^{jk}E_{mn}E^{nl}\\
&+\tfrac{2}{3}\varepsilon^{jklm}\gamma_b\Lambda^nE_{mn}E_{il}T^{ba}{}_{jk}-\tfrac{1}{2}\varepsilon_{jklm}\gamma_b\Lambda^lE_{ni}E^{nm}T^{bajk}+2\,\gamma^c\Lambda^lE_{li}T^{ab}{}_{jk}T_{cb}{}^{jk}\\
&+2\,\varepsilon_{ijkl}\varepsilon_{mnpq}\gamma^c\Lambda^mE^{ln}T^{abpq}T_{cb}{}^{jk}+\tfrac{1}{2}\varepsilon_{ijkl}\varepsilon_{mnpq}\gamma^a\Lambda^nE^{ml}T^{pq}\cdot T^{jk}\\
&-\tfrac{23}{3}\gamma^c\Lambda^lE_{lj}T^{ab}{}_{ik}T_{cb}{}^{jk}+2\,\gamma^c\Lambda^jE_{ik}T^{ab}{}_{lj}T_{cb}{}^{lk}+\tfrac{4}{3}\varepsilon^{jklm}\Lambda_lE_{im}T^{ab}{}_{jk}P_b\\
&-\tfrac{2}{3}\varepsilon_{ijkl}\gamma_{cb}\Lambda_mE^{ml}T^{abjk}P^c-2\,\varepsilon^{jklm}\gamma_{cb}\Lambda_mE_{il}T^{ab}{}_{jk}P^c+\tfrac{1}{3}\varepsilon_{ijkl}\gamma^{bc}\Lambda_mE^{ml}T_{bc}{}^{jk}P^a\\
&+\tfrac{46}{3}\Lambda_kT^{abkj}T_{bcji}P^c+\tfrac{28}{3}\Lambda_iT^{abjk}T_{bcjk}P^c+\tfrac{34}{3}\gamma^{cd}\Lambda_kT^{abkj}T_{bcji}P_d\\
&+\tfrac{4}{3}\gamma^{cd}\Lambda_iT^{abjk}T_{bcjk}P_d+\tfrac{14}{3}\gamma\cdot T^{kj}\Lambda_kT^{ab}{}_{ji}P_b-\tfrac{2}{3}\gamma\cdot T^{jk}\Lambda_iT^{ab}{}_{jk}P_b\\
&-2\,\varepsilon_{jklm}\gamma_b\Lambda^lT^{jk}\cdot T^{mn}T^{ab}{}_{in}+\varepsilon_{ijkl}\gamma_b\Lambda^lT^{jk}\cdot T^{mn}T^{ab}{}_{mn}\\
&+8\,\varepsilon_{jklm}\gamma_d\Lambda^lT^{ab}{}_{in}T_{bc}{}^{jk}T^{cdmn}\,.
\end{aligned}
\qquad (7.6)
$$

The total Lagrangian of $N=4$ conformal supergravity is constructed in the magical and extremal form on current equation

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\mathrm{T}}=\mathcal{H}\Big[&\bar{R}(Q)^i_{ab}\,R(S)^{ab}_i-\tfrac{3}{4}\bar{\chi}^{ij}{}_kD\chi_{ij}{}^k-\tfrac{1}{4}\bar{\chi}_{ij}{}^kD\chi^{ij}{}_k-\tfrac{3}{8}\bar{\Lambda}^i\left(DD^2+D^2D-D^3\right)\Lambda_i\\
&-\tfrac{1}{8}\bar{\Lambda}_i\left(DD^2+D^2D-D^3\right)\Lambda^i-\tfrac{1}{8}\bar{\chi}^{lm}{}_k\gamma\cdot DT^{ij}\Lambda^k\varepsilon_{ijlm}-\tfrac{1}{8}\bar{\chi}^k{}_{lm}\gamma\cdot DT_{ij}\Lambda_k\,\varepsilon^{ijlm}\\
&+\tfrac{1}{8}\bar{\chi}^{lm}{}_k\gamma\cdot T^{ij}D\Lambda^k\varepsilon_{ijlm}+\tfrac{1}{8}\bar{\chi}^k{}_{lm}\gamma\cdot T_{ij}D\Lambda_k\varepsilon^{ijlm}-\tfrac{1}{4}E^{ij}\bar{\chi}^{kl}{}_i\chi^{mn}{}_j\varepsilon_{klmn}\\
&-\tfrac{1}{4}E_{ij}\,\bar{\chi}^i{}_{kl}\chi^j{}_{mn}\,\varepsilon^{klmn}-\tfrac{1}{2}\bar{R}(Q)^{ab\,i}DT_{ab}{}^{kl}\Lambda^j\varepsilon_{ijkl}-\tfrac{1}{2}\bar{R}(Q)^{ab}_iDT_{ab\,kl}\,\Lambda_j\,\varepsilon^{ijkl}\\
&-D_c\bar{R}(Q)^{ab\,i}\gamma_c\Lambda^jT_{ab}{}^{kl}\varepsilon_{ijkl}-\tfrac{1}{2}D_c\bar{R}(Q)^{ab}_i\gamma_c\Lambda_jT_{ab\,kl}\,\varepsilon^{ijkl}+2\,D_a\bar{\Lambda}_iR(Q)^{ab\,i}P_b\\
&+\tfrac{1}{2}T_{ij}\cdot T_{kl}\,\bar{\Lambda}_m\chi^{kl}{}_n\,\varepsilon^{ijmn}+\tfrac{1}{2}T^{ij}\cdot T^{kl}\bar{\Lambda}^m\chi_{kl}{}^n\varepsilon_{ijmn}+\tfrac{1}{2}P^a\bar{\Lambda}_k\gamma\cdot T^{ij}\gamma_a\chi^k{}_{ij}\\
&+\tfrac{1}{2}\bar{P}^a\bar{\Lambda}^k\gamma\cdot T_{ij}\gamma_a\chi^{ij}{}_k-\tfrac{1}{12}E^{ij}E^{kl}\bar{\Lambda}_i\chi^{mn}{}_k\,\varepsilon_{jlmn}-\tfrac{1}{12}E_{ij}E_{kl}\bar{\Lambda}^i\chi^k{}_{mn}\,\varepsilon^{jlmn}\\
&-\tfrac{1}{6}E^{ij}\,\bar{P}^a\bar{\Lambda}^k\gamma_a\chi^{lm}{}_i\,\varepsilon_{jklm}-\tfrac{1}{6}E_{ij}P^a\bar{\Lambda}_m\gamma_a\chi^i{}_{kl}\,\varepsilon^{jklm}-\tfrac{1}{6}E^{ij}D_aE_{jk}\,\bar{\Lambda}_i\gamma^a\Lambda^k\\
&-\tfrac{1}{48}E^{ij}D_aE_{ij}\,\bar{\Lambda}_k\gamma^a\Lambda^k-\tfrac{1}{48}E_{ij}D_aE^{ij}\bar{\Lambda}_k\gamma^a\Lambda^k-\tfrac{1}{16}E^{ij}E_{ij}\bar{\Lambda}_k\gamma^aD_a\Lambda^k\\
&+\tfrac{1}{48}E^{ij}E_{ij}\,D_a\bar{\Lambda}_k\gamma^a\Lambda^k+\tfrac{1}{12}E^{ik}E_{ij}\,\bar{\Lambda}_k\gamma^aD_a\Lambda^j-\tfrac{1}{4}E_{ik}E^{ij}D_a\bar{\Lambda}_j\gamma^a\Lambda^k\\
&+\tfrac{5}{12}D_aE^{ij}P^a\bar{\Lambda}_i\Lambda_j+\tfrac{5}{12}D_aE_{ij}\bar{P}^a\bar{\Lambda}^i\Lambda^j-\tfrac{1}{3}E^{ij}P^a\bar{\Lambda}_i\gamma_{ab}D_b\Lambda_j+\tfrac{1}{3}E^{ij}D_aP^a\bar{\Lambda}_i\Lambda_j\\
&+\tfrac{1}{3}E_{ij}\,D_a\bar{P}^a\bar{\Lambda}^i\Lambda^j-\tfrac{1}{4}\bar{\Lambda}^i\gamma^a\Lambda_iD_b\bar{P}_aP^b-\tfrac{5}{12}\bar{\Lambda}^i\gamma^a\Lambda_iD_bP_a\bar{P}^b+\tfrac{2}{3}\bar{\Lambda}^i\gamma^aD_b\Lambda_i\bar{P}_aP^b\\
&-\tfrac{2}{3}D_a\bar{\Lambda}^i\gamma^b\Lambda_i\bar{P}^aP^b+\tfrac{1}{12}\bar{\Lambda}^i\gamma^b\Lambda_iD_a\bar{P}^aP^b-\tfrac{1}{12}\bar{\Lambda}^i\gamma^b\Lambda_iD_aP^a\bar{P}^b-\tfrac{2}{3}D_a\bar{\Lambda}^i\gamma^a\Lambda_i\bar{P}^bP_b\\
&+\tfrac{4}{3}\bar{\Lambda}_i\gamma_aD_b\Lambda^i\bar{P}_cP_d\varepsilon^{abcd}-2T_{ab\,ij}T^{ac\,ik}\bar{\Lambda}_k\gamma^bD_c\Lambda^j-2T_{ab\,ij}D_cT^{ac\,ik}\bar{\Lambda}_k\gamma^b\Lambda^j
\end{aligned}
$$

$$
\begin{aligned}
&- 2\, T_{abij}T^{acij}D_c\bar\Lambda_k\gamma^b\Lambda^k - 2\, D_cT_{abij}T^{acij}\bar\Lambda_k\gamma^b\Lambda^k + \tfrac13 P_aT^{ab}{}_{ij}\bar\Lambda_kD_b\Lambda_l\varepsilon^{ijkl}\\
&+ \tfrac13\bar P_aT^{ab\,ij}\bar\Lambda^kD_b\Lambda^l\varepsilon_{ijkl} - \tfrac12 P_cD^bT_{ab\,ij}\bar\Lambda_k\gamma^{ac}\Lambda_l\varepsilon^{ijkl} - \tfrac12\bar P^cD_bT^{ab\,ij}\bar\Lambda^k\gamma_{ac}\Lambda^l\varepsilon_{ijkl}\\
&- \tfrac13 D^cP_bT^{ab}{}_{ij}\bar\Lambda_k\gamma_{ac}\Lambda_l\varepsilon^{ijkl} - \tfrac13 D^c\bar P_bT^{ab\,ij}\bar\Lambda^k\gamma_{ac}\Lambda^l\varepsilon_{ijkl} + \tfrac16 E^{ij}D_bT^{ab\,lm}\bar\Lambda_i\gamma_a\Lambda^k\varepsilon_{jklm}\\
&- \tfrac16 E_{ij}D_bT^{ab}{}_{lm}\bar\Lambda_k\gamma_a\Lambda^i\varepsilon^{jklm} - \tfrac13 E_{ij}T^{ab}{}_{kl}D_b\bar\Lambda_m\gamma_a\Lambda^i\varepsilon^{jklm} + \tfrac13 E^{ij}T^{ab\,kl}\bar\Lambda_i\gamma_aD_b\Lambda^m\varepsilon_{jklm}\\
&- \tfrac12 D_bE^{km}T^{ab\,ij}\bar\Lambda_k\gamma_a\Lambda^l\varepsilon_{ijlm} + D_bE_{km}T^{ab}{}_{ij}\bar\Lambda_l\gamma_a\Lambda^k\varepsilon^{ijlm} - \tfrac1{24}E^{ij}T_{ab\,kl}T^{ab}{}_{mn}\bar\Lambda_i\Lambda_j\varepsilon^{klmn}\\
&- \tfrac1{24}E_{ij}T_{ab}{}^{kl}T^{ab\,mn}\bar\Lambda^i\Lambda^j\varepsilon_{klmn} - \tfrac13 E_{ij}P^aT_{ab}{}^{ik}\bar\Lambda_k\gamma^b\Lambda^j + \tfrac13 E^{ij}P^aT_{ab\,ik}\bar\Lambda_j\gamma^b\Lambda^k\\
&- \tfrac16 P^cP_cT_{ab}{}^{ij}\bar\Lambda_i\gamma^{ab}\Lambda_j - \tfrac16\bar P^c\bar P_cT_{abij}\bar\Lambda^i\gamma^{ab}\Lambda^j - \tfrac1{24}T_{ab}{}^{ij}T^{ab\,kl}\bar\Lambda_m\gamma_c\Lambda^mP^c\varepsilon_{ijkl}\Big]\\
&+ \mathcal{DH}\Big[\tfrac14\bar\chi^{ij}{}_k\gamma_a\chi^k{}_{ij}\bar P^a - \tfrac12\bar\chi^{ik}{}_l\chi^{jl}{}_kE_{ij} - \tfrac14\bar\chi^{kl}{}_m\gamma^{ab}\chi^{mn}{}_kT_{ab}{}^{ij}\varepsilon_{ijln} - \tfrac12 E_{ij}\bar R(Q)^i_{ab}R(Q)^{ab\,j}\\
&- \tfrac34 T_{ab}{}^{ij}\bar R(Q)^{ab\,k}\chi^{lm}{}_k\varepsilon_{ijlm} - \tfrac12 D^{ij}{}_{kl}\bar\Lambda_i\chi^{kl}{}_j - \tfrac12 R(V)_{ab}{}^j{}_k\bar\Lambda_i\gamma^{ab}\chi^{ik}{}_j - \tfrac14 E_{ij}E^{ik}\bar\Lambda_l\chi^{jl}{}_k\\
&- \tfrac14 E_{ij}\bar\Lambda_kD\chi^i{}_{lm}\varepsilon^{jklm} + \tfrac1{24}E^{ij}T_{ab}{}^{kl}\bar\Lambda_i\gamma^{ab}\chi^{mn}{}_j\varepsilon_{klmn} + \tfrac18 E^{ij}T_{ab}{}^{kl}\bar\Lambda_m\gamma^{ab}\chi^{mn}{}_i\varepsilon_{jkln}\\
&+ \tfrac14\bar\Lambda_i\gamma\cdot T^{jk}D\chi^i{}_{jk} - \tfrac1{12}\bar\chi^{lm}{}_k\gamma\cdot T^{ij}\bar P\Lambda^k\varepsilon_{ijlm} + \tfrac18\bar\chi^k{}_{lm}\gamma\cdot T_{ij}\bar P\Lambda_k\varepsilon^{ijlm} - \tfrac38\bar\Lambda^i\gamma_aD^2\Lambda_i\bar P^a\\
&+ \tfrac13 E^{ij}T_{ab}{}^{kl}\bar\Lambda_iR(Q)^{ab\,m}\varepsilon_{jklm} + \tfrac16 T^{ab}{}_{ij}\bar P^c\bar\Lambda_k\gamma_cR(Q)_{abl}\varepsilon^{ijkl} - \tfrac34 T_{ab}{}^{ij}\bar P^c\bar R(Q)^{ab\,k}\gamma_c\Lambda^l\varepsilon_{ijkl}\\
&+ \tfrac13\bar\Lambda_iR(Q)^i_{ab}\bar P^aP^b - \tfrac76\bar\Lambda_iR(Q)^{ab\,j}R(V)_{ab}{}^i{}_j - \tfrac32\bar\Lambda_iT^{ij}\cdot R(S)_j + \tfrac14\bar\Lambda_i\gamma^{cd}R(Q)^{ab\,i}R(M)_{abcd}\\
&+ \tfrac18\bar\Lambda^i\gamma_d\Lambda_jR(V)_{ab}{}^j{}_i\bar P_c\varepsilon^{abcd} + \tfrac16\bar\Lambda_k\Lambda_lT^{ab}{}_{ij}R(V)_{ab}{}^k{}_m\varepsilon^{ijlm} + \tfrac83\bar\Lambda^i\gamma^b\Lambda_iP^a\bar P_a\bar P_b\\
&- \tfrac7{12}\bar\Lambda^i\gamma^bD_aD_b\Lambda_i\bar P^a - \tfrac14\bar\Lambda^iD\Lambda_iD_a\bar P^a - \tfrac12\bar\Lambda^i\gamma^aD_b\Lambda_iD_a\bar P^b + \tfrac1{24}\bar\Lambda_i\gamma^aD^2\Lambda^i\bar P_a\\
&+ \tfrac1{24}D^b\bar\Lambda_i\gamma^aD_b\Lambda^i\bar P_a - \tfrac1{12}\bar\Lambda_i\gamma^bD_aD_b\Lambda^i\bar P^a - \tfrac1{12}\bar\Lambda_iD\Lambda^iD_a\bar P^a - \tfrac16\bar\Lambda_i\gamma^aD_b\Lambda^iD_a\bar P^b\\
&- \tfrac18 D_a\bar\Lambda_iD\Lambda^i\bar P^a + \tfrac18 D_a\bar\Lambda^iD\Lambda_i\bar P^a - \tfrac16\bar\Lambda^i\gamma^a\Lambda_iD_aD_b\bar P^b - \tfrac{17}{12}\bar\Lambda^i\gamma^a\Lambda_iP_a\bar P^b\bar P_b\\
&+ \tfrac56 E_{ij}T_{ab\,kl}\bar\Lambda_m\gamma^b\Lambda^i\bar P^a\varepsilon^{jklm} + \tfrac1{18}E^{ij}T_{ab}{}^{kl}\bar\Lambda_i\gamma^b\Lambda^m\bar P^a\varepsilon_{jklm} + \tfrac16 E_{ij}T_{ab}{}^{ik}D^b\bar\Lambda_k\gamma^a\Lambda^j\\
&+ \tfrac16 D^bE_{jk}T_{ab}{}^{ij}\bar\Lambda_i\gamma^a\Lambda^k + \tfrac13 E_{ij}T_{ab}{}^{ik}\bar\Lambda_k\gamma^aD^b\Lambda^j - \tfrac13 E_{ij}D^bT_{ab}{}^{ik}\bar\Lambda_k\gamma^a\Lambda^j\\
&- T_{abij}\bar\Lambda_k\gamma^{ac}D^bD_c\Lambda_l\varepsilon^{ijkl} + \tfrac16 D_aD^cT_{bc\,kl}\bar\Lambda_i\gamma^{ab}\Lambda_j\varepsilon^{ijkl} - \tfrac16 T^{ab}{}_{ij}D_a\bar\Lambda_kD_b\Lambda_l\varepsilon^{ijkl}\\
&- \tfrac16 D_aT^{ab}{}_{ij}\bar\Lambda_kD_b\Lambda_l\varepsilon^{ijkl} + \tfrac1{12}D_aT_{bc\,ij}\bar\Lambda_k\gamma^{ab}D^c\Lambda_l\varepsilon^{ijkl} - \tfrac1{12}D^cT_{ac\,ij}\bar\Lambda_k\gamma^{ab}D_b\Lambda_l\varepsilon^{ijkl}\\
&- \tfrac13 T_{ab\,ij}D^b\bar\Lambda_k\gamma^{ac}D_c\Lambda_l\varepsilon^{ijkl} + \tfrac23 P_aT^{ab\,ij}\bar\Lambda_iD_b\Lambda_j - \tfrac12 D^cT_{bc}{}^{ij}P_a\bar\Lambda_i\gamma^{ab}\Lambda_j\\
&- \tfrac1{24}P^aD_aT_{bc}{}^{ij}\bar\Lambda_i\gamma^{bc}\Lambda_j + \tfrac13 P_aT_{bc}{}^{ij}\bar\Lambda_i\gamma^{ab}D^c\Lambda_j + \tfrac5{12}P^aT_{bc}{}^{ij}\bar\Lambda_i\gamma^{bc}D_a\Lambda_j\\
&- \tfrac59 T_{ac}{}^{ij}\bar P^a\bar P_b\bar\Lambda^k\gamma^{bc}\Lambda^l\varepsilon_{ijkl} + \tfrac7{12}T_{abij}\bar P^bP_c\bar\Lambda_k\gamma^{ac}\Lambda_l\varepsilon^{ijkl} - \tfrac13 E_{ij}E^{ik}\bar\Lambda_k\gamma_a\Lambda^j\bar P^a\\
&+ \tfrac1{48}E_{ij}E^{ij}\bar\Lambda_k\gamma^a\Lambda^k\bar P_a - \tfrac23 T_{ac}{}^{kl}T^{abij}\bar\Lambda_k\gamma_bD^c\Lambda^m\varepsilon_{ijlm} + \tfrac18 T^{kl}\cdot T^{ij}\bar\Lambda_mD\Lambda^m\varepsilon_{ijkl}\\
&+ \tfrac13 T_{ac}{}^{kl}T^{abij}D^c\bar\Lambda_k\gamma_b\Lambda^m\varepsilon_{ijlm} - \tfrac1{24}T^{kl}\cdot T^{ij}\bar\Lambda^mD\Lambda_m\varepsilon_{ijkl} - \tfrac13 T_{ac}{}^{ij}D_bT^{abkl}\bar\Lambda_k\gamma^c\Lambda^m\varepsilon_{ijlm}\\
&+ \tfrac16 T_{ac}{}^{ij}D_bT^{abkl}\bar\Lambda_m\gamma^c\Lambda^m\varepsilon_{ijkl} + \tfrac13 T_{ab}{}^{ij}D_cT^{abkl}\bar\Lambda_k\gamma^c\Lambda^m\varepsilon_{ijlm} + \tfrac13 T_{ab}{}^{ij}D_cT^{abkl}\bar\Lambda_m\gamma^c\Lambda^m\varepsilon_{ijkl}\\
&+ \tfrac18 D^2E^{ij}\bar\Lambda_i\Lambda_j + \tfrac12\bar P^a\bar P_aE_{ij}\bar\Lambda^i\Lambda^j - \tfrac16\bar P^aP_aE^{ij}\bar\Lambda_i\Lambda_j - \tfrac1{24}\bar\Lambda_i\gamma\cdot R(V)^k{}_j\Lambda_kE^{ij}\\
&+ \tfrac1{24}E_{ij}E^{ij}E^{kl}\bar\Lambda_k\Lambda_l - \tfrac1{16}E_{ij}E^{ik}E^{jl}\bar\Lambda_k\Lambda_l - \tfrac7{288}E^{ij}E^{kl}T_{ab}{}^{mn}\bar\Lambda_i\gamma^{ab}\Lambda_k\varepsilon_{jlmn}\\
&+ \tfrac1{16}T_{ab\,ij}T^{ab}{}_{kl}T_{cd}{}^{ij}\bar\Lambda_m\gamma^{cd}\Lambda_n\varepsilon^{klmn} + \tfrac1{16}T_{ab\,ij}T^{ab}{}_{kl}T_{cd}{}^{kl}\bar\Lambda_m\gamma^{cd}\Lambda_n\varepsilon^{ijmn}\Big]\\
&+ \mathcal D^2\mathcal H\Big[\tfrac14 E_{ij}T_{ab}{}^{ik}\bar\Lambda_l\gamma^{ab}\chi^{jl}{}_k - \tfrac58 E_{ij}T_{ab}{}^{ik}\bar\Lambda_kR(Q)^{ab\,j} - \tfrac18 E_{ij}E_{kl}\bar\Lambda_m\chi^{ik}{}_n\varepsilon^{jlmn}\\
&+ \tfrac14\bar\Lambda_k\chi^{mn}{}_lT^{ij}\cdot T^{kl}\varepsilon_{ijmn} + \tfrac14\bar\Lambda_m\gamma^b{}_c\chi^{mn}{}_iT_{ab}{}^{ij}T^{ac\,kl}\varepsilon_{jkln} + \tfrac5{64}\bar\Lambda_m\gamma^{cd}R(Q)^m_{ab}T^{ab\,ij}T_{cd}{}^{kl}\varepsilon_{ijkl}
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{5}{8}\bar\Lambda_i R(Q)_{bc}^{m}\, T^{ab\,ij} T_a{}^{c\,kl}\varepsilon_{jklm} - \tfrac{3}{4}\bar\Lambda^i\gamma^a D_b\Lambda_i \bar P_a \bar P^b - \tfrac{5}{24}\bar\Lambda^i D\Lambda_i \bar P^a \bar P_a - \tfrac{5}{12}\bar\Lambda^i\gamma_a\Lambda_i \bar P^a D_b \bar P^b \\
&- \tfrac{1}{2}\bar\Lambda^i\gamma^b\Lambda_i \bar P^a D_a \bar P_b + \tfrac{1}{12}\bar\Lambda_i\gamma^a D_b\Lambda^i \bar P_a \bar P^b - \tfrac{1}{8}\bar\Lambda_i D\Lambda^i \bar P^a \bar P_a - \tfrac{1}{6}E_{ij}\bar P^b \bar\Lambda_k\gamma^a\Lambda^i T_{ab}{}^{jk} \\
&- \tfrac{1}{6}T_{ab\,ij}\bar P^a \bar\Lambda_k D^b\Lambda_l \varepsilon^{ijkl} + \tfrac{1}{24}\bar P_a D^c T_{bcij}\bar\Lambda_k\gamma^{ab}\Lambda_l\varepsilon^{ijkl} - \tfrac{7}{12}T_{ab\,ij}\bar P_c \bar\Lambda_k\gamma^{ac}D^b\Lambda_l\varepsilon^{ijkl} \\
&- \tfrac{1}{8}T_{ab\,ij}D_c\bar P^b \bar\Lambda_k\gamma^{ac}\Lambda_l\varepsilon^{ijkl} + \tfrac{1}{6}T_{bc}{}^{ij}P^a\bar P_a\bar\Lambda_i\gamma^{bc}\Lambda_j + \tfrac{1}{48}T_{ab}{}^{ij}T^{abkl}\bar P_c\bar\Lambda_m\gamma^c\Lambda^m\varepsilon_{ijkl} \\
&- \tfrac{1}{2}\bar\Lambda_i\gamma_b{}^c\Lambda_k R(V)_{ac}{}^k{}_j T^{ab\,ij} + \tfrac{1}{2}\bar\Lambda_i\Lambda_k R(V)_{ab}{}^k{}_j T^{ab\,ij} + \tfrac{1}{16}\bar\Lambda_k\gamma^{ab}\Lambda_l R(V)_{ab}{}^i{}_m E_{ij}\varepsilon^{jklm} \\
&+ \tfrac{1}{8}\bar\Lambda_i\gamma^{cd}\Lambda_j R(M)_{abcd}T^{ab\,ij} - \tfrac{1}{16}\bar\Lambda_i\gamma^{ab}\Lambda_j T_{ab}{}^{kl}D^{ij}{}_{kl} + \tfrac{1}{16}\bar\Lambda_j\Lambda_n E_{im}D^{ij}{}_{kl}\varepsilon^{klmn} \\
&- \tfrac{1}{48}E_{ij}E^{ij}T_{ab}{}^{kl}\bar\Lambda_k\gamma^{ab}\Lambda_l + \tfrac{1}{24}E_{ij}E^{ik}T_{ab}{}^{jl}\bar\Lambda_k\gamma^{ab}\Lambda_l - \tfrac{1}{48}E^{ij}T^{ab\,kl}T_{ab}{}^{mn}\bar\Lambda_i\Lambda_j\varepsilon_{klmn}\Big] \\
&+ \mathcal{D}^3\mathcal{H}\Big[\tfrac{1}{8}T_{ab\,ij}\bar P^a\bar P_c\bar\Lambda_k\gamma^{bc}\Lambda_l\varepsilon^{ijkl} - \tfrac{1}{64}E_{ij}E_{kl}T_{ab}{}^{ik}\bar\Lambda_m\gamma^{ab}\Lambda_n\varepsilon^{jlmn} - \tfrac{1}{12}T^{ab\,ij}\Lambda_m\varepsilon_{jkln} \\
&+ \tfrac{1}{64}T^{ab\,ij}T_{ab}{}^{kl}T_{cd}{}^{mn}\bar\Lambda_i\gamma^{cd}\Lambda_j\varepsilon_{klmn} - \tfrac{1}{32}T_{ab}{}^{ij}T^{ac\,kl}T_{cd}{}^{mn}\bar\Lambda_i\gamma^{bd}\Lambda_j\varepsilon_{klmn} \\
&+ \tfrac{1}{192}E_{ij}E_{kl}E_{mn}\bar\Lambda_p\Lambda_q\varepsilon^{ikmp}\varepsilon^{jlnq} - \tfrac{3}{8}\bar\Lambda^i\gamma_b\Lambda_i\bar P_a\bar P^a\bar P^b\Big] + \text{h.c.}
\end{aligned}
\tag{7.7}
$$

The Lagrangians derived above can be directly incorporated into Poincaré supergravity as a four-derivative coupling following the same construction carried out originally, by including the superconformal Lagrangian of this Letter before proceeding to the standard gauge choices. The SU(1, 1) symmetry will then become entangled with an electric-magnetic duality transformation in the vector-multiplet sector. Alternatively this Lagrangian can be obtained in one step by applying the superform method directly based on an extended field configuration consisting of at least six (on-shell) vector supermultiplets and the Weyl supermultiplet, where one must bear in mind that the supersymmetry algebra for this field representation will no longer close off shell. It is then possible to compare the corresponding expressions with the R^2-couplings for $N = 4$ Poincaré supergravity, which may also depend non-trivially on the coset fields. In the context of Poincaré supergravity the higher-derivative couplings are primarily studied as potential counter-terms that could render the theory finite. At this moment there is agreement that this theory is not finite at the four-loop level. The presence of a U(1) anomaly and the non-trivial dependence on the coset fields plays an important role in this discussion, as was extensively discussed. As should be clear from the iterative procedure that was used in this paper, the consistency of each term in our result relies on the consistency of many other terms. Therefore, our computation passes a multitude of crosschecks. It should also be noted that all the terms in our result correspond to possible Feynman diagrams of the gauge theory with divergent contributions.

8 Conclusions and outlook

In this paper, we have explicitly constructed an $\mathcal{N} = 4$ density formula using the superform method. Invariance under the local $\mathcal{N} = 4$ superconformal symmetries is ensured provided the lowest Weyl weight fields satisfy the set of constraints, and that the remaining fields are defined via the supersymmetry transformation rules. We then showed that, by expressing these fields in terms of those of the $\mathcal{N} = 4$ Weyl multiplet such that the constraints are satisfied, the density formula leads to a class of $\mathcal{N} = 4$ conformal supergravity actions parametrized by a holomorphic function of the coset scalars. Based on the uniqueness of the $\mathcal{N} = 4$ supercurrent, we further argued that this must correspond to the most general class of maximal conformal supergravity actions. We presented its expression up to terms which are quadratic in the covariant fermion fields. A stringent check of this result is that when the function is set to a constant, it recovers. For ergonomic considerations, the complete action is given explicitly in an addendum file. As a second application of the density formula, we also re-derived an on-shell sector of the action constructed for a vector multiplet in a background of conformal supergravity. An intriguing feature of the density formula we employed is that it seems it could be derived from a single superfield $\Phi^{ij}{}_{kl}$ and a constant $c^{\boldsymbol{\alpha\beta}}$. The properties of this superfield resemble those of a G-analytic superfield in $(4,2,2)$ superspace, but it cannot be a Lagrangian in that superspace because it has the wrong dimension. Perhaps it can be used to build an action principle in $(4,2,2)$ superspace along the lines. The construction of the full class of $\mathcal{N} = 4$ conformal supergravity actions opens up various perspectives on the higher-derivative structure of the Poincaré theory. As was shown already long ago in the research literature, $\mathcal{N} = 4$ Poincaré supergravity at the two-derivative level can be described as a system of six vector multiplets coupled to conformal supergravity. The standard Poincaré action is recovered after gauge fixing the conformal symmetries and integrating out the various auxiliary fields of the Weyl multiplet. It is now possible to consider the class of actions constructed in this paper as a deformation of the two-derivative conformal setup. In this case the transition to the Poincaré theory is non-trivial as the field equations of the auxiliary fields have now become non-linear. This requires to integrate out the fields through an iterative procedure, which will result in an infinite power series of the spin-1 field strengths and their derivatives. We will show in an upcoming paper that this procedure can be carried out consistently and leads to a class of supersymmetric higher-derivative Poincaré invariants which depends on the holomorphic function of the coset scalars. The procedure can also be applied to describe Poincaré supergravity coupled to vector multiplets. These higher-derivative Poincaré couplings are relevant from several point views. When considered on-shell, they could be directly compared with the results obtained. It would also be interesting to see if they could be embedded in the formalism where higher-derivative corrections are described as deformations of the twisted self-duality constraint relating the spin-1 field strengths to their magnetic duals. Another application concerns the matching of subleading corrections to the microscopic entropy of $\mathcal{N} = 4$ black holes obtained via state counting. From the supergravity point of view, some of these corrections are known to originate from the class of couplings considered in this paper, and could be calculated by considering the induced modifications to the area law, or perhaps by using more recent localization techniques along the lines. These approaches have so far relied on a truncated $\mathcal{N} = 2$ setting and it should be interesting to reconsider these results in a fully $\mathcal{N} = 4$ supersymmetric formalism. Finally, these invariants might clarify the ultraviolet properties of the Poincaré theory. Explicit loop computations have revealed a divergence at four loops which is believed to be connected to the presence of a potential anomaly in the duality symmetry of the theory. It was however shown recently that there exists a finite counterterm, whose leading term includes the square of the Riemann tensor multiplied by a holomorphic function of the coset scalars, and which cancels the anomalous contribution of the graphs up to two loops. The consequences of this counterterm for the finiteness of the Poincaré theory at four loops however remain to be explored. While these amplitude computations rely on a description of the counterterm via the double copy construction, its explicit supersymmetric expression should follow from the class of invariants constructed in this paper, provided the correct holomorphic function is chosen. In the context of Poincaré supergravity the higher-derivative couplings are primarily studied as potential counterterms that could render the theory finite. At this moment there is agreement that this theory is not finite at the four-loop level . The presence of a U(1) anomaly and the non-trivial dependence on the coset fields plays an important role in this discussion, as was extensively discussed. As we mentioned earlier, another possible application of the result of this Letter concerns the calculation of the corrections to $N{=}4$ supersymmetric black hole entropy that are known to originate from precisely this class of Lagrangians. Both these approaches have only been applied so far to $N{=}2$ supersymmetric truncations. It should be very interesting to understand these results in the context of $N{=}4$ supersymmetric formulation.

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