ON THE TWIN PRIME CONJECTURE

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ABSTRACT. Every prime number $p \geq 5$ has the form $6x-1$ or $6x+1$. We call x the **generator** of p. Twin primes are distinguished by a **common generator** for each pair. Therefore it makes sense to search for the Twin Primes on the level of their generators. This paper presents a new approach to prove the **Twin Prime Conjecture** by a method to extract all Twin Primes on the level of the Twin Prime Generators. We define the ω_{p_n}–numbers x as numbers for that holds that $6x-1$ and $6x+1$ are coprime to the primes $5, 7, \dots, p_n$. By dint of the average size $\bar{\delta}(p_n)$ of the ω_{p_n}–gaps we can prove the **Twin Prime Conjecture**..

1. Introduction

The question on the infinity of the twin primes keeps busy many mathematicians for a long time. 1919 V. Brun [3] had proved that the series of the inverted twin primes converges while he had tried to prove the Twin Prime Conjecture. Several authors worked on bounds for the length of prime gaps (see f.i. [4, 5, 6]). In 2008 B. Green and T. Tao [7] succeeded in proving that there are arbitrarily long arithmetic progressions containing only prime numbers. 2014 Y. Zhang [8] obtained a great attention with his proof that there are infinitely many consecutive primes with a gap of $70,000,000$ at most. With the project "PolyMath8", in particular forced by T. Tao [9], this bound could be lessened down to 246 respectively to 12 assuming the validity of the Elliott–Halberstam Conjecture [10].

In [11] an attempt to prove the Twin Prime Conjecture only by elementary instruments is shown. Another approach as in the most works on this topic is presented there. The looking for twin primes is transfered to the level of their generators because each twin prime has a common generator. However the proof based on a vague assumption over the distribution of the ω_p–numbers. With this new paper this gap will be closed. For a better understanding some definitions, statements and results from [11] are repeated in the next two sections.

2. Twin Prime Generators

It is well known that every prime number $p \geq 5$ has the form $6x-1$ or $6x+1$. We will call x the **generator** of p. Twin primes are distinguished by a common generator for each pair.

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Definition 2.1. Let

$\mathbb{N}$ be the set of the positve integers,

$\mathbb{P}$ the set of the prime numbers, $\mathbb{P}^*$ primes ≥ 5,

$\mathbb{P}_- = \{p \in \mathbb{P}^* \mid p \equiv -1 (\bmod 6)\}$, $\mathbb{P}_+ = \{p \in \mathbb{P}^* \mid p \equiv +1 (\bmod 6)\}$

and

$$\kappa(n) := \left\lfloor \frac{n+1}{6} \right\rfloor \text{ for } n \in \mathbb{N} \tag{2.1}$$

the **generator** function of the pair $(6\kappa(n) - 1, 6\kappa(n) + 1)$. If a pair $(6x - 1, 6x + 1)$ is a twin prime, then we call x as a **twin prime generator** and

$$\mathbb{G} := \{x \in \mathbb{N} \mid 6x - 1 \in \mathbb{P}_-, 6x + 1 \in \mathbb{P}_+\}$$

is the set of all twin prime generators. Hence the pair $(5, 7)$ is the least twin prime in our consideration.

In order to transfer the searching for twin primes to the level of their generators we need a criterion for checking a natural number to be a twin prime generator.

Theorem 2.2. *A number x is a **twin prime generator**, a member of $\mathbb{G}$, if and only if there is **no** $p \in \mathbb{P}^*$ with $p < 6x - 1$ that one of the following congruences fulfills:*

$$x \equiv -\kappa(p) (\bmod p) \tag{2.2}$$

$$x \equiv +\kappa(p) (\bmod p) \tag{2.3}$$

Proof. At first we assume that there is a prime $p \in \mathbb{P}^*$ with $p < 6x - 1$ such that (2.2) or (2.3) is valid. There are two cases.

(1) $p \in \mathbb{P}_-$, what means $p = 6\kappa(p) - 1$:
If (2.2) is true, then there is an $n \in \mathbb{N}$ with

$$\begin{aligned} x &= -\kappa(p) + n \cdot (6\kappa(p) - 1) \\ 6x &= -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) \\ 6x + 1 &= -6\kappa(p) + 6n \cdot (6\kappa(p) - 1) + 1 \\ &= (6n - 1)(6\kappa(p) - 1) \\ &= (6n - 1) \cdot p \\ \Longrightarrow \; 6x + 1 &\equiv 0 (\bmod p) \implies x \notin \mathbb{G} \end{aligned}$$

For (2.3) the proof will be done with $6x - 1$:

$$\begin{aligned} 6x - 1 &= 6\kappa(p) + 6n \cdot (6\kappa(p) - 1) - 1 \\ &= (6n + 1)(6\kappa(p) - 1) \\ &= (6n + 1) \cdot p \\ \Longrightarrow \; 6x - 1 &\equiv 0 (\bmod p) \implies x \notin \mathbb{G} \end{aligned}$$

(2) $p \in \mathbb{P}_+$, what means $p = 6\kappa(p) + 1$:
We go the same way with (2.2) and $6x - 1$ as well as (2.3) and $6x + 1$:

$$6x - 1 = (6n - 1)(6\kappa(p) + 1) \implies 6x - 1 \equiv 0 (\bmod p)$$

$$6x + 1 = (6n + 1)(6\kappa(p) + 1) \implies 6x + 1 \equiv 0 (\bmod p)$$

With these it's shown that $x \notin \mathbb{G}$, if the congruences (2.2) or (2.3) are valid. They cannot be true both because they exclude each other.

If on the other hand $x \notin \mathbb{G}$, then $6x-1$ or $6x+1$ is no prime. Let be $6x-1 \equiv 0 (\operatorname{mod} p)$ for any $p \in \mathbb{P}_-$. Then we have

$$\begin{aligned} 6x-1 &\equiv 0 (\operatorname{mod} p) \equiv p (\operatorname{mod} p) \\ &\equiv (6\kappa(p)-1)(\operatorname{mod} p) \\ 6x &\equiv 6\kappa(p)(\operatorname{mod} p) \\ x &\equiv \kappa(p)(\operatorname{mod} p) \implies (2.3). \end{aligned}$$

For any $p \in \mathbb{P}_+$ we have

$$\begin{aligned} 6x-1 &\equiv -p (\operatorname{mod} p) \\ &\equiv -(6\kappa(p)+1)(\operatorname{mod} p) \\ 6x &\equiv -6\kappa(p)(\operatorname{mod} p) \\ x &\equiv -\kappa(p)(\operatorname{mod} p) \implies (2.2). \end{aligned}$$

The other both cases we can handle in the same way. Therefore either (2.2) or (2.3) is valid if $x \notin \mathbb{G}$. □

If we consider that the least proper divisor of a number $6x+1$ is less or equal to $\sqrt{6x+1}$, then p in the congruences (2.2) and (2.3) can be limited by

$$\hat{p}(x) = \max(p \in \mathbb{P}^* \mid p \leq \sqrt{6x+1}).$$

Remark 2.3. Henceforth we will use the letter p for a general prime number and p_n if we describe an element of a sequence of primes.

With p_n as the n-the prime number[1] and $\pi(z)$ as the number of primes $\leq z$ we have with

$$\begin{gathered} x \equiv -\kappa(p_n)(\operatorname{mod} p_n) \text{ or } x \equiv +\kappa(p_n)(\operatorname{mod} p_n) \\ \text{for} \\ 3 \leq n \leq \pi\left(\hat{p}(x)\right) \end{gathered} \tag{2.4}$$

a provable system of criteria to exclude all numbers $x \geq 4$ being no twin prime generators. It is not usable for $x < 4$, because in this case would be $\hat{p}(x) < p_3 = 5$. Although the positions $1, 2, 3$ evidently are members of $\mathbb{G}$.

Since the modules are primes the criteria are independent among each other.

Hence we can square the congruences (2.4) and get

$$x^2 \equiv \kappa(p_n)^2 (\operatorname{mod} p_n) \text{ for } 3 \leq n \leq \pi\left(\hat{p}(x)\right). \tag{2.5}$$

This results in a system of **indicator functions** $\psi(x, p_n)$ for that holds for $3 \leq n \leq \pi\left(\hat{p}(x)\right)$

$$\begin{aligned} x^2 - \kappa(p_n)^2 &\equiv \psi(x, p_n)(\operatorname{mod} p_n) \text{ respectively} \\ \psi(x, p_n) &= \left(x^2 - \kappa(p_n)^2\right) \operatorname{Mod} p_n. \end{aligned} \tag{2.6}$$

[1]It is $p_1 = 2$

Obviously holds that if $\psi(x,p) = 0$ for any $p \le \hat{p}(x)$, then x cannot be a twin prime generator[2] and inversed, x is a twin prime generator, if for all $p \in \mathbb{P}^*$ with $p \le \hat{p}(x)$ holds $\psi(x,p) > 0$. Due to *modulo* the indicator function $\psi(x,p)$ is periodical in x with a period length of p. With the indicator functions for all modules $p_3, \dots, p_n$ we build the *aggregate* indicator functions

$$\Psi(x, p_n) = \prod_{i=3}^{n} \frac{\psi(x, p_i)}{p_i}$$
$$\text{and} \tag{2.7}$$
$$\widehat{\Psi}(x) = \Psi(x, \hat{p}(x)).$$

Because the co-domain of $\psi(x,p)$ consists of integers between 0 and $p-1$, the aggregate functions $\Psi(x,p)$ and $\widehat{\Psi}(x)$ have rational values between 0 and < 1.

Lemma 2.4. *For $x \in \mathbb{N} \mid x \le \kappa(p)$ holds*

$$\Psi(x,p) = 0$$

even if x is a twin prime generator.

Proof. If x is a twin prime generator, then holds $q = 6x-1 \in \mathbb{P}^*$ and $q+2 = 6x+1 \in \mathbb{P}^*$. Then is $x = \kappa(q)$ and $\frac{\psi(\kappa(q),q)}{q}$ is a factor of the product $\Psi(x,p)$, because $\kappa(q) \le \kappa(p)$ and therefore $q \le p$. And since $\psi(\kappa(q), q) = 0$ we get $\Psi(x,p) = 0$. □

Therefore the aggregate indicator function can only be used for $x > \kappa(p)$. This we will call as the *κ-condition.*

Definition 2.5. Let be

$$\xi_n := \min(x \in \mathbb{N} \mid \hat{p}(x) = p_n). \tag{2.8}$$

It is the first integer x for that the modul p_n could be a prime factor of $6x \pm 1$. Therefore we denote it as the **origin** of the modul p_n.

Up from ξ_n in every ψ–period there are just $p_n - 2$ positions with $\psi(x, p_n) > 0$ and two positions with $\psi(x, p_n) = 0$, once if (2.2) and on the other hand if (2.3) holds. Obviously the distance between these both positions is $2\kappa(p_n)$ respectively $4\kappa(p_n) \pm 1$ since the period length is $6\kappa(p_n) \pm 1$.

Let be $p_n \le \hat{p}(x) \le \sqrt{6x+1}$ and therefore $p_n^2 \le 6x+1$. Then $\frac{p_n^2-1}{6}$ is the least number that meets this relation. Comparing with (2.8) we get

$$\xi_n = \frac{p_n^2 - 1}{6}. \tag{2.9}$$

It is easy to prove that for every integer $n \ge 3$ holds that ξ_n is an integer divisible by 4.

Lemma 2.6. *At the origin ξ_n cannot be a twin prime generator since we have $\psi(\xi_n, p_n) = 0$.*

[2]For $x < \kappa(p)$ we use $\psi(x,p) = \big(p + x^2 - \kappa(p)^2\big) \operatorname{Mod} p$. Therefore holds $\psi(x,p) \ge 0$

Proof. We substitute p_n by $6\kappa(p_n) \pm 1$. With this and (2.9) holds

$$\begin{aligned}\xi_n &= \frac{(6\kappa(p_n) \pm 1)^2 - 1}{6} \\ &= \frac{6\kappa(p_n)\,(6\kappa(p_n) \pm 2)}{6} \\ &= \kappa(p_n)\,(6\kappa(p_n) \pm 1) \pm \kappa(p_n) \\ &= \kappa(p_n) \cdot p_n \pm \kappa(p_n) \\ &\equiv \pm\kappa(p_n) (\operatorname{mod} p_n) \implies \psi(\xi_n, p_n) = 0.\end{aligned}$$

□

In the proof we have seen that for every prime p holds that $p^2 - 1$ is an integer divisible by 6.

For every $x \geq \xi_n$ the local position relative to the period start[3] can be determined by the **position function** $\tau(x, p_n)$:

$$\begin{aligned}x + \kappa(p_n) &\equiv \tau(x, p_n) (\operatorname{mod} p_n) \text{ respectively} \\ \tau(x, p_n) &= (x + \kappa(p_n)) \operatorname{Mod} p_n.\end{aligned} \tag{2.10}$$

Hence the co-domain of the position function $\tau(x, p)$ are the integers $0, 1, \ldots, p-1$ and it has in x a period length of p. Between the indicator function $\psi(x, p)$ and the position function $\tau(x, p)$ there is the following relationship:

$$\begin{aligned}\psi(x, p) &= (x^2 - \kappa(p)^2) \operatorname{Mod} p \\ &= (x + \kappa(p))(x - \kappa(p)) \operatorname{Mod} p \\ &= \tau(x, p) \cdot (x - \kappa(p)) \operatorname{Mod} p \\ &= \tau(x, p) \cdot (\tau(x, p) - 2\kappa(p)) \operatorname{Mod} p.\end{aligned} \tag{2.11}$$

Obviously holds $\psi(x, p) = 0$ if and only if $\tau(x, p) = 0$ or $\tau(x, p) = 2\kappa(p)$. From (2.10) we see that the values of the position function $\tau(x, p)$ consists exactly of the p values $0, 1, 2, \ldots, p-1$. Because the values 0 and $2\kappa(p)$ indicate that x cannot be a twin prime generator (see Theorem 2.2) we will call them as τ*–bad values* and the others as τ*–good values*. Hence there are two τ–bad values and $p - 2$ τ–good values for each modul p.

3. The ω_{p_n}–numbers

For every natural number x in the interval

$$\mathcal{A}_n := \{x \in \mathbb{N} \mid \xi_n \leq x < \xi_{n+1}\} \tag{3.1}$$

$\hat{p}(x)$ persists constant on the value p_n. The length of this interval will be denoted as d_n. It is depending on the distance between successive primes. Since they can

[3] For $p_n \in \mathbb{P}_-$ the period start is ξ_n and else it is $\xi_n - 2\kappa(p_n)$.

only be even, we have with $a = 2, 4, 6, \ldots$

$$\begin{aligned} d_n &= \frac{(p_n + a)^2 - 1}{6} - \frac{p_n^2 - 1}{6} \\ &= \frac{2ap_n + a^2}{6} \\ &= \frac{a}{3}(p_n + \frac{a}{2}) \\ &\geq \frac{2}{3}(p_n + 1). \end{aligned} \tag{3.2}$$

On the other hand we obtain because of $p_{n+1} < 2p_n$ (see [2], p. 188)

$$\begin{aligned} d_n &= \frac{p_{n+1}^2 - 1}{6} - \frac{p_n^2 - 1}{6} \\ &= \frac{p_{n+1}^2 - p_n^2}{6} \\ &= \frac{(p_{n+1} + p_n)(p_{n+1} - p_n)}{6} \\ &< \frac{3p_n \cdot p_n}{6} = \frac{p_n^2}{2}. \end{aligned}$$

And since d_n is an integer and p_n^2 is odd it holds

$$d_n \leq \frac{p_n^2 - 1}{2} = 3\xi_n \text{ and hence} \tag{3.3}$$

$$\xi_{n+1} \leq 4\xi_n. \tag{3.4}$$

For all $x \geq \xi_n$ the κ-*condition* (see Lemma 2.4) is satisfied since

$$\begin{aligned} \xi_n &= \frac{p_n^2 - 1}{6} \\ &= \frac{(6\kappa(p_n) \pm 1)^2 - 1}{6} \\ &= \kappa(p_n)(6\kappa(p_n) \pm 2) \\ &> \kappa(p_n). \end{aligned}$$

The congruences in (2.10)

$$x + \kappa(p_i) \equiv \tau(x, p_i) (\bmod p_i), \quad 3 \leq i \leq n \tag{3.5}$$

fulfill the requirements of the Chinese Remainder Theorem (see [1], p. 89). Therefore it is modulo $5 \cdot 7 \cdot \ldots \cdot p_n$ uniquely resolvable. With

$$p_n\#_5 := \prod_{i=3}^{n} p_i = 5 \cdot 7 \cdot \ldots \cdot p_n \tag{3.6}$$

it's $(\bmod\, p_n\#_5)$ [4] uniquely resolvable. Therefore the aggregate indicator function $\Psi(x, p_n)$ has the period length $p_n\#_5$ and it holds:

$$\Psi(x + a \cdot p_n\#_5, p_n) = \Psi(x, p_n) \mid a \in \mathbb{N}.$$

[4] It is $p_n\#_5 = \frac{p_n\#}{6}$, with the primorial $p_n\#$.

Definition 3.1. A positive integer x will be called as an ω_{p_n}–**number**, if both $6x-1$ and $6x+1$ are coprime [5] to $p_n\#_5$. Then is $\Psi(x,p_n)>0$.

Corollary 3.2. *Because of the periodicity of the aggregate indicator function $\Psi(x,p_n)$ in x there are infinitely many ω_{p_n}–numbers.*

Definition 3.3. Let be

$$\mathcal{P}_n := \{x \in \mathbb{N} \mid \xi_n \leq x < \xi_n + p_n\#_5\}$$

the interval of one period of the aggregate indicator function $\Psi(x,p_n)$. We'll denote it henceforth as **period section**. Evidently is $\mathcal{A}_n \subset \mathcal{P}_n$ for all $n \geq 3$.

Proposition 3.4. *The period section $\mathcal{P}_n$ contains*

$$\phi(p_n) := \prod_{k=3}^{n}(p_k-2) \tag{3.7}$$

ω_{p_n}–numbers.

Proof. Due to Definition 3.3 the period section $\mathcal{P}_n$ contains $p_n\#_5$ successive natural numbers $\xi_n, \xi_n+1, \ldots, \xi_n+p_n\#_5-1$. Each sequence of p_k successive members of them for $3 \leq k \leq n$ contains due to (2.11) two members $x_{\mp}$ with the two τ–bad values

$$\tau(x_-,p_k)=0 \text{ and } \tau(x_+,p_k)=2\kappa(p_k)$$

and p_k-2 members y with the p_k-2 τ–good values. Each ω_{p_n}–number $y \in \mathcal{P}_n$ is represented exactly by one m–tuple with $m=n-2$

$$(\tau_3, \tau_4, \ldots, \tau_n),$$

where all values $\tau_k=\tau(y,p_k)$ are τ–good values and since by virtue of (2.11) for only τ–good values holds

$$\Psi(y,p_n)=\prod_{k=3}^{n}\frac{\tau_k(\tau_k-2\kappa(p_k)\,\mathrm{Mod}\,p_k}{p_k}>0.$$

Because the primes $5,7,\ldots,p_n$ are independent, all the m–tuples are different and their number is

$$\prod_{k=3}^{n}(p_k-2).$$

Since each such m–tuple represents exactly one different ω_{p_n}–number in $\mathcal{P}_n$ it holds

$$\phi(p_n)=\prod_{k=3}^{n}(p_k-2).$$

□

Conclusion 3.5. *Let x be an ω_{p_n}–number as a member of $\mathcal{A}_n$. Then x is a* ***twin prime generators*** *because by virtue of Definition 3.1 $6x-1$ as well as $6x+1$ are prime to $5,7,\ldots,p_n$ and it holds $\Psi(x,p_n)=\widehat{\Psi}(x)>0$.*

Vice versa an ω_{p_n}–number as member of $\mathcal{A}_m$ by $m>n$ is not necessarily a twin prime generator.

[5] Then is $\gcd(36x^2-1, p_n\#_5)=1$.

The ratio between (3.7) and the period length by virtue of (3.6) results in

$$\eta(p_n) := \frac{\phi(p_n)}{p_n\#_5} = \prod_{i=3}^{n} \frac{p_i - 2}{p_i}, \tag{3.8}$$

as the **average density** of the ω_{p_n}–numbers in $\mathcal{P}_n$. Obviously $\eta(p)$ is a strictly monotonic decreasing function.

Proposition 3.6. *The average density function $\eta(p_n)$ is double-sided bounded by*

$$\frac{3}{p_n} < \eta(p_n) < \frac{3}{\log p_n}.$$

Proof.

A) At first we prove the left inequality.
Because all primes > 2 are odd numbers, the count of primes $\leq p_n$ is less than the count of odd numbers in the same range. All factors of $\eta(p_n)$ are less than 1. Thus we get with $m = \frac{p_n+1}{2}$

$$\begin{aligned}\eta(p_n) = \prod_{k=3}^{n} \frac{p_k - 2}{p_k} &> \prod_{k=3}^{m} \frac{(2k-1) - 2}{2k-1} \\ &= \frac{3}{5} \cdot \frac{5}{7} \cdot \frac{7}{9} \cdot \ldots \cdot \frac{p_n - 4}{p_n - 2} \cdot \frac{p_n - 2}{p_n} \\ &= \frac{3}{p_n}.\end{aligned}$$

B) It's well known that (see [1], p. 40 above)

$$\log x < \prod_{p \leq x} \left(1 - \frac{1}{p}\right)^{-1} = \prod_{k=1}^{\pi(x)} \left(\frac{p_k - 1}{p_k}\right)^{-1}.$$

Therefore is

$$\begin{aligned}\frac{1}{\log p_n} &> \prod_{k=1}^{n} \frac{p_k - 1}{p_k} \\ &= \frac{1}{2} \cdot \frac{2}{3} \cdot \prod_{k=3}^{n} \frac{p_k - 1}{p_k} = \frac{1}{3} \cdot \prod_{k=3}^{n} \frac{p_k - 1}{p_k} \\ &> \frac{1}{3} \cdot \prod_{k=3}^{n} \frac{p_k - 2}{p_k} = \frac{\eta(p_n)}{3}.\end{aligned}$$

Hence we get for the right inequality $\eta(p_n) < \dfrac{3}{\log p_n}$.

With it the proof is completed. □

Corollary 3.7. *Since both bounds in Proposition 3.6 go to zero holds*

$$\lim_{n\to\infty} \eta(p_n) = 0.$$

This means that the ω_{p_n}–numbers have an asymptotic zero–density. Therefore the twin prime generators as subset of the ω_{p_n}–numbers have an asymptotic zero–density too. This result is in accordance with the the fact that also the set of the primes has an asymptotic zero–density.

4. The ω_{p_n}–Gaps

Definition 4.1. The inverse of $\eta(p_n)$ means the **average distance** between two immediately successive ω_{p_n}–numbers in the period section $\mathcal{P}_n$

$$\bar{\delta}(p_n) := \frac{1}{\eta(p_n)}, \tag{4.1}$$

the average size of the so called ω_{p_n}–**gaps**.

Corollary 4.2. *From Proposition 3.6 follows*

$$\lim_{n\to\infty} \bar{\delta}(p_n) = \infty.$$

Theorem 4.3. *For $p_n > 200$ the square of the average size of the ω_{p_n}–gaps is less than the length of the A-section $\mathcal{A}_n$*

$$\bar{\delta}(p_n)^2 < d_n \textit{ for } p_n > 200.$$

Proof. At first we prove that

$$u(p_n) := p_n \eta(p_n)^2$$

is an increasing function by trend. We consider their properties for two cases:

A) $p_{n+1} \geq p_n + 4$:

$$\begin{aligned} u(p_{n+1}) - u(p_n) &= \eta(p_n)^2 \left(p_{n+1} \frac{(p_{n+1}-2)^2}{p_{n+1}^2} - p_n \right) \\ &= \eta(p_n)^2 \left(\frac{p_{n+1}(p_{n+1}-4)+4}{p_{n+1}} - p_n \right) \\ &= \eta(p_n)^2 \left(p_{n+1} - 4 - p_n + \frac{4}{p_{n+1}} \right) \\ &\geq \frac{4\eta(p_n)^2}{p_{n+1}} > 0. \end{aligned}$$

Hence it holds in this case $u(p_{n+1}) > u(p_n)$.

B) $p_{n+1} = p_n + 2$:

$$u(p_{n+1}) - u(p_n) = \eta(p_n)^2 \left(p_{n+1} \frac{(p_{n+1}-2)^2}{p_{n+1}^2} - p_n \right)$$

and since $p_n = p_{n+1} - 2$

$$\begin{aligned} &= p_n \eta(p_n)^2 \left(\frac{p_n}{p_{n+1}} - 1 \right) \\ &= p_n \eta(p_n)^2 \cdot \left(\frac{p_n - p_{n+1}}{p_{n+1}} \right) \\ &= -\frac{2p_n}{p_n+2} \eta(p_n)^2 < 0. \end{aligned}$$

Hence holds $u(p_{n+1}) < u(p_n)$. Now we set $u(p_{n+1}) = u(p_n) - v(p_n)$ with the "*loss function*"

$$v(p_n) := \frac{2p_n}{p_n+2} \eta(p_n)^2.$$

At first we'll look for the behavior of $u(p_{n+2})$ depending on the prime distance

$$a := p_{n+2} - p_{n+1} = p_{n+2} - p_n - 2 \text{ for } a = 4, 6, 10, 12, 16, \ldots$$

With it we get

$$\begin{aligned} u(p_{n+2}) &= p_{n+2} \cdot \eta(p_{n+2})^2 \\ &= (p_n + a + 2) \cdot \eta(p_n + a + 2)^2 \\ &= (p_n + a + 2) \cdot \frac{(p_n + a)^2}{(p_n + a + 2)^2} \cdot \frac{p_n^2}{(p_n + 2)^2} \cdot \eta(p_n)^2 \\ &= u(p_n) \cdot \frac{(p_n + a)^2 \cdot p_n}{(p_n + a + 2)(p_n + 2)^2}. \end{aligned}$$

We consider the difference between nominator and denominator of the fraction

$$\begin{aligned} &(p_n + a)^2 \cdot p_n - (p_n + a + 2)(p_n + 2)^2 \\ &\qquad = (p_n^2 + 2ap_n + a^2)p_n - (p_n + a + 2)(p_n^2 + 4p_n + 4) \\ &\qquad = p_n^3 + 2ap_n^2 + a^2 p_n - p_n^3 - 4p_n^2 - 4p_n - ap_n^2 - \\ &\qquad - 4ap_n - 4a - 2p_n^2 - 8p_n - 8 \\ &\qquad = (a - 6)p_n^2 + (a^2 - 4a - 12)p_n - 4a - 8 \end{aligned}$$

and get for

$$\begin{aligned} &a = 4 \ \rightarrow \ -2p^2 - 12p - 24 < 0 \\ &a = 6 \ \rightarrow \ -32 < 0 \\ &a = 10 \ \rightarrow \ 4p^2 + 48p - 48 > 0 \mid p \geq 2. \end{aligned}$$

This means that $u(p_{n+2}) < u(p_n)$ for $a = 4, 6$ and $u(p_{n+2}) > u(p_n)$ for $a \geq 10$.

With an analogous procedure we can demonstrate even for the case $p_{n+3} = p_{n+2} + 2$ that also holds

$$u(p_{n+3}) > u(p_n),$$

if $a \geq 10$. Also for the case $a = 4$ and $p_{n+3} = p_n + 2 + a + b$ with $b > 2$. It seems to be important to emphasize that all these results hold for the case $p_{n+1} - p_n = 2$.

The loss function $v(p_n)$ is strictly monotonic decreasing, because for two twin primes $p_n, p_n + 2$ und $p_n + 2 + d, p_n + 4 + d$ with the distance $d \geq 4$

holds

$$\begin{aligned} v(p_n+2+d) &= \frac{2(p_n+2+d)}{p_n+4+d}\eta(p_n+2+d)^2 \\ &\quad \text{and since } \eta(p) \text{ is a decreasing function} \\ &\le \frac{2}{p_n+4+d}\cdot\frac{(p_n+d)^2}{p_n+2+d}\eta(p_n+2)^2 \\ &= \frac{2}{p_n+4+d}\cdot\frac{(p_n+d)^2}{p_n+2+d}\cdot\frac{p_n^2}{(p_n+2)^2}\eta(p_n)^2 \\ &= v(p_n)\cdot\frac{p_n}{p_n+2}\cdot\frac{p_n+d}{p_n+2+d}\cdot\frac{p_n+d}{p_n+4+d} < v(p_n). \end{aligned}$$

From twin prime to twin prime the loss function $v(p_n)$ monotonically decreases for each twin distance d.

As upshot we see that in the majority of cases $u(p)$ is an increasing function while the loss function $v(p)$ from twin prime to twin prime decreases. The function $u(p)$ tends to result in an increasing function.

The greatest twin prime < 200 is $(197, 199)$. For the next prime number 211 holds $u(211) > 1.5159 = \frac{3}{2} + 0.0159$. Since the next prime after 211 follows only at 223 therefore for no prime $p_n > 211$ is $u(p_n) < u(211)$. On the other hand since $v(p_n)$ is monotonically decreasing, is $v(197) < 0.0148 < 0.0159$ and all further $v(p_{n+k})$ are more less. Therefore we get for $p_n > 200$ with (3.2) and (4.1)

$$u(p_n) = p_n\eta(p_n)^2 > \frac{3}{2} \longrightarrow \bar{\delta}(p_n)^2 < \frac{2}{3}p_n < \frac{2}{3}(p_n+1) \le d_n.$$

This completes the proof. □

Corollary 4.4. *Since $u(p)$ is an increasing function by trend there is always a prime p and a number $c_p > 1$ such that holds*

$$c_p \cdot \bar{\delta}(p_n)^2 < d_n \textit{ for } p_n > p.$$

For instance we get for

$$\begin{aligned} p_n > 1,277 &\to 2\bar{\delta}(p_n)^2 < d_n, \\ p_n > 25,561 &\to 10\bar{\delta}(p_n)^2 < d_n, \\ p_n > 77,291 &\to 20\bar{\delta}(p_n)^2 < d_n \text{ or for} \\ p_n > 830,293 &\to 100\bar{\delta}(p_n)^2 < d_n. \end{aligned}$$

5. The Distribution of the ω_{p_n}–Numbers

5.1. **The Symmetry of the ω_{p_n}–Numbers.** In the period section $\mathcal{P}_n$ the element

$$x_n^{(0)} := p_n\#_5$$

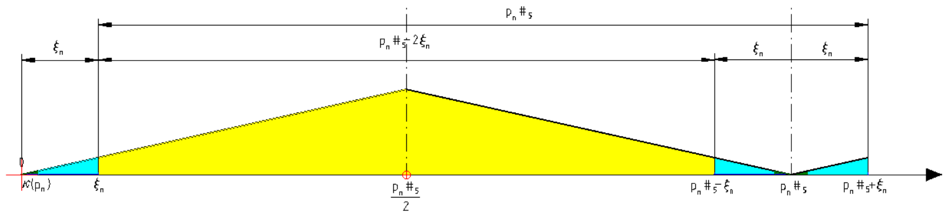

FIGURE 1. Symmetry in a period section $\mathcal{P}_n$ (not to scale)

has a particular importance. Because $p_n\#_5$ is divisible by all primes between 5 and p_n it holds

$$p_n\#_5 \equiv 0 (\bmod p_m) \mid 3 \leq m \leq n, p_m \in \mathbb{P}^*$$
$$\text{and hence}$$
$$x_n^{(0)} \not\equiv \pm\kappa(p_m) (\bmod p_m) \mid 3 \leq m \leq n, p_m \in \mathbb{P}^*.$$

Hence $x_n^{(0)}$ is an ω_{p_n}–number and it holds

$$\Psi(p_n\#_5, p_n) > 0. \tag{5.1}$$

Because of $\xi_n < p_n\#_5 < \xi_n + p_n\#_5$ the number $x_n^{(0)}$ is in the inner of $\mathcal{P}_n$ but near to the end.

Theorem 5.1. *The ω_{p_n}–numbers are symmetrically distributed around the axis $x_n^{(0)}$*

$$\Psi(x_n^{(0)} - a, p_n) = \Psi(x_n^{(0)} + a, p_n)$$

for any positive integer $a < x_n^{(0)}$.

Proof. Since by virtue of (2.7) the aggregate indicator function $\Psi(x, p_n)$ consists of the product of the indicator functions $\psi(x, 5), \ldots, \psi(x, p_n)$, the ω_{p_n}–numbers are symmetrically arranged around the axis $x_n^{(0)}$ if and only if holds for $m = 3, \ldots, n$ [6]

$$\psi(x_n^{(0)} - a, p_m) = \psi(x_n^{(0)} + a, p_m) \tag{5.2}$$

with any number $a < x_n^{(0)}$. From (2.6) we have

$$\psi(x, p_m) = \left(x^2 - \kappa(p_m)^2\right) \operatorname{Mod} p_m.$$

With $x_n^{(0)} \pm a$ for x we get

$$\begin{aligned}
\psi(x_n^{(0)} \pm a, p_m) &= \Big((x_n^{(0)} \pm a)^2 - \kappa(p_m)^2\Big) \operatorname{Mod} p_m \\
&= \Big(x_n^{(0)}(x_n^{(0)} \pm 2a) + a^2 - \kappa(p_m)^2\Big) \operatorname{Mod} p_m \\
&\text{and since } x_n^{(0)} \equiv 0 (\bmod p_m) \\
&= \left(a^2 - \kappa(p_m)^2\right) \operatorname{Mod} p_m. \qquad (5.3)
\end{aligned}$$

Since $\psi(x_n^{(0)} \pm a, p_m)$ result in a common value it follows (5.2). □

[6] Unless otherwise specified the use of the variable p_m means below $p_m \in \mathbb{P}^* \mid 3 \leq m \leq n$.

Additionally from (5.3) we see that

$$\psi(x_n^{(0)} \pm a, p_m) = 0 \text{ for } a = \kappa(p_m).$$

This means that at these positions there cannot be ω_{p_n}–numbers. Around the axis $x_n^{(0)}$ there are ω_{p_n}–gaps with the length $\kappa(p_n)$ but an ω_{p_n}–gap cannot reach over $x_n^{(0)}$.

If we limit our consideration to the period section $\mathcal{P}_n$, then there is a section of symmetry with a length of $2\xi_n$ at the end of $\mathcal{P}_n$ around on $x_n^{(0)} = p_n\#_5$. But in the remaining of the period section there is symmetry too (see Figure 1). Let

$$x_n^{(1)} := \frac{p_n\#_5}{2}$$

be a rational number as the middle between the integers

$$x_n^{(1-)} := \frac{p_n\#_5 - 1}{2} \text{ and } x_n^{(1+)} := \frac{p_n\#_5 + 1}{2}. \tag{5.4}$$

Theorem 5.2. *The ω_{p_n}–numbers are symmetrically distributed around the axis $x_n^{(1)}$*

$$\Psi(x_n^{(1-)} - a, p_n) = \Psi(x_n^{(1+)} + a, p_n)$$

for any positive integer $a < x_n^{(1)}$.

Proof. Because of the periodicity of the aggregate indicator function we have

$$\begin{aligned} \Psi(\xi_n + a, p_n) &= \Psi(p_n\#_5 + \xi_n + a, p_n) \text{ and because of Theorem 5.1} \\ &= \Psi(p_n\#_5 - \xi_n - a, p_n). \end{aligned} \tag{5.5}$$

Therefore there is symmetry around

$$\frac{\xi_n + a + p_n\#_5 - \xi_n - a}{2} = \frac{p_n\#_5}{2} = x_n^{(1)}.$$

We set $x_n^{(1+)}$ instead of ξ_n in (5.5) and get

$$\begin{aligned} \Psi\left(x_n^{(1+)} + a, p_m\right) &= \Psi\left(p_n\#_5 - (x_n^{(1+)} + a), p_m\right) \\ &= \Psi\left(p_n\#_5 - \frac{p_n\#_5 + 1}{2} - a, p_m\right) \\ &= \Psi\left(\frac{p_n\#_5 - 1}{2} - a, p_m\right) \\ &= \Psi\left(x_n^{(1-)} - a, p_m\right) \end{aligned}$$

□

The section around $x_n^{(1)}$ has a length of $p_n\#_5 - 2\xi_n$. It increases more quickly than the length $2\xi_n$ of the section around $x_n^{(0)}$ because $p_n\#_5$ increases more quickly than ξ_n.

Corollary 5.3. *Since by virtue of Theorems 5.1 and 5.2 the ω_{p_n}–numbers are symmetrically distributed around the axes $x_n^{(0)} = p_n\#_5$ as well as $x_n^{(1)} = \dfrac{p_n\#_5}{2}$ therefore the ω_{p_n}–gaps are symmetrically distributed with respect to their sizes and positions.*

Lemma 5.4. *At the positions* $x_n^{(1-)}$ *and* $x_n^{(1+)}$ *(see* (5.4)*) are* ω_{p_n}*–numbers.*

Proof. Since $4 \not\equiv 0 (\mathrm{mod}\, p_n\#_5)$ we can multiply (2.6) by 4 and have for $3 \le m \le n$

$$4\psi(x, p_m) \,\mathrm{Mod}\, p_m = \big((2x)^2 - (2\kappa(p_m))^2\big) \,\mathrm{Mod}\, p_m$$

and get for $x = \dfrac{p_n\#_5 \pm 1}{2}$

$$\begin{aligned} 4\psi(\frac{p_n\#_5 \pm 1}{2}, p_m) \,\mathrm{Mod}\, p_m &= \big((p_n\#_5 \pm 1)^2 - 4\kappa(p_m)^2\big) \,\mathrm{Mod}\, p_m \\ &= \big(p_n\#_5(p_n\#_5 \pm 2) + 1 - 4\kappa(p_m)^2\big) \,\mathrm{Mod}\, p_m \\ &= \big(1 - 4\kappa(p_m)^2\big) \,\mathrm{Mod}\, p_m. \end{aligned}$$

Because this never can be zero therefore at the positions $x_n^{(1-)}$ and $x_n^{(1+)}$ are ω_{p_n}–numbers. □

Hence no ω_{p_n}–gap can reach over the symmetry axis $x_n^{(1)}$. Each ω_{p_n}–gap has in $\mathcal{P}_n$ always a symmetry partner, each ω_{p_n}–gap occurs twice in $\mathcal{P}_n$.

With the transition $p_n \to p_{n+1}$ the symmetry of the ω_{p_n}–numbers in their period sections $\mathcal{P}_n$ repeats p_{n+1}–times oneself in $\mathcal{P}_{n+1}$, disturbed by the

$$\begin{aligned} p_{n+1} \cdot \phi(p_n) - \phi(p_{n+1}) &= p_{n+1} \cdot \phi(p_n) - \phi(p_n) \cdot (p_{n+1} - 2) \\ &= 2\phi(p_n) \end{aligned} \tag{5.6}$$

excludings since the indicator function $\psi(x, p_{n+1})$ becomes zero. These excludings x are characterized by $\psi(x, p_n) = 0$, while $\psi(x, p_m) > 0$ for $m < n$.

5.2. **The Overlapping of the period sections.** The intervals $\mathcal{A}_n, n \ge 3$ defined by (3.1) cover the positive integers ≥ 4 gapless and densely. It is

$$\mathbb{N} = \{1, 2, 3\} \cup \bigcup_{n=3}^{\infty} \mathcal{A}_n \text{ and } \bigcap_{n=3}^{\infty} \mathcal{A}_n = \emptyset.$$

They are the beginnings of the period sections $\mathcal{P}_n$ of the ω_{p_n}–numbers. Hereafter let's say **A-sections** to the intervals $\mathcal{A}_n$. Every ω_{p_n}–number that lies in an A-section is a twin prime generator (see Conclusion 3.5). In contrast to the A-sections the period sections $\mathcal{P}_n$ overlap each other very closely. So the period section $\mathcal{P}_9$ reachs over 1739 A-sections up to the beginning of the period section $\mathcal{P}_{1748}$ and the next $\mathcal{P}_{10}$ over 7863 A-sections up to the beginning of $\mathcal{P}_{7873}$.

Lemma 5.5. *Each origin* ξ_n *cannot be located at the beginning* $\xi_m + a \cdot p_m\#_5$ *of any period of the aggregate indicator function* $\Psi(x, p_m)$ *for* $n > m$ *and* $a \in \mathbb{N}$. *Therefore it holds for* $n > m$

$$\xi_n \not\equiv \xi_m (\mathrm{mod}\, p_m\#_5).$$

Proof. The equation

$$\frac{p_m^2 - 1}{6} + a \cdot p_m\#_5 = \frac{p_n^2 - 1}{6} \text{ and hence } p_m(p_m + a \cdot p_{m-1}\#) = p_n^2$$

is for no prime $p_n > p_m$ solvable since holds $\gcd(p_n, p_m) = 1$. □

Vice versa every period section $\mathcal{P}_{n+1}$ starts always inside of the previous period section $\mathcal{P}_n$ nearby to its origin because (see (3.3) too)

$$\xi_{n+1} = \xi_n + d_n \text{ and}$$
$$d_n < \frac{p_n^2}{2} \ll \frac{p_n\#_5}{2}.$$

Lemma 5.6. *Even with modul* p_m *instead of* $p_m\#_5$ *holds*

$$\xi_n \not\equiv \xi_m (\operatorname{mod} p_m) \textit{ for all } n > m.$$

Proof. We assume contrarily $\xi_n = \xi_m + a \cdot p_m$ for any $a \in \mathbb{N}$. Analogously to the proof of Lemma 5.5 we multiply by 6 and get finally

$$p_n^2 = p_m(p_m + 6a).$$

Also this equation is since $\gcd(p_m, p_n) = 1$ for no $p_m \neq p_n \in \mathbb{P}$ solvable. □

Conclusion 5.7. *The shown overlapping of the period sections* $\mathcal{P}_n$ *and the symmetry and periodicity of the distribution of the* ω_{p_n}*–numbers avoid the formation of extreme nonuniform distribution of the* ω_{p_n}*–numbers.*

6. Proof of the Twin Prime Conjecture

Proof. The proof will be done by contradiction. We assume contrarily that there is only a finite number of twin primes and therefore only a finite number of twin prime generators. Let y_o be the greatest one. It lies in the A-section $\mathcal{A}_{n_o}$ with $n_o = \pi(\hat{p}(y_o))$, the beginning of the period section $\mathcal{P}_{n_o}$. W.l.o.g. we can assume that $n_o > 200$. In the successive A-sections $\mathcal{A}_t$ with $t > n_o$ consequently there cannot be any twin prime generators and hence by virtue of Corollary 3.5 no ω_{p_t}–numbers. But then we have ω_{p_t}–gaps with sizes $> d_t$ in **all** (infinitely many) period sections $\mathcal{P}_t$ for $t > n_o$. Additionally for $p \geq p_t$ the function $u(p) = p \cdot \eta(p)^2$ is a strictly monotonic increasing function (see Theorem 4.3).
But because

- the squared average size of the ω_{p_t}–gaps $\bar{\delta}(p_t)^2$ is **less** than a fraction of the size of the A-section $\mathcal{A}_t$
- and since Conclusion 5.7,

therefore it is not possible to have for **all** $t \geq n_o$ only A-sections $\mathcal{A}_t$ with ω_{p_t}–gaps that all are **greater** than d_t. Hence the assumption is wrong and therefore the

Twin Prime Conjecture is true.

□

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