Theorems on power signals in electrical circuits with sinusoidal voltage and current signals

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Abstract

In this paper four theorems are stated: the first one is on expressions for a power signal in which the definition of the associated power (Π) is introduced. The second one is on the uniqueness of impedances underlying an electrical power signal. The third one is on some properties of the power signal with respect to the active power P and the reactive power Q. The fourth one deals with the conservation of power in a cicuit. Numerical examples provided illustrate the scope of the theorems.

Theorem 1

Let $u(t) \coloneqq |U| \cdot \cos\big(\omega \cdot t + \arg(U)\big)$ and $i(t) \coloneqq |I| \cdot \cos\big(\omega \cdot t + \arg(I)\big)$ be a voltage signal and a current signal respectively, in which the complex-valued voltage U and the complex-valued current I are related through the impedance $Z = U/I$.

Then the power signal $p(t) \coloneqq u(t) \cdot i(t)$ can be expressed as:

(1) $p(t) = \mathrm{Re}\big(S + \Pi \cdot e^{2 \cdot j \cdot \omega \cdot t}\big)$
where $S \coloneqq \frac{1}{2} \cdot U \cdot I^* = \frac{1}{2} \cdot |I|^2 \cdot Z$ and $\Pi \coloneqq \frac{1}{2} \cdot U \cdot I = \frac{1}{2} \cdot I^2 \cdot Z$

or

(2) $p(t) = \mathrm{Re}\left(S \cdot \left(1 - e^{2 \cdot j \cdot \omega \cdot (t - t_0)}\right)\right)$
where $t_0 \coloneqq \frac{T}{4} \cdot \left(1 + \frac{1}{\pi} \cdot \arg\left(\frac{S}{\Pi}\right)\right) \in \left[0, \frac{1}{2} T\right]$, $p(t_0) = 0$

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Proof

$$
\begin{aligned}
p(t) = u(t)\cdot i(t) &= |U|\cdot\cos\big(\omega\cdot t+\arg(I)+\arg(Z)\big)\cdot|I|\cdot\cos\big(\omega\cdot t+\arg(I)\big)\\
&= \tfrac{1}{2}\cdot|U|\cdot|I|\cdot\Big(\cos\big(\arg(Z)\big)+\cos\Big(2\cdot\big(\omega\cdot t+\arg(I)\big)+\arg(Z)\Big)\Big)\\
&= |S|\cdot\Big(\cos\big(\arg(S)\big)+\cos\Big(2\cdot\big(\omega\cdot t+\arg(I)\big)+\arg(S)\Big)\Big)\\
&= \mathrm{Re}\big(S+S\cdot e^{2\cdot j\cdot(\omega\cdot t+\arg(I))}\big)\\
&= \mathrm{Re}\big(S+e^{2\cdot j\cdot\omega\cdot t}\cdot S\cdot e^{2\cdot j\cdot\arg(I)}\big)\\
&= \mathrm{Re}\Big(S+e^{2\cdot j\cdot\omega\cdot t}\cdot\tfrac{1}{2}\cdot U\cdot I^{*}\cdot e^{2\cdot j\cdot\arg(I)}\Big)\\
&= \mathrm{Re}\Big(S+e^{2\cdot j\cdot\omega\cdot t}\cdot\tfrac{1}{2}\cdot U\cdot I\Big)\\
&= \mathrm{Re}\big(S+\Pi\cdot e^{2\cdot j\cdot\omega\cdot t}\big)
\end{aligned}
$$

$$
\begin{aligned}
p(t_0) &= \mathrm{Re}\left(S+\Pi\cdot e^{2\cdot j\cdot\frac{2\pi}{T}\cdot\frac{T}{4}\cdot\left(1+\frac{1}{\pi}\cdot\arg\left(\frac{S}{\Pi}\right)\right)}\right) = \mathrm{Re}\left(S+\Pi\cdot e^{j\cdot\pi\cdot\left(1+\frac{1}{\pi}\cdot\arg\left(\frac{S}{\Pi}\right)\right)}\right)\\
&= \mathrm{Re}\left(S+e^{j\cdot\pi}\cdot\Pi\cdot e^{j\cdot arg\left(\frac{S}{\Pi}\right)}\right) = \mathrm{Re}\left(S-\Pi\cdot\frac{S}{\Pi}\right) = \mathrm{Re}(S-S) = 0
\end{aligned}
$$

□

We will call the new complex-valued quantity Π the “associated” power.

The following theorem states that a power signal determines the underlying impedance up to conjugacy and a real-valued scaling factor ρ.

Theorem 2

Let $p(t) \coloneqq \mathrm{Re}\big(S+\Pi\cdot e^{2\cdot j\cdot\omega\cdot t}\big)$ and $\tilde{p}(t) \coloneqq \mathrm{Re}\big(\tilde{S}+\tilde{\Pi}\cdot e^{2\cdot j\cdot\omega\cdot t}\big)$. If $\tilde{p}(t)=p(t)$, then

(1)

$$\mathrm{Re}\big(\tilde{S}\big)=\mathrm{Re}(S) \;\wedge\; \tilde{\Pi}=\Pi$$

(2) Let $\tilde{Z} \coloneqq \rho^2\cdot Z^{*}, \rho\in\mathbb{R}\backslash\{0\}$, then

$$\begin{cases}\tilde{I}=\frac{1}{\rho}\cdot I\cdot\frac{Z}{|Z|}\\ \tilde{U}=\rho\cdot U\cdot\frac{Z^{*}}{|Z^{*}|}\end{cases}$$

(3) Let $\tilde{Z} \coloneqq \rho^2\cdot Z, \rho\in\mathbb{R}\backslash\{0\}$, then

$$\begin{cases}\tilde{I}=\frac{1}{\rho}\cdot I\\ \tilde{U}=\rho\cdot U\end{cases}$$

Proof

$$
\begin{aligned}
&\tilde{p}(t)=p(t)\\
\Leftrightarrow\;& \mathrm{Re}\big(\tilde{S}+\tilde{\Pi}\cdot e^{2\cdot j\cdot\omega\cdot t}\big)=\mathrm{Re}\big(S+\Pi\cdot e^{2\cdot j\cdot\omega\cdot t}\big)\\
\Leftrightarrow\;& \mathrm{Re}\big(\tilde{S}-S\big)+\mathrm{Re}\Big(\big(\tilde{\Pi}-\Pi\big)\cdot e^{2\cdot j\cdot\omega\cdot t}\Big)=0\\
\Leftrightarrow\;& \mathrm{Re}\big(\tilde{S}-S\big)=0 \;\wedge\; \tilde{\Pi}-\Pi=0\\
\Leftrightarrow\;& \mathrm{Re}\big(\tilde{S}\big)=\mathrm{Re}(S) \;\wedge\; \tilde{\Pi}=\Pi
\end{aligned}
$$

$$\tilde{\Pi} = \Pi$$
$$\Leftrightarrow \quad \tilde{I}^2 \cdot \tilde{Z} = I^2 \cdot Z$$
$$\Leftrightarrow \quad \tilde{I}^2 = I^2 \cdot \frac{Z}{\tilde{Z}}$$

$$\mathrm{Re}(\tilde{S}) = \mathrm{Re}(S)$$
$$\Leftrightarrow \quad \mathrm{Re}\left(\left|\tilde{I}\right|^2 \cdot \tilde{Z}\right) = \mathrm{Re}(|I|^2 \cdot Z)$$
$$\Leftrightarrow \quad \mathrm{Re}\left(\frac{\tilde{Z}}{|\tilde{Z}|}\right) = \mathrm{Re}\left(\frac{Z}{|Z|}\right) = \mathrm{Re}\left(\frac{Z^*}{|Z^*|}\right)$$
$$\Leftrightarrow \quad \rho^2 \coloneqq \frac{\tilde{Z}}{Z} = \frac{|\tilde{Z}|}{|Z|} \quad \vee \quad \rho^2 \coloneqq \frac{\tilde{Z}}{Z^*} = \frac{|\tilde{Z}|}{|Z^*|}, \quad \rho \in \mathbb{R}\backslash\{0\}$$
$$\Leftrightarrow \quad \tilde{Z} = \rho^2 \cdot Z \quad \vee \quad \tilde{Z} = \rho^2 \cdot Z^*$$
$$\Leftrightarrow \quad \tilde{I}^2 = \frac{I^2}{\rho^2} \qquad \vee \quad \tilde{I}^2 = \frac{I^2}{\rho^2} \cdot \frac{Z^2}{|Z|^2}$$
$$\Leftrightarrow \quad \tilde{I} = \frac{1}{\rho} \cdot I \qquad \vee \quad \tilde{I} = \frac{1}{\rho} \cdot I \cdot \frac{Z}{|Z|}$$
$$\Leftrightarrow \quad \tilde{U} = \rho \cdot U \quad \vee \quad \tilde{U} = \rho \cdot U \cdot \frac{Z^*}{|Z^*|}$$

□

The following theorem provides some properties of the power signal with respect to the active power P and the reactive power Q.

Theorem 3

Let $t_0 \coloneqq \frac{T}{4} \cdot \left(1 + \frac{1}{\pi} \cdot \arg\left(\frac{S}{\Pi}\right)\right)$. Then

(1) Let $t_1 \coloneqq t_0 + \frac{T}{4\pi}$, then

$$\dot{p}(t_0) \cdot (t_1 - t_0) = Q$$

(2) Let $t_2 \coloneqq t_0 + \frac{T}{8}$, then

$$p(t_2) = P + Q$$

(3) Let $t_3 \coloneqq t_0 + \frac{T}{4}$, then

$$p(t_3) = 2 \cdot P$$

Proof

Using part (2) of Theorem 1

$$\dot{p}(t) = 2 \cdot \omega \cdot \mathrm{Im}\left(S \cdot e^{2 \cdot j \cdot \omega \cdot (t - t_0)}\right), \quad \omega \coloneqq \frac{2\pi}{T}$$

$$\dot{p}(t_0) \cdot (t_1 - t_0) = \frac{T}{4\pi} \cdot 2 \cdot \frac{2\pi}{T} \cdot \mathrm{Im}\left(S \cdot e^{2 \cdot j \cdot \omega \cdot 0}\right) = \mathrm{Im}(S \cdot 1) = Q$$

$$p(t_2) = \mathrm{Re}\left(S \cdot \left(1 - e^{2 \cdot j \cdot \frac{2\pi}{T} \cdot \frac{T}{8}}\right)\right) = \mathrm{Re}\left(S \cdot \left(1 - e^{j \cdot \frac{\pi}{2}}\right)\right) = \mathrm{Re}\big(S \cdot (1 - j)\big) = P + Q$$

$$p(t_3) = \mathrm{Re}\left(S \cdot \left(1 - e^{2 \cdot j \cdot \frac{2\pi}{T} \cdot \frac{T}{4}}\right)\right) = \mathrm{Re}\left(S \cdot \left(1 - e^{j \cdot \pi}\right)\right) = \mathrm{Re}\big(S \cdot (1 + 1)\big) = 2 \cdot P$$

□

Example

Given the source voltage signal $u(t) := 100 \cdot \cos\left(1000t + \arccos\left(\frac{-1}{\sqrt{5}}\right)\right)$ V and the impedances $Z_1 := 3\ \Omega$, $Z_2 := 4j\ \Omega$ of the circuit shown in Figure 1

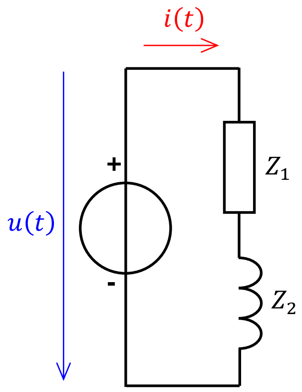


Figure 1

The equivalent impedance of the circuit equals

$$Z := Z_1 + Z_2 = 3 + j \cdot 4 = 5 \cdot e^{j \cdot \arccos(0.6)}\ \Omega$$

The complex-valued source voltage equals

$$U = 100 \cdot e^{j \cdot \left(\arccos\left(\frac{-1}{\sqrt{5}}\right)\right)}\ \mathrm{V}$$

The complex-valued source current equals

$$I = \frac{U}{Z} = 20 \cdot e^{j \cdot \arccos\left(\frac{1}{\sqrt{5}}\right)}\ \mathrm{A}$$

The source current signal equals

$$i(t) := 20 \cdot \cos\left(1000t + \arccos\left(\frac{1}{\sqrt{5}}\right)\right)\ \mathrm{A}$$

The source voltage signal and the source current signal of the circuit shown in Figure 1 are displayed in Figure 2 and Figure 3 respectively

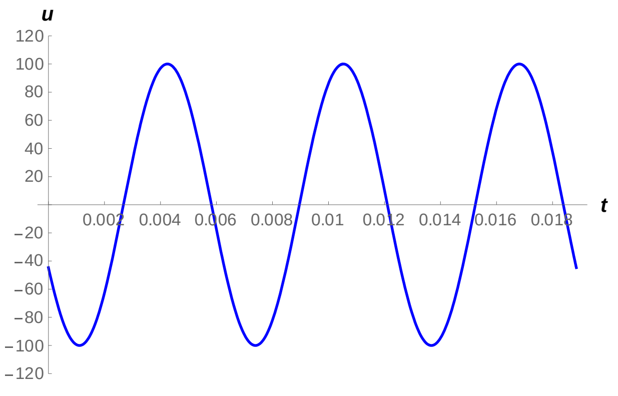


Figure 2

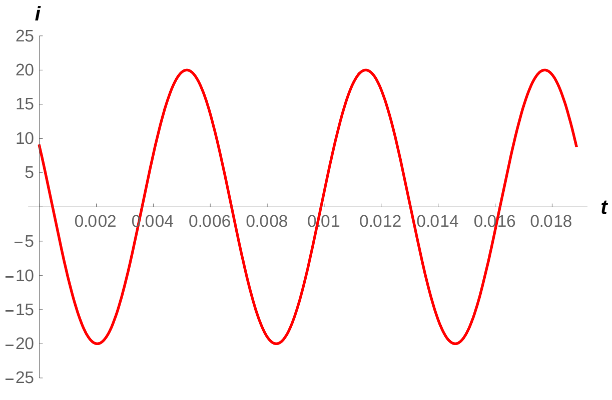
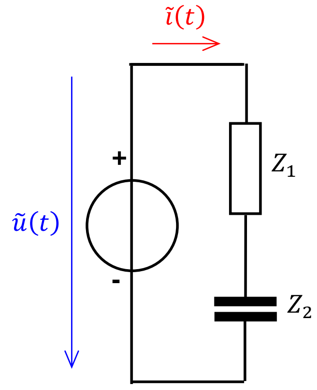


Figure 3

The complex-valued source voltage and equivalent impedance of the circuit shown in Figure 4 are constructed from part (2) of Theorem 2 for $\rho = 1$



Figuur 4

$$\tilde{U} := \rho \cdot U \cdot \frac{Z^*}{|Z^*|} = 100 \cdot e^{j \cdot \arccos\left(\frac{1}{\sqrt{5}}\right)} \text{ V}$$

$$\tilde{Z} := \rho^2 \cdot Z^* = 3 - j \cdot 4 \ \Omega$$

Remark: note that $\tilde{Z} = Z_1 + Z_2$ for the given impedances $Z_1 := 3\ \Omega,\ Z_2 := -4j\ \Omega$

The source voltage signal equals

$$\tilde{u}(t) = 100 \cdot \cos\left(1000t + \arccos\left(\frac{1}{\sqrt{5}}\right)\right) \text{V}$$

The complex-valued source current equals

$$\tilde{I} := \frac{\tilde{U}}{\tilde{Z}} = 20 \cdot e^{j \cdot \arccos\left(\frac{-1}{\sqrt{5}}\right)} \text{ A}$$

The source current signal equals

$$\tilde{\imath}(t) = 20 \cdot \cos\left(1000t + \arccos\left(\frac{-1}{\sqrt{5}}\right)\right) \text{A}$$

The source voltage signal and the source current signal of the circuit shown in Figure 4 are displayed in Figure 5 and Figure 6 respectively

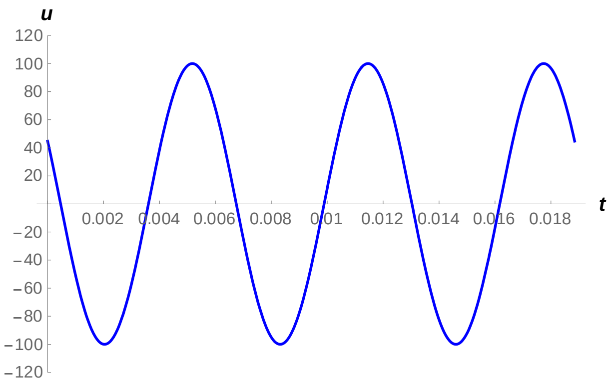


Figure 5

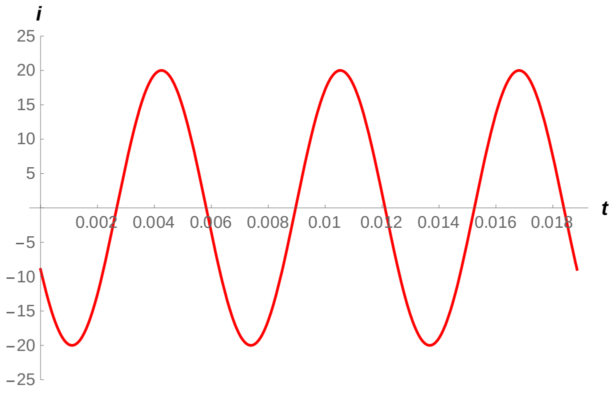


Figure 6

The source power signals of the circuits shown in Figure 1 and Figure 2 are according to Theorem 2 equal to

$$p(t) := u(t) \cdot i(t) =$$
$$\tilde{p}(t) := \tilde{u}(t) \cdot \tilde{\imath}(t) =$$
$$2000 \cdot \cos\left(1000t + \arccos\left(\frac{1}{\sqrt{5}}\right)\right) \cdot \cos\left(1000t + \arccos\left(\frac{-1}{\sqrt{5}}\right)\right) \text{ W}$$

The source power signal $p(t) = \tilde{p}(t)$ is displayed in Figure 7.

The time instance t_0 for which $p(t_0) = 0$ VA can be computed applying Theorem 3

$$S = \frac{1}{2} \cdot |I|^2 \cdot Z = 600 + j \cdot 800 \text{ VA}$$

$$\Pi = \frac{1}{2} \cdot I^2 \cdot Z = -1000 + j \cdot 0 \text{ VA}$$

$$t_0 := \frac{T}{4} \cdot \left(1 + \frac{1}{\pi} \cdot \arg\left(\frac{S}{\Pi}\right)\right) = 0.0004636 \text{ s}$$

$$P = \mathrm{Re}(S) = 600 \text{ W}$$

$$Q = \mathrm{Im}(S) = 800 \text{ VAR}$$

$$\dot{p}(t_0) \cdot (t_1 - t_0) = Q = 800 \text{ VAR}$$

$$p(t_2) = P + Q = 1400 \text{ VA}$$

$$p(t_3) = 2P = 1200 \text{ W}$$

An example of the time instances t_0, t_1, t_2 and t_3 introduced in Theorem 3 is displayed in Figure 7.

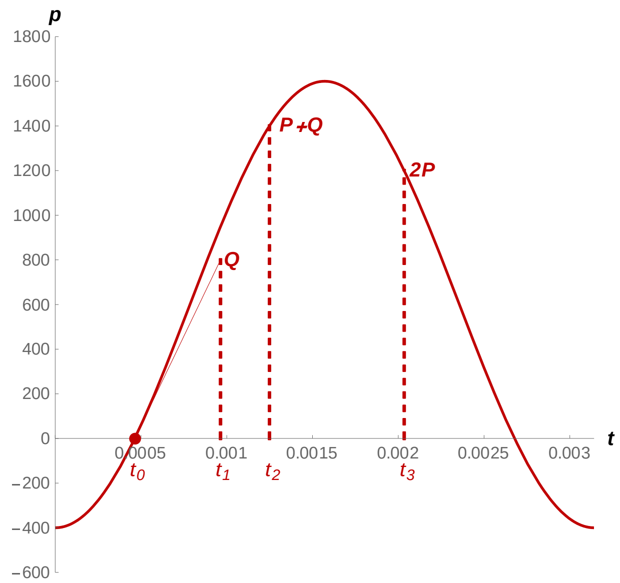


Figure 7: The power signal of the circuits shown in Figure 1 and Figure 4. The time instances t_1, t_2 and t_3 with their corresponding values expressed through the active power P and the reactive power Q. By definition $p(t_0) = 0$.

The complex-valued source power S equals the sum of the powers of the underlying impedances in a circuit

$$S = \sum_n S_n$$

The domain of the real-valued variables U and I can be extended to the complex plane, to obtain an analytic continuation of the real-valued function $U \cdot I$. Therefore, the complex-valued power Π is an analytic continuation to the complex plane of the real-valued function $\frac{1}{2} \cdot U \cdot I$. Then, similar to the complex-valued source power S

$$\Pi = \sum_n \Pi_n$$

The following theorem states a similar property on conservation for power signals in a circuit.

Theorem 4

Let $p(t)$ a source power signal and let $p_1(t), p_2(t)$, ... the power signals of the underlying impedances in a circuit, then

$$p(t) = \sum_n p_n(t)$$

Proof

$$\begin{aligned}
\sum_n p_n(t) &= \sum_n \mathrm{Re}\left(S_n + \Pi_n \cdot e^{2 \cdot j \cdot \omega \cdot t}\right) \\
&= \mathrm{Re}\left(\sum_n \left(S_n + \Pi_n \cdot e^{2 \cdot j \cdot \omega \cdot t}\right)\right) \\
&= \mathrm{Re}\left(\sum_n S_n + e^{2 \cdot j \cdot \omega \cdot t} \cdot \sum_n \Pi_n\right) \\
&= \mathrm{Re}\left(S + \Pi \cdot e^{2 \cdot j \cdot \omega \cdot t}\right) \\
&= p(t)
\end{aligned}$$

□

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