Constructive QFT : a condensed math point of view II

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Abstract

In this series, we show a road map to prove the existence of ϕ^4 quantum field theory over 4-d spacetime. We suggest a new axiomatic approach on constructive quantum field theory via condensed mathematics. The goal is to extend this new approach to cover the mathematical viability of field theories of standard model and beyond.

II. Reminder of QFT from path integral perspective

1 QFT in a nutshell

Note that the following sections contain reminders on physics. In order to make readers from other side feel safe without entire background on physics, we make qualitative statements without engaging every details. Also it is for reminding those who already know the whole story. In the end, it is about why we deal with Schwinger functions and how it can be interpreted in *condensed* language. So if it is OK with the former, then one can go to the next note which contains the background of *condensed* mathematics.

Quantum mechanics was developed as operator algebras on Hilbert space. Since then, classical mechanics was required to be translated into quantum mechanics via the procedure called quantization. There are several methods to do it, but we focus on the Feynman's path integral method in this work. It is an equivalent approach with canonical quantization, when Hamiltonian is at most quadratic. And it is based on Lagrangian, so it is easier to handle when one wants to manipulate symmetry of the theory.

Field theory is also a classical physics. So it was required to quantize field theory in order to read quantum world. After quantizing field theory, fields on space has different meaning than before. Fields now are defined as operator valued distributions. It is composed of creation and annihilation operators. Theory now has a quantized particles when the field is excited. For example, Electromagnetic field, when it is quantized, has photons to be produced and disappeared. EM field has $U(1)$ local symmetry, this is called gauge symmetry, and quantum particle of it is called gauge bosons.[1]

Quantum particles in nature can be represented by various fields. EM fields and other gauge fields are vector fields. Fermions are represented by spinors. Higgs boson is by scalar fields. For mathematical approach, we deal with scalar field theory only. Standard model is combining all the fields together, with each free terms and each interacting terms in a Lagrangian. Having a viable scalar field theory is a first step to achieve viability of other natural field theories.

What it means to quantize fields and calculate scattering matrix is somewhat blur in it's original formulation. But after Wick rotation, which changes Minkowski spacetime to Euclidean spacetime, the picture is intuitionally closer to statistical mechanical view point. In statistical mechanics what one calculates is, via Gaussican measures, the mean and variance of the observables. Accordingly, QFT over Euclidean spacetime gives similar formalism as statistical case, and it gives intuition of what one calculates via path integral perturbative method.

Namely, in statistical mechanics, if one is given partition function which contains the state of the system, one extract statistical data such as entropy or energy, by manipulating the partition functions. Similarly in QFT over Euclidean spacetime, given generating functional, one extracts quantum mechanical amplitudes by manipulating generating functional. By this intuition, one can rationalize what one calculates via Feynman diagrams. Next section is based on this intuitional set-up over 4-d Euclidean spacetime, and the foundation is this picture throughout the series.

The contents of reminders are the ground that we build a new constructive field theory on.[2] For convenience, notations used are ephemeral. We permit notations overlap here and there at least in reminder note, instead of employing entire alphabets and Greeks, etc. The terminology is already overlapped such as *condensed*, *solid* and *liquid*, but the situation is worse in notations. Some notations, for example p is obviously momentum for physicists, but obviously prime number for mathematician. The notations will be carefully chosen at later stage.[3]

Also the goal of this series is to show the existence of certain field theory with algebraic method. So we reduce quantitative details, and focus on the qualitative definitions.

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[1] bosons named by the spin-staticstic of the particle.

[2] Ideas develops from plowing this ground.

[3] Already in math or physics, notations bring some confusions at it's own volume. there will require whole note for straighten the notations used in this series.

2 Feynman's formalism on quantum mechanics

We show a point of view from statistical formalism first. Let's start with a partition function

$$Z_{stat} = \sum_i e^{-\beta E_i}$$

Once partition function of a system is known, one can derive statistical information out of it. For example expected value of total energy of any system is

$$\langle E \rangle \;=\; -\frac{1}{Z}\frac{\partial Z}{\partial \beta} = -\frac{\partial \ln Z}{\partial \beta}$$

Probabilistically saying, partition function can be read as statistical information generator. If one is interested in a specific microstate s of a system then one calculates

$$P_s = \frac{1}{Z} e^{-\beta E_s}$$

By having this short argument in mind, one can approach the probabilistic nature of quantum states of field theory as follows.

Let's start with the Feynman's path integral formula for a free particle first, before talking about fields. Note that it is always over 4-d Euclidean space, where this statistical argument is appropriate with real number amplitudes. And we set $\hbar = c = 1$.

Then

$$Z_0 = \int \mathcal{D}\mathbf{x}\; e^{-S(x,\dot{x},t)}$$

Where S is defined as Euclidean actions such as

$$S = \int L\; dt, \quad L = T + V$$

This can be read as a partition function for a particle's states, initiated from all the points to finished at all the points. If one wants to calculate specific states s initiated certain point to end at certain point then,

$$Z_s = \int_{\mathbf{x}(0)=x} \mathcal{D}\mathbf{x}\; e^{-S(x,\dot{x},t)} \psi_0(\mathbf{x}, t)$$

$\psi_0(\mathbf{x}, t)$ is a test function, which is distribution that only have meaning at integrand. The test function at integrand sorts out the specific states out of entire information of a system. Here Z_s is the partition function for certain states starting from ψ_0 and ending at any x. So we have a wavefunction via path integral formalism,

$$\psi(x, t) = \frac{1}{Z_0} \int_{\mathbf{x}(0)=x} \mathcal{D}\mathbf{x}\; e^{-S(x,\dot{x},t)} \psi_0(\mathbf{x}, t)$$

In a measure theoretic point of view, there is the Wiener measure which supports such definition on the paths space when $L = T$. It works as if the system is analogous to random walks or Brownian motions. The advantage of working with 4-d Euclidean space is the usability of such classical results. So the existence of QFT is studied in this background. It should be supported on the countable number of degrees of freedom. As we will see in the next note, to avoid uncountably many degrees of freedom regularization is required before constructing continuum field theory. We postpone regularization procedure to the next note. The definition of *condensed* field theory will be constructed in the next note.

This approach on path integral quantization is equivalent to the original formalism of Feynman integral via canonical quantization, which is

$$\langle x, t | x', t' \rangle = \int_x^{x'} \mathcal{D}x \, e^{-S}$$

Axiomatically saying, Feynman postulated[4]

1. each path has weight e^{-S}.
2. The sum of all weights on paths between initial point to final point contribute to the end result $\langle x, t | x', t' \rangle$.
3. probability of an event is $| \langle x, t | x', t' \rangle |^2$.

Even though our formalism is statistical version, we consider the background axioms for path integral quantization as this axiom set.

3 Schwinger functions

Now, let's talk about fields. The observables are field configurations and multiple particles can be annihilated and created in a QFT system. Unlike the path integral of particle formalism above, the existence of such a measure on every field configurations, creating and annihilating every possible way, is not established in general. This is what we hope to achieve via the *condensed* measures.

The partition function of a free scalar field ϕ is

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]}$$

Now the action is over fields, so $S[\phi]$ is called action functional,

$$S[\phi] = \int \mathcal{L}_{free} \, d^4x, \quad \mathcal{L}_{free} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2$$

Statistical information of such quantized fields is extracted as the following general formulations.

[4]This is Euclidean version.

$$\langle F \rangle = \frac{\int \mathcal{D}\phi \; F[\phi] \; e^{-S[\phi]}}{\int \mathcal{D}\phi \; e^{-S[\phi]}}$$

Here $F[\phi]$ means time ordered operators that specify certain states out of whole system. Now we define Schwinger function as a probabilistic amplitudes on the spacetime via the above formula.

$$S_n(x_1, x_2, \cdots, x_n) \equiv \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n) \rangle \quad x_k \in \mathbb{R}^d$$

This function has a rather concrete physical meaning when we rely on the statistical mechanical argument. In quantum field theory, field is the source of particles and particle has field realization. So given a scalar field ϕ, the operator $\phi(x_1)$ means a particle created or annihilated at spacetime x_1. Time ordered operators mean that these events occur in a time ordered manner.[5] So S_n is an expectation value of n-particles correlated as such conditions. Or one just reads it as a probability amplitudes on a n-pointed test function space. In this sense, it is also called n-points correlators.

The existence of QFT is whether this correlators behave well or not. Axiomatic approach on Schwinger functions will appear at step 2. Then let's focus on more about how to dissect this correlators.

Now let's define generating functional version of Schwinger functions. From

$$\begin{aligned} S_n(x_1, x_2, \cdots, x_n) &\equiv \langle \phi(x_1)\phi(x_2)\cdots\phi(x_n) \rangle \\ &= \frac{\int \mathcal{D}\phi \; \phi(x_1)\phi(x_2)\cdots\phi(x_n) \; e^{-S[\phi]}}{\int \mathcal{D}\phi \; e^{-S[\phi]}} \end{aligned}$$

We have generating functionals.

$$Z[J] \equiv \int \mathcal{D}\phi \; e^{-(S[\phi] + \int dx \, J(x)\phi(x))}$$

With J for source field. Then,

$$\begin{aligned} Z[0] = Z &= \int \mathcal{D}\phi \; e^{-S[\phi]} \\ \left((-1)^n \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)} \right)_{J=0} &= \int \mathcal{D}\phi \; \phi(x_1)\phi(x_2)\cdots\phi(x_n) \; e^{-S[\phi]} \end{aligned}$$

Now one can rewrite Schwinger functions with $Z[J]$.

$$S_n(x_1, x_2, \cdots, x_n) = \frac{1}{Z[0]} \left((-1)^n \frac{\delta^n Z[J]}{\delta J(x_1)\delta J(x_2)\ldots\delta J(x_n)} \right)_{J=0}$$

As a result, we can obtain Schwinger functions by manipulating only generating functionals.

[5]From right to left, or the other way around up to notations.

4 Propagators

When there is no interactions, created particles would just freely propagate and detected. This amplitude of free particle is called propagator. And without interactions, n-point Schwinger functions can be calculated just by combinatorial manipulations of propagators. In other words, two point correlators can be amalgamated to produce $2n$-point correlators. In this case, the existence holds relying on the exact solution of the propagator.

Propagators can be calculated by Green's functions. Green's function is a solution of the equation,

$$(\Box + m^2)G(x,y) = \delta(x-y)$$

In the end, in a concise form via Fourier transform to momentum space,

$$G(x,y) = \frac{1}{(2\pi)^4}\int \frac{e^{ik(x-y)}}{k^2+m^2}\, d^4k$$

This integral requires regularization process to have rigorous meaning. It is an important step to achieve a mathematically legitimate theories.[6]

However when interactions exist, one has to encounter every possible events between particles that really or virtually created and annihilated at singular points. Even, a propagator has a possible self interaction loops for entire configurations. Feynman diagram is a bookkeeping device to fulfill the calculations of interacting theory.

5 Feynman diagrams

Feynman diagram is a graphical method to sort out the contributions of weights according to Feynman axiom 2. One has to sum all the contributions out of possible path configurations between initial and final state degree by degree. Now, the Lagrangian has interacting term. So,

$$\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$$

Basically $\mathcal{L}_{int}$ can be any degree, but for our purpose let's stick to the ϕ^4 interactions. So,

$$\mathcal{L} = (\partial\phi)^2/2 - m_B^2\phi^2/2 - \lambda_B\phi^4/4!$$

This Lagrangian is the only theory we concern from now on.[7]

So, Schwinger functions of this Lagrangian becomes

$$S_n(x_1, x_2, \cdots, x_n) = \frac{\int \mathcal{D}\phi\ \phi(x_1)\phi(x_2)\cdots\phi(x_n)\ e^{-(S[\phi]_{free}+S[\phi]_{int})}}{\int \mathcal{D}\phi\ e^{-(S[\phi]_{free}+S[\phi]_{int})}}$$

[6] Lattice regularization is also the starting point of replacing the analytic definition of field theory with algebraic definition. It will be reminded to give a background for the *condensed* approach in the next note.

[7] B is for bare. This theory needs to be regularized and renormalized to get finite answer. In the end, mass and coupling should be modified as renormalization procedure is done.

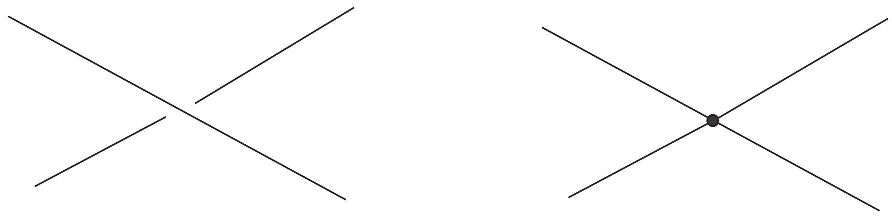

Fig. 1: These are contributions to four point correlators. First is a free diagram. Second is a first degree ϕ^4 interacting diagram.

By taking power series on $e^{-S[\phi]_{int}}$,

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\int \mathcal{D}\phi \; \phi(x_1)\phi(x_2)\cdots\phi(x_n) \; e^{-S[\phi]_{free}} \, (S[\phi]_{int})^n}{\int \mathcal{D}\phi \; e^{-S[\phi]_{free}} \, (S[\phi]_{int})^n}$$

By perturbation series, the calculation becomes a sum of free theories up to each degree of interaction part. Feynman diagram helps one keep track of which contributions to calculate. For example, free theory has diagrams just as straight lines. While ϕ^4 interaction theory should involve possible encounter from four lines into one vertex. Once a diagram is given, there is Feynman rules to calculate the contribution. Schematically, for momentum space calculations,[8]

- external lines for input and output momentum 4-vectors.
- propagators for internal lines
- divide combinatorial factors
- λ dependent contribution for each 4-point vertex
- renormalized factors at each internal interactions

In this calculation, the availability of perturbation depends on the coupling constant λ.[9] The term should be less than 1, so that higher degree terms vanish. Also, it works only when the coupling is dimensionless.[10] Only then renormalization of constant is possible.

For large coupling case, the theory is non-perturbative. We want to provide a tool to study the layered structure of real scalar QFT in any counplings. We want to stack up the existence of correlators by liquid measures even for strong couplings. This process, unlike Feynman diagram, has no calculational intuitions, since it doesn't slice the process degree by degree but by uncorrelated ones.

[8]Since our focus is on ϕ^4 interaction theory, we just deal with those diagrams for ϕ^4 interactions.
[9]The mass m can also be manipulated to depend on λ
[10]It means the number doesn't have physical units.

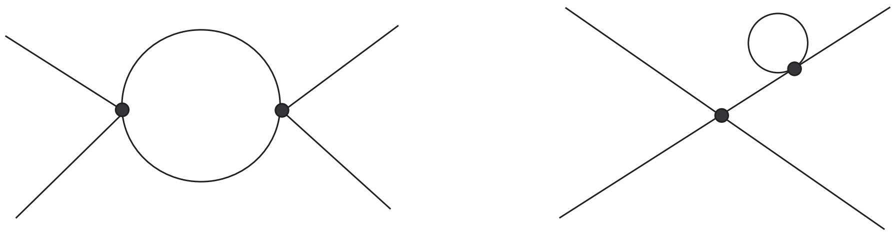

Fig. 2: These are examples of degree two contribution to four point correlators. Note that the interacting points are distinguishable by definitions.

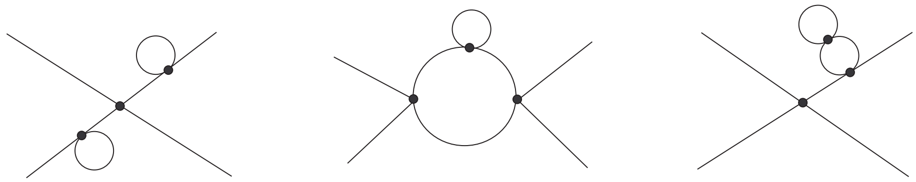

Fig. 3: These are examples of degree three contributions to four point correlators.

Even with small coupling case, calculating loop diagrams embarks divergence problems. There are two different infinities. One is UV problem which is by the virtual particles with unlimited momentum. The other is IR problem which is by the zero mass particles involved in a loop. To resolve UV problem, whole package of technics called renormalization is developed. To resolve IR problem one can let particles to have non-zero mass then limit to zero mass. But it does not resolve the issue from a theory with strong coupling such as QCD. We suggest a *globalization* technics to resolve IR problem via *condensed* treatment.

6 Toward *condensed* lattice field theory

Theory built so far is a physical version of QFT. In order to get mathematically rigorous theory, one has to take constructive approaches. Especially, the path integral formalism requires an important step. Since any field theory over continuum spacetime brings uncountably many degrees of freedom, one should regularize the spacetime by lattice for formality.[11] CQFT is following steps as

1. Translate QFT to Euclidean spacetime by Wick rotation.
2. Regularize Euclidean spacetime by finite lattice Λ^4. Renormalize parameters up to scales.

[11] This is also important for practical reason. Non-perturbative theory is approximated in lattice field theory.

3. Take the continuum limit, and the infinite volume limit of the lattice.

4. Analytically continue back to Minkowski spacetime.

Osterwalder-Schrader theorem guarantees that Schwinger functions satisfying (E) can be brought back to Wightman distribution satisfying (W). So if we have a QFT satisfying axioms (E) or (OS) after the process 2,3, then the viability holds. Our task is to redefine process 2,3 by *condensed* way with a new axiom set. In the next note, we will remind lattice field theory and start to define *condensed* version of Schwinger functions.

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