PRIMITIVES PYTHAGOREAN TRIPLES, with **m** and **n** both odd

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ABSTRACT

The formula with which all the primitive Pythagorean triples are obtained, provided by Euclid around 300 BC, attributes to m, n alternative odd, even values, provided that m and n are coprime, i.e. without common divisors and that m> n. The same formula in cases where m, n, are both odd or both even provides only derived Pythagorean triples. After a specific search I found an alternative form to the algorithm which allows to obtain all the primitive Pythagorean triples $a^2+b^2=c^2$ by assigning both odd positive integer values to m, n and to obtain primitive Pythagorean triples but with mixed values (integers and decimals) to a,b,c, assigning to m, n alternative odd, even values.

The Pythagorean theorem states that "the sum of the perfect squares built on the legs of any right triangle is equivalent to the perfect square built on the hypotenuse". The indication that there are infinite Pythagorean triples formed by positive integer natural numbers a, b, c, such that $a^2+b^2 = c^2$, was provided in 300, BC. from Euclid, who in his Elements wrote the formula able to find all the Primitives Pythagorean triples: with m> n,

$a= m^2 - n^2$;
$b= 2mn$;
$c= m^2 + n^2$.

By attributing to m, n alternative values odd, even, (for example m=7, b=4) or even, odd, (for example m=8, n=5) we obtain primitive Pythagorean triples, provided that m and n are coprime, that is, without common divisors; instead, if m and n have common divisors (example: m=12 and n=9, both divisible by 3) or if m, n are both odd or both even, we obtain derived Pythagorean triples, i.e. multiples of primitive Pythagorean triples (e.g. 3k, 4k, 5k ⇢ 9, 12, 15 or 12, 16, 20, …).

Therefore, the algorithms pointed out by Euclid, provide the primitives Pythagorean triples only by simultaneously attributing alternative odd, even, values to m, n; instead if m and n are both odd or both even, the algorithms provide only derived Pythagorean triples.
Intrigued by this, I did a search to verify the possibility of obtaining primitive Pythagorean triples through algorithms different from the known ones, finding alternative formulas that allow us to obtain all primitive Pythagorean triples by assigning to m, n both odd values, and, on the contrary , to obtain derived Pythagorean triples, also composed of positive integers, assigning to m, n both even values.

Instead, by assigning odd-even values to m, n or vice versa, the formulas produce algorithms whose products are equivalent to primitive Pythagorean triples formed by one leg corresponding to a positive integer while the other leg and the hypotenuse correspond to decimal numbers, such that each of the sides of the triangle is equivalent to half the sides of the primitive Pythagorean triples obtainable through the algorithms cited by Euclid.

Therefore, from this I deduced that:

There exists at least a triad of algorithms a, b, c, formed by positive integers m, n both odd, such that the equation $a^2+b^2=c^2$, provides infinite primitive Pythagorean triples.

In fact, assigning both n and m odd values, if m and n have no common divisor, and $m>n$, the following algorithms
$a = mn$;
$b = [(n^2+m^2)/2]-n^2$;
$c = [(n^2+m^2)/2]$.
they always form a primitive Pythagorean triple $a^2+b^2=c^2$.

PRIMITIVES PYTHAGOREAN TRIPLES, with m and n both odd

m-value	n-value		a= mn;		b = [(m^2+n^2)/2]-n^2;		c = [(m^2+n^2)/2].	MCD
3	1		3		4		5	
5	3		15		8		17	
7	5		35		12		37	
9	7		63		16		65	
11	9		99		20		101	
5	1		5		12		13	
7	3		21		20		29	
9	5		45		28		53	
11	7		77		36		85	
13	9		117		44		125	
7	1		7		24		25	
9	3		27		36		45	**9**
11	5		55		48		73	
13	7		91		60		109	
15	9		135		72		153	**9**
9	1		9		40		41	
11	3		33		56		65	
13	5		65		72		97	
15	7		105		88		137	
17	9		153		104		185	
11	1		11		60		61	
13	3		39		80		89	
15	5		75		100		125	**25**
17	7		119		120		169	
19	9		171		140		221	

The table confirms the fact that the Pythagorean triples obtained are mainly primitive, with the exception of the cases in which m and n have divisors in common and therefore the triples, in such cases, are derived from other primitive triples, as highlighted in the cases marked in bold . Among other things, the arrangement of the table helps to better understand that there are infinite series of homogeneous primitive Pythagorean triples (where m and n always have the same difference of values between them 2, or 4, or 6, etc.) each of which, in turn, produces infinite Pythagorean triples.

DERIVED PYTHAGOREAN TRIPLES, with m, n both even

$a = mn;$
$b = [(n^2+m^2)/2]-n^2;$
$c = [(n^2+m^2)/2].$

m-value	n-value		a= mn;		b = $[(m^2+n^2)/2]-n^2$;		c = $[(m^2+n^2)/2]$.	GCD
4	2		8		6		10	2
6	4		24		10		26	2
8	6		48		14		50	2
10	8		80		18		82	2
12	10		120		22		122	2
14	12		168		26		170	2
16	14		224		30		226	2
6	2		12		16		20	4
8	4		32		24		40	8
10	6		60		32		68	4
12	8		96		40		104	8
14	10		140		48		148	4
16	12		192		56		200	8
8	2		16		30		34	2
10	4		40		42		58	2
12	6		72		54		90	18
14	8		112		66		130	2
10	2		20		48		52	4
12	4		48		64		80	16
14	6		84		80		116	4
16	8		128		96		160	32
12	2		24		70		74	2
14	4		56		90		106	2
14	2		28		96		100	4
16	4		64		120		136	8

The table confirms the fact that the Pythagorean triples obtained are all derivatives (the GCD of each triple is indicated in the last column).

PYTHAGOREAN TRIPLES with m even, n odd
and, alternatively, m odd, n even

a= mn;
b = [(n^2+m^2)/2]-n^2;
c = [(n^2+m^2)/2].

m-value	n-value		a: mn		b: [(m^2+n^2)/2]-n^2		c: [(m^2+n^2)/2]	a^2	b^2	c^2
2	1		2		1,5		2,5	4	2,25	6,25
3	2		6		2,5		6,5	36	6,25	42,25
4	3		12		3,5		12,5	144	12,25	156,25
5	4		20		4,5		20,5	400	20,25	420,25
6	5		30		5,5		30,5	900	30,25	930,25
7	6		42		6,5		42,5	1764	42,25	1806,25
8	7		56		7,5		56,5	3136	56,25	3192,25
9	8		72		8,5		72,5	5184	72,25	5256,25
10	9		90		9,5		90,5	8100	90,25	8190,25
4	1		4		7,5		8,5	16	56,25	72,25
5	2		10		10,5		14,5	100	110,25	210,25
6	3		18		13,5		22,5	324	182,25	506,25
7	4		28		16,5		32,5	784	272,25	1056,25
8	5		40		19,5		42,5	1600	380,25	1806,25
9	6		54		22,5		56,5	2916	506,25	3192,25
10	7		70		25,5		72,5	4900	650,25	5256,25
6	1		6		17,5		18,5	36	306,25	342,25
7	2		14		22,5		26,5	196	506,25	702,25
8	3		24		27,5		36,5	576	756,25	1332,25
9	4		36		32,5		48,5	1296	1056,25	2352,25
10	5		50		37,5		62,5	2500	1406,25	3906,25
8	1		8		31,5		32,5	64	992,25	1056,25
9	2		18		38,5		42,5	324	1482,25	1806,25
10	3		30		45,5		54,5	900	2070,25	2970,25
10	1		10		49,5		50,5	100	2450,25	2550,25

In these cases, even if only one of the three sides of the triangle is an integer while the other two sides (a cathetus and the hypotenuse) are numbers with the decimal part always corresponding to 0.50, the sum of the three sides of the triangle corresponds to the half the sum of a primitive Pythagorean triple and, obviously, the sum of the perfect squares of the legs is equal to the perfect square of the hypotenuse $a^2+b^2=c^2$.