Perfect fluid model with g = 2

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Abstract

A perfect fluid model with a shell of charge is presented which yields g = 2 for low angular velocity. This model is not intended to represent a classical model of the electron but to show that a simple model based on equations consistent with special relativity can yield a value of g = 2.

I. Introduction

An electron takes on a value of g = 2 as predicted by the Dirac equation and is often taken as consequence of quantum mechanics, (for example see Sakauri[1]). If we have a matter distribution where the mass density is proportional to the charge density then a value of g = 1 is found (for example see Singh and Raghuvanshi[2]) and g = 1 is often taken as the classical value of g, however different mass and charge distributions can yield different g values. These do not include the electromagnetic field contribution to the mass and angular momentum which will affect the value of g. Numerous attempts have been made to find a classical electron model with a value of g = 2, (for a review see Lorentz[3], Rohrlich[4], and Spohn[5]).

We will consider a shell of charge model. The electrostatic mass of a spherical shell of charge q and radius R is given by $\frac{1}{2}q^2\frac{1}{R}$, and if this is used and there is no mechanical mass then g takes on the value of $\frac{3}{2}$, (for example see Norvik[6]). However if we define the electrostatic mass using the non-relativistic radiation reaction from an accelerated charge then the electrostatic mass is given the value $\frac{2}{3}q^2\frac{1}{R}$ (for example see Jackson[7]), and if this value is used then we obtain g = 2 (for example see Crisp[8]).

In this paper we will consider a perfect fluid inside a shell of charge and show that for low angular velocity we obtain g = 2 using the electrostatic mass $\frac{1}{2}q^2\frac{1}{R}$ as given in Norvik[6]. Bialynicki-Birula[9] also considers a perfect fluid model of the electron but does not use a shell of charge. Horwitz and Katz[10] consider an perfect fluid with no rotation and use a general charge distribution. Giulini[11] considers a rotating spherical shell with stresses inside the shell to counterbalance the electrostatic forces. He only considers low angular velocity as we do, but does not consider a fluid inside the shell. Jimenez and Campos[12] use a perfect fluid as we do but do not write down equations of motion in the rest frame.

Our model is not an attempt to make a classical electron model but just to show that g = 2 is obtainable using the normal definition of electrostatic mass and equations consistent with special relativity with a solution in the low angular velocity limit.

We are taking the speed of light c to be one, and using the Einstein summation convention where repeated indices represent a summation. Greek indices indicate space-time indices with 0 representing time.

II. Equations of Motion

Following Misner, Thorne and Wheeler (herein MTW) [13] we have for an prefect fluid a mechanical stress-energy tensor $S^{\alpha\beta}$ expressed in the form

$$S^{\alpha\beta} = pg^{\alpha\beta} + (\rho + p)u^{\alpha}u^{\beta} \qquad (1)$$

where p and ρ are the pressure and mass density in the rest frame of the fluid element. u^{α} is the 4-velocity of the fluid, and $g^{\alpha\beta}$ is the metric of the coordinate system. Jimenez and Campos[12] also uses this form for the stress-energy tensor but with the opposite signed metric.

Following MTW[13] we have the equations of motion in general coordinates in the following form

$$S^{\alpha\beta}{}_{;\beta} = F^{\alpha}{}_{\beta}\, j^{\beta} \tag{2}$$

where the ; represents a covariant derivative. j^{α} is the 4-current density and $F^{\alpha}{}_{\beta}$ is the electromagnetic field . We are taking the system to be cylindrically symmetric and stationary.

Using cylindrical coordinates t, r, z, and ϕ take the system to be rotating at a constant angular velocity ω. We can then write the 4-velocity as

$$u^{0} = (1 - r^{2}\omega^{2})^{-1/2}\ ,\ u^{\phi} = \omega u^{0}\ ,\ u^{r} = u^{z} = 0 \tag{3}$$

with $j^{\alpha} = \sigma u^{\alpha}$ where σ is the charge density in the rest frame of the fluid.

In this way eq. (1) reduces to

$$S^{00} = (pr^{2}\omega^{2} + \rho)(1 - r^{2}\omega^{2})^{-1} \tag{4a}$$

$$S^{0\phi} = (p + \rho)\omega(1 - r^{2}\omega^{2})^{-1} \tag{4b}$$

$$S^{\phi\phi} = (pr^{-2} + \rho\omega^{2})(1 - r^{2}\omega^{2})^{-1} \tag{4c}$$

$$S^{rr} = S^{zz} = p \tag{4d}$$

with the rest of the $S^{\alpha\beta}$ terms being zero.

Then in terms of cylindrical coordinates eq. (2) reduces to the two equations

$$p,_r - (p+\rho)r\omega^2(1-r^2\omega^2)^{-1} = \sigma(F^r{}_0 + F^r{}_\phi\omega)u^0 \qquad (5a)$$

$$p,_z = \sigma(F^z{}_0 + F^z{}_\phi\omega)u^0 \qquad (5b)$$

along with the condition that $\sigma F^0{}_\phi = 0$. $F^0{}_\phi$ is zero since the charge distribution is independent of ϕ.

III. **Boundary Condition**

In the interior take the charge density σ to be zero and the pressure constant so setting $p = p_0$ on the inside, from eq. (5a) we need $\rho = -p_0$. This leads to a negative pressure, which Jimenez and Campos[12] also use to counterbalance the electrostatic forces. There is only constant pressure in the interior of the shell, not in the shell itself which we take to have a width d.

All of the charge is concentrated in the outside shell which we take to have the unit normal

$$\mathbf{n} = n_r\mathbf{r} + n_z\mathbf{z} \qquad (6)$$

pointing outward. **r** and **z** are unit normals pointing in the r and z directions. If n is a coordinate of unit length in the normal direction then

$$\frac{\partial p}{\partial n} = \nabla p \bullet \mathbf{n} = p_r n_r + p_z n_z \qquad (7)$$

which with the use of eqs. (5a) and (5b) becomes

$$\frac{\partial p}{\partial n} = (p+\rho)r\omega^2 n_r (1-r^2\omega^2)^{-1} + \sigma\{(F^r{}_0 + F^r{}_\phi \omega)n_r + (F^z{}_0 + F^z{}_\phi \omega)n_z\}u^0 \tag{8}$$

Integrating eq. (8) from the inside of the shell to the outside yields

$$0 - p_0 = \rho = \int_0^d [(p+\rho)r\omega^2 n_r (1-r^2\omega^2)^{-1} + \sigma\{(F^r{}_0 + F^r{}_\phi \omega)n_r + (F^z{}_0 + F^z{}_\phi \omega)n_z\}u^0]dn \tag{9}$$

Now consider eq. (9) as the width of the shell d goes to zero. Since ρ and p are finite, in this limit the $(p+\rho)r\omega^2 n_r(1-r^2\omega^2)^{-1}$ term gives no contribution, and eq. (9) reduces to

$$\rho = \int_0^d \sigma\{(F^r{}_0 + F^r{}_\phi \omega)n_r + (F^z{}_0 + F^z{}_\phi \omega)n_z\}u^0 dn \tag{10}$$

Since $F^r{}_0$, $F^r{}_\phi$, $F^z{}_0$ and $F^z{}_\phi$ can be expressed in terms of the charge and current density we should in principle be able to solve this. Since ρ is constant, the right hand side of eq. (10) has to be independent of position, and this requirement may determine the shape of the surface.

IV. **Low angular velocity approximation**

In general eq. (10) will not be an easy problem to solve. Since the current density is linear in ω, $F^r{}_\phi$ and $F^z{}_\phi$ will also be linear in ω. If we only consider low angular velocity so that we can ignore second order ω terms, then eq. (10) becomes

$$\rho = \int_0^d \sigma(F^r{}_0 n_r + F^z{}_0 n_z)dn \qquad (11)$$

and thus we can approximate the surface as spherical with a constant charge density. We can set $F^r{}_0 = En_r$ and $F^z{}_0 = En_z$ where E is the magnitude of the electric field normal to the surface so that eq. (11) becomes

$$\rho = \int_0^d \sigma E dn \qquad (12)$$

since $n_r{}^2 + n_z{}^2 = 1$.

From Gauss's law we have

$$4\pi r^2 E = 4\pi \int dv\sigma \qquad (13)$$

where the integral is over the volume inside a radius r, and again ignoring ω^2 terms.

Taking the charge density σ to be evenly distributed from 0 to d, eq. (13) reduces to

$$E = \frac{1}{r^2}\int dv\sigma = \frac{4\pi\sigma}{r^2}\int_{R-d}^{r} r^2 dr = \frac{4\pi\sigma}{3r^2}(r^3 - (R-d)^3) \qquad (14)$$

and eq. (12) becomes

$$\rho = \sigma \int_{R-d}^{R} E dr = \frac{4\pi\sigma^2}{3R}(\frac{3}{2}Rd^2 - d^3) \qquad (15)$$

Thus in the limit of d going to zero, $\rho = 2\pi\sigma^2 d^2$ and the total charge is $q = 4\pi R^2 \sigma d$.

Our boundary condition eq. (15) then becomes

$$\rho = \frac{1}{8\pi}\frac{q^2}{R^4} \tag{16}$$

V. g = 2 calculation

The total mechanical mass inside the shell is then

$$m_m = \int dv S^{00} = 4\pi\int_0^R r^2 S^{00} dr = 4\pi\int_0^R r^2 \frac{pr^2\omega^2 + \rho}{1 - r^2\omega^2} dr \tag{17a}$$

$$= 4\pi\int_0^R r^2 \rho dr = \frac{4}{3}\pi R^3 \rho = \frac{1}{6}\frac{q^2}{R} \tag{17b}$$

since $p = -\rho$ on the inside of the shell. Howitz and Katz[10] also obtain this value.

Following Crisp[8] and Nodvik[6] the electromagnetic mass can be written as

$$m_{em} = \frac{1}{2} q^2 \frac{1}{R} + \frac{1}{2} I_{em} \omega^2 \tag{18}$$

where

$$I_{em} = \frac{64\pi^2}{9}\int_0^\infty r^4 \sigma(r) \int_r^\infty r' \sigma(r') dr' dr \tag{19}$$

In the case of a shell of charge eq. (19) becomes

$$I_{em} = \frac{2}{9} q^2 R \tag{20}$$

so that eq. (18) reduces to

$$m_{em} = \frac{1}{2} q^2 (\frac{1}{R} + \frac{2}{9} R\omega^2) = \frac{1}{2} q^2 \frac{1}{R} \tag{21}$$

again neglecting ω^2 terms. The total mass is then

$$m = m_m + m_{em} = \frac{2}{3} q^2 \frac{1}{R} \tag{22}$$

The mechanical mass has no angular momentum since $S^{0\phi} = 0$, so the total angular momentum is

all electromagnetic which, again following Crisp[8], is

$$L = I_{em}\omega = \frac{2}{9} q^2 \omega R \tag{23}$$

Following Singh and Raghuvanshi[2] the magnetic moment is

$$\mu = \frac{1}{3} q\omega R^2 \tag{24}$$

and from Jackson[7] we have the relationship

$$\mu = \frac{g}{2}\frac{q}{m}L \qquad (25)$$

so that we need $g = 2$.

VI. Assumptions and possible Electron model

The only assumptions that have gone into this are

(1) shell of charge

(2) Perfect fluid inside with constant angular velocity

(3) constant pressure inside

(4) low angular velocity.

It is interesting to see how the properties of an electron apply to this model. Using $L = \sqrt{\frac{3}{4}}\hbar$ along with the mass and charge of the electron we obtain $R = 1.9 * 10^{-13}\,\mathrm{cm}$ and $\omega = 8.5 * 10^{25}\,\mathrm{rad/s}$. These values make the velocity at the radius R much greater than the speed of light. Thus as a model of the electron our assumption of neglecting ω^2 terms is not valid.

VII. Conclusion

This model is not intended to be a classical model of the electron but to show that a simple model with equations consistent with special relativity can yield $g = 2$ in the low angular velocity limit.

It would be interesting to try to solve this model for a more general value of ω and to see if that model would also yield $g = 2$ and whether the speed of the shell was less than light when the properties of the electron were applied. The idea of a negative pressure is also not realistic but it is needed to make the model work.

VIII. References

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