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ON THE THEORY OF OLIGOPOLY

In een heterogeen oligopolie met perfecte informatie is de keuze van instrument, prijs of hoeveelheid, door ondernemingen wezenlijk willekeurig. De oplossing mag daarom niet van de instrumentkeuze afhangen. Deze eis van invariantie sluit het Cournot evenwicht en het Bertrand evenwicht en, meer algemeen, elk niet-coöperatief evenwicht volgens Nash uit als een geldige, dat wil zeggen op rationeel gedrag berustende oplossing. Wel in aanmerking komen oligopolie-modellen met conjecturele variaties. Conjecturele variaties hebben veel verwarring gesticht. Maar als ze op de juiste manier worden geïnterpreteerd, dan blijkt de oplossingsverzameling niets anders te zijn dan de contract curve, een coöperatief evenwichtsbegrip. De oplossingsverzameling kan worden ingeperkt door een begrip van economische kracht uitgeoefend door de agenten te introduceren naar analogie van de zwaartekracht uitgeoefend door massieve objecten. Dit leidt tot de conclusie dat het evenwicht wordt gevormd door het punt waar de gezamenlijke winst, inclusief de winst van consumenten, maximaal is. In het geval van prijsnemend gedrag valt de oplossing samen met het competitieve evenwicht. Bij "prijsmakend" gedrag hangt de oplossing (mede) af van cardinale eigenschappen van de nutsfuncties van de consumenten.

Abstract

It is argued, in the context of a heterogeneous oligopoly with perfect information, that rational behaviour should lead to an equilibrium that is invariant to the choice of instrument, price or quantity, by firms. The Cournot oligopoly and the Bertrand oligopoly fail this test; as both solutions are considered an example of a Nash noncooperative equilibrium, this result casts doubt on the appropriateness of the latter concept. The conjectural variations oligopoly passes the test. A natural consistency condition on the conjectural variations implies that the solution is in fact the contract (hyper)surface. The non-uniqueness of the equilibrium can be resolved by introducing a concept of economic force exerted by agents analogous to the force of gravity exerted by massive objects. This leads to the conclusion that the joint profit maximizing point is the equilibrium, where the joint profit includes the profits of the consumers. The equilibrium depends on cardinal properties of the consumers' utility functions. The emphasis is on the exposition of ideas rather than on technical detail.

Keywords: Conjectural variation, Contract curve, Force, Game, Nash noncooperative equilibrium, Oligopoly.

ON THE THEORY OF OLIGOPOLY

1 Introduction

Oligopoly theory concerns the partial equilibrium analysis of economies where agents exert some influence on the price of the good (or service) they sell in an environment of price taking buyers. Usually, the problem is stated in terms of *firms* producing and selling goods and *households* buying and consuming those goods, and that is what we shall do here. The fact that firms may influence the price of the good they sell is thought to derive from their being few in number. Here, "few" means anything between the polar cases of only one firm (monopoly, a quite uninteresting case) and of many firms (polypoly) who all act as price takers.

Ever since Cournot first studied a model with few (in fact two) sellers, economists have shown a keen interest in the case. One reason is its supposed empirical relevance: firms often do not seem to act as price takers. But no doubt, another reason is the intellectual challenge of formulating an apt description of firm behaviour in such situations. This has proved to be a hard task, even to the extent that one has come to speak of the "oligopoly problem". There are several solution concepts which yield different outcomes in general. That the problem still is very much alive, is evident from the recent outburst of papers on "consistency of conjectures" (a conjecture being the opinion of one firm on the reaction of another firm to its own action). Consistency of conjectures also plays a role in the solution we propose, but it is a different concept of consistency than considered hitherto.

The content of this paper is as follows. In the next section we present the general features of the model and introduce most of the notation. In order to provide some background material we review and discuss, in Section 3, several well-known noncooperative solution concepts: those of Cournot and Bertrand, (Chamberlinian) monopolistic competition, Bowley's conjectural variations oligopoly (including its modern version with consistent conjectures), and the Nash equilibrium for noncooperative games. In Section 4 we concentrate on the conjectural variations oligopoly as it seems to be the only solution concept that meets a mild requirement of rationality on the part of firms. We show that with some natural, simple restrictions on the conjectural variations the set of equilibrium points is the contract (hyper)surface. A stronger equilibrium concept is needed. The obvious choice is the joint pay-off (profit) maximizing point (that lies on the contract surface). In Section 5 we suggest a rationale for this choice: the joint pay-off maximum may be regarded as a stationary point in a field of economic forces in action space. Application to the oligopoly model requires that the pay-off to consumers and firms be expressed in the same units. Therefore we describe consumers as profit maximizers in Section 6. Our formulation is such that in the case of price taking behaviour by all agents the joint profit maximum is the competitive equilibrium. If firms are price makers, then the joint profit maximum depends on cardinal properties of the consumers' utility functions. Section 7 concludes.

2 Assumptions, notation, and terminology

We consider a world with I different, imperfectly substitutable products, each one produced by one single firm. All consumers are price taking, budget-constrained utility maximizers. Firms maximize profits. However, firms are not infinitesimal relative to the size of the market and consequently they have some market power; they do not take prices as given.

Let x_i be the quantity of product i and p_i its price. We employ two equivalent representations of the demand side in the model. One is a set of well-behaved, differentiable (*ordinary*) market demand functions giving quantities demanded as a function of prices:

(2.1) $$x_i = g_i(p_1, \ldots, p_I), \qquad i = 1, \ldots, I,$$

where we have deleted aggregate consumer income Y from the list of arguments; this makes it convenient to regard the prices p_i as normalized prices $(p_i := p_i^*/Y)$. We define

(2.2) $$E := [e_{ij}] = [\partial \ln g_i / \partial \ln p_j],$$

the matrix of (uncompensated) price elasticities of quantities demanded. The other is a set of *inverse* market demand functions giving prices offered as a function of quantities:

(2.3) $$p_i = f_i(x_1, \ldots, x_I), \qquad i = 1, \ldots, I.$$

We define

(2.4) $$D := [d_{ij}] = [\partial \ln f_i / \partial \ln x_j],$$

the matrix of (uncompensated) quantity elasticities of prices offered. D and E are inverses of one another.

Both the ordinary and the inverse demand functions are polar cases of the more general class of *mixed* demand functions, which have a mixture of prices and quantities, one from each pair (p_i, x_i), as left hand variables. We assume that these functions exist for all mixtures and that they are differentiable. (An individual's mixed demand functions have been studied by Chavas (1984).)

For the greater part of the paper, these assumptions on the demand side are all we need. The implications of optimizing behaviour by consumers do not play a role until Section 6. There, we will describe consumers as profit maximizing agents.

Firms maximize profits in two stages. In the first stage firm i minimizes its production cost for a given but arbitrary production level x_i. Minimal cost C_i^* is a function of x_i and other variables (prices of variable production factors, quantities of fixed production factors). These latter variables are deleted from the list of arguments since they play no role in our partial equilibrium model. Thus we have

$$C_i = \gamma_i(x_i), \qquad i = 1, \ldots, I,$$

where $C_i := C_i^*/Y$. By partial differentiation we obtain the marginal cost functions,

$$(2.5) \qquad MC_i := \partial\gamma_i/\partial x_i, \qquad i = 1, \ldots, I.$$

MC_i is a positive but not necessarily increasing function of x_i.

The profit of firm i can now be written as

$$(2.6) \qquad \Pi_i = p_i x_i - \gamma_i(x_i), \qquad i = 1, \ldots, I.$$

In the second stage each firm determines the quantity and price of its product such that profits are maximal. By assumption, firms are perfectly informed. They know the (inverse) demand functions and all cost functions. They behave rationally. *The problem we address is what rational behaviour amounts to in situations where firms have market power.*

By using the inverse market demand functions (2.3) all firms can formulate their decision problem as one of choosing the volume of production. Alternatively, they can all formulate it as one of choosing their product price by substituting the market demand functions (2.1) for x_i in the profit definition (2.6). There also is a number of intermediate cases where some firms use quantity and other firms use price as instrument; these formulations involve the use of mixed demand functions. *These are all informationally equivalent versions of one and the same problem.*

Two assumptions are crucial in reducing the dynamic real life decision problem full of uncertainty that firms face to the static mathematical problem without uncertainty that we study. One assumption is that information is costless, or that the speed of information is infinite; the other is that adjustment is costless, or that the speed of adjustment is infinite.

In drawing the logical conclusions from these assumptions we must stay within the confines of the model. This means that we cannot study processes, for processes take time and there is no real time in our model. Nevertheless, in thinking about the oligopoly problem one is naturally led to perform thought experiments involving the reaction of some firm to the action of another firm. Therefore terms like "conjectural variation" and "reaction function" occur in the literature on the oligopoly problem. These terms are misleading because of their dynamic flavor. Actually all we can do is study states of nature. In particular, we are interested in the existence of states with certain desirable properties, called "equilibria". This requires the statement of equilibrium conditions based on the premise that firms are rational profit maximizing agents. To this we now turn.

3 Some noncooperative equilibrium concepts

In this section we review several well-known noncooperative equilibrium concepts and make a number of comments on them.

3.1 Cournot oligopoly

Cournot considered an oligopoly with one homogeneous good and assumed that each oligopolist chooses his *production level* taking the *quantities* of his rivals as given. We shall designate the natural generalization of this model to a world with differentiated products by *Cournot oligopoly*. Under the assumption of "quantity taking" we can easily derive the first order condition for a profit maximum of firm i by substituting the *inverse* demand function (2.3) into the profit definition (2.6) and then taking the *partial* derivative with respect to x_i. We can rewrite the resulting expression to

(3.1) $$MC_i = p_i(1 + d_{ii}), \qquad i = 1,\ldots, I.$$

Under certain appropriate regularity conditions a (possibly unique) solution to equations (2.3) and (3.1) exists, see Friedman (1977).

3.2 Bertrand oligopoly

Bertrand criticized Cournot by arguing that firms choose *prices* rather than quantities, taking the *prices* of their rivals as given. We shall designate the natural generalization of this model to a world with differentiated products by *Bertrand oligopoly*, while acknowledging the contribution of Edgeworth.

Under the assumption of "price taking" we can easily derive the first order condition for a profit maximum of firm i by substituting the *(ordinary)* demand function (2.1) into the profit definition (2.6) and then taking the *partial* derivative with respect to p_i. We can rewrite the resulting expression to

$$(3.2)\quad MC_i = p_i(1 + e_{ii}^{-1}), \qquad i = 1,\ldots, I.$$

Again, under certain regularity conditions a (possibly unique) solution to equations (2.1) and (3.2) exists.

3.3 Monopolistic competition

Chamberlin added an element of the theory of monopoly -the assumption that firms may set the price of their own product- to the theory of perfect competition -where firms are small and many- to obtain the theory of monopolistic competition. Presumably, *product differentiation* is the feature that bestows consistency upon this model. It does not belong to the theory of oligopoly proper but to the more general field of imperfect competition. We discuss it here for later reference.

The assumption that firms are small and many allows one to ignore the income effect of a single price change because it has the order of magnitude of 1/I. Thus one obtains the equilibrium conditions

$$(3.3)\quad MC_i = p_i\, [1 + (e_{ii}^*)^{-1}], \qquad i = 1,\ldots, I,$$

where e_{ii}^* is the *compensated* price elasticity of quantity demanded. The formal derivation proceeds by substituting the compensated (ordinary) demand function into the profit definition and then taking the partial derivative with respect to p_i.

Usually, compensated demand functions of the following special form are postulated in the model of monopolistic competition:

(3.4) $x_i = c_i(x, p_i/p)$, $i = 1, \ldots, I$,

where x is total (industry) demand and p is the price of the composite good. Dixit and Stiglitz (1977) take x to be a Constant Elasticity of Substitution (CES) function with parameter σ. In that case (3.4) specializes to

(3.4') $x_i = x.(p_i/p)^{-\sigma}$, $i = 1, \ldots, I$,

with p a CES function of the individual prices p_i. However, in deriving the first order condition for a profit maximum of firm i they neglect the effect of p_i on p. Thus they obtain

(3.5) $MC_i = p_i(1 - \sigma^{-1})$.

In comparison to (3.3) another term with order of magnitude of 1/I has been ignored.

Bertrand oligopoly and Cournot oligopoly are dual versions of one another. Although it does not occur in the literature, there also exists a dual version of Chamberlinian monopolistic competition, a version where firms use quantity as instrument instead of price. The first order conditions for an equilibrium then read

(3.6) $MC_i = p_i(1 + d^*_{ii})$, $i = 1, \ldots, I$,

where d^*_{ii} is the *compensated* quantity elasticity of price offered. A special form of the compensated inverse demand functions analogous to (3.4) is

(3.7) $p_i = d_i(p, x_i/x)$, $i = 1, \ldots, I$.

If we take p to be a CES function with parameter $\tau = 1/\sigma$, then

(3.7') $p_i = p.(x_i/x)^{-1/\sigma}$, $i = 1, \ldots, I$.

Neglecting the effect of x_i on x we obtain the equilibrium conditions (3.5). This is a remarkable result.

3.4 Conjectural variations oligopoly

According to Bowley, firms take the impact of their actions on their rivals' actions into account. Mathematically this means that we obtain the appropriate first order conditions for profit maximization by taking *total* rather than partial derivatives. In case firms use *quantity* as instrument we can write these conditions as

$$(3.8)\quad MC_i = p_i(1 + \delta^i_{ii}) = p_i(1 + \sum_j d_{ij}X^i_{ji}), \qquad i = 1,\ldots, I,$$

where $X^i_{ji} = (d\ln x_j/d\ln x_i)^i$. In accordance with the usual parlance in the literature on oligopoly we may call X^i_{ji} (i, j = 1,..., I) the (quantity-quantity) *conjectural variation elasticity*: it is the relative change in the quantity produced by firm j that firm i (superscript) thinks is induced by an infinitesimal relative change in its own quantity.

In case firms use *price* as instrument the first order conditions take the form

$$(3.9)\quad MC_i = p_i[1 + (\epsilon^i_{ii})^{-1}] = p_i[1 + (\sum_j e_{ij}\phi^i_{ji})^{-1}], \qquad i = 1,\ldots, I,$$

where $\phi^i_{ji} = (d\ln p_j/d\ln p_i)^i$. We may call ϕ^i_{ji} the (price-price) conjectural variation elasticity: it is the relative change in the price charged by firm j that firm i thinks is induced by an infinitesimal relative change in the price it charges for its own product.

There also is a number of intermediate cases where some firms use quantity and other firms use price as instrument. If we want to (but we don't), we can write down the appropriate first order conditions by making use of mixed demand functions. The expressions would involve price-quantity and quantity-price conjectural variation elasticities.

A problem that has puzzled economists for a long time, is: how are the conjectural variation elasticities to be determined? In the (apparent) absence of any clear guiding principle -it is not obvious which functions are being differentiated- the students of the oligopoly problem have frequently resorted to the assumption of these elasticities being zero, thus essentially falling back on the concept of Cournot oligopoly (or Bertrand oligopoly, as the case may be). Recently however, several authors have suggested that the conjectural variations must satisfy some *consistency condition;* see Bresnahan (1981), Laitner (1980), Perry (1982), and Kamien and Schwartz (1983). Below we point out the sense in which, according to these authors, the conjectural variations should be consistent.

Condition (3.8), which states that in equilibrium for each firm marginal cost equals (perceived) marginal revenue (MR), can be thought of as an implicit function giving x_i in terms of the other quantities. Let r_i be the function that solves (3.8) for x_i in terms of those quantities:

(3.10) $x_i = r_i(x_1,\dots, x_{i-1}, x_{i+1},\dots, x_I)$ solves (3.3),

for $i = 1,\dots, I$. The function r_i is usually called the *reaction function* of firm i. From (3.10) it follows

$$\left.\frac{d\ln x_i}{d\ln x_j}\right|_{MC_i = MR_i} = \sum_{k\neq i} \frac{\partial \ln r_i}{\partial \ln x_k} \cdot \frac{d\ln x_k}{d\ln x_j} .$$

According to the authors mentioned above, conjectures are consistent if

$$(3.11) \quad \chi^j_{ij} = \left. \frac{d\ln x_i}{d\ln x_j} \right|_{MC_i = MR_i} .$$

A solution to the set of equations (3.10) -or equivalently, (3.8)- and (3.11) is called a *consistent conjectures equilibrium* (provided, of course, that the second order conditions for profit maximization are fulfilled). For special cases of the (inverse) demand functions and marginal cost functions such an equilibrium has been shown to exist, see Bresnahan (1981).

3.5 Nash equilibrium

The final solution concept we consider has its origin in game theory: the *noncooperative equilibrium according to Nash*. Each player i $(i = 1,\ldots, I)$ chooses an action a_i from his action possibility set A_i such that the value of his pay-off or utility ω_i,

$$(3.12) \quad \omega_i = w_i(a_1,\ldots, a_I),$$

is as high as possible. The players are in a situation of interdependence because the outcome for each player in general depends on the actions taken by other players as well as on his own action. Each player i decides on his action a_i without communicating with other players, but fully informed about the action possibility sets and preferences of all players. A Nash non-cooperative equilibrium $a_1^*, \ldots, a_I^*$ is defined by

$$(3.13) \quad w_i(a_1^*,\ldots, a_i^*,\ldots, a_I^*) = \max_{a_i \in A_i} w_i(a_1^*,\ldots, a_i,\ldots, a_I^*), \qquad i = 1,\ldots, I.$$

Johansen (1982) has argued that the Nash noncooperative equilibrium is a fundamental solution concept in situations of noncooperative interaction: if we take for granted that there is a natural solution concept in such situations, then it must be the Nash noncooperative equilibrium; it incorporates fully rational behaviour on the part of each player; player i must consider all the decisions $a_1, \dots, a_I$ simultaneously when deciding on his own action a_i, *i.e.* he must consider the full problem of determining $a_1^*, \dots, a_I^*$ according to the conditions of definition (3.13).

There are two obvious ways in which the concept of Nash equilibrium can be applied to the oligopoly problem. In both versions, the pay-off for firm i is identified with its profit. In one version, a decision is identified with the choice of quantity by each firm; in the other, a decision is identified with the choice of price by each firm. (There also is a number of intermediate cases where some firms choose a quantity and other firms set a price for their product.)

Note that the definition of a Nash equilibrium given above is not formulated in terms of a set of I equations, one for each firm, representing the first order conditions for profit maximization. The decision situation in which the players find themselves is described by Van Neumann and Morgenstern as follows: "Thus each participant attempts to maximize a function of which he does not control all variables. This is certainly no maximum problem, but a peculiar and disconcerting mixture of several conflicting maximum problems. Every participant is guided by another principle and neither determines all variables which affect his interest. This kind of problem is nowhere dealt with in classical mathematics." In the next subsection we shall return to this point.

3.6 Discussion

Many authors give misleading explanations of what a Nash noncooperative equilibrium is; for a sample of quotations, see Johansen (1982). Frequently, it is simply stated that a Nash equilibrium is a set of decisions in which each player's decision is optimal *given the decisions of all other players*. But player i does not know the actions of the other players. He only knows the action possibility sets and the pay-off functions, and he must figure out what the other decisions will be. Whatever the appropriate wording is, in the differentiable case (3.13) is mathematically equivalent to a definition in terms of first order conditions obtained by partial differentiation of each player's pay-off function with respect to his decision variable. Thus, in the case of quantity (price) setting oligopolists, one arrives at the Cournot (Bertrand) solution which may then be called the Cournot (Bertrand)-Nash equilibrium.

Now, by assumption all firms have full knowledge of the (inverse) demand functions and thus they are able to translate any price strategy into a quantity strategy, and *vice versa*. Hence the choice of instrument is essentially arbitrary. *Therefore, a minimal requirement of rationality (on the part of each firm) would seem to be that the solution is the same, no matter whether firms use quantity or price as instrument*. If a Nash equilibrium assumes fully rational behaviour by firms as Johansen (1982) argues it does, then it must fulfill this requirement. But a Cournot oligopoly and a Bertrand oligopoly are not equivalent, *i.e.* their equilibrium points do not generally coincide.[1] In our view, the most objectionable feature of the Cournot and Bertrand oligopoly is this dependence of the ensuing equilibrium on the *arbitrary choice of instruments*.

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[1]See Hathaway and Rickard (1979).

In Section 3.3 we have encountered an example of equivalence of the price and quantity version of monopolistic competition, but that result holds only approximately for a special form of the (compensated) demand and inverse demand functions.

Now we have seen that the Cournot and Bertrand solution -and, more generally, every Nash noncooperative equilibrium- must be discarded, we are faced with the question how to obtain a set of first order conditions for profit maximization that corresponds to rational behaviour.

The natural route is to take the *total* derivative of each firm's profit function with respect to its own decision variable. This possibility corresponds, of course, to the conjectural variations oligopoly in its various forms. For one thing, we observe that this class of models is rich enough to meet the requirement that the solution to the output game coincides with the solution to the price game. In the case of a duopoly, this requirement leads to the following relation between the price-price and quantity-quantity conjectural variation elasticities:

$$(3.14) \qquad X^j_{ij} = (e_{ij} + e_{ii}\,\phi^j_{ij}) \,/\, (e_{jj} + e_{ji}\,\phi^j_{ij}), \qquad i, j = 1, 2.$$

It follows from equating the right hand sides of (3.8) and (3.9). A similar derivation in the case of more than two firms fails because of the abundance of the number of conjectural variation elasticities relative to the number of equality restrictions. But by using the fact that D and E are inverses of one another, we can easily prove that the obvious generalization of (3.14),

$$(3.15) \qquad X^j_{ij} = \sum_{\ell} e_{i\ell}\,\phi^j_{\ell j} \,/\, \sum_{\ell} e_{j\ell}\,\phi^j_{\ell j}, \qquad i, j = 1,\ldots, I,$$

continues to guarantee the equivalence of the price and quantity version of the game.

Some students may argue, like Daughety (1985, p. 374) does, that there is no place for (non-zero) conjectural variations in a static model. They are wrong. There is nothing inherently dynamic in the mathematical concept of a derivative, be it a partial or a total derivative.

That non-zero conjectural variations need not, indeed *cannot* be ruled out *a priori* is evident from (3.14) and (3.15), which show that generally to zero conjectural variations in price space there correspond non-zero conjectural variations in quantity space. Of course, the converse holds good as well.

Confining ourselves to conjectural variations oligopolies we face the problem of determining the conjectures. Do natural restrictions exist that these conjectures must obey? Preferably, such restrictions (if there are any) must *imply* the equivalence of the various (price, quantity, mixed) versions of the game. One possibility that comes to mind is the assumption of consistency set out in Section 3.4 above.[2] We regard this solution as unsatisfactory for reasons to be explained in the next section.

So far we have not paid any attention to two well-known models, Stackelberg's *leader-follower oligopoly* and Sweezy's *kinked demand curve oligopoly*. In the latter case, a specific asymmetry is introduced in firms' conjectures: competitors are assumed to follow a price cut but not to follow a price increase at all. In the former case, firms may be *followers*, taking the quantity (price) of their rivals as given, or *leaders*, assuming that their rivals are followers and exploiting this to their advantage. In both cases, arbitrary assumptions are made about the conjectural variations.[3] In our opi-

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[2]In the duopoly case Salant (1984), *confining* his attention to quantity-quantity and price-price conjectures that yield the same outcome, showed that to a consistent conjectures equilibrium in quantities there corresponds a consistent conjectures equilibrium in prices, and *vice versa*. Of course, this does not *prove* that consistency (of conjectures) implies equivalence (of the price and quantity version).

[3]In the Stackelberg oligopoly the arbitrariness is frequently shifted one stage back by postulating a size distribution of firms (which actually is endogeneous) and linking size with behaviour, assuming large firms to be leaders and small ones to be followers.

nion, however, the (properties of the) conjectures must be derived as part of the solution to the oligopoly problem rather than assumed *a priori*.[4] To this we turn in the next section.

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[4]Stigler (1964, p. 44) takes a similar position when he writes: "A satisfactory theory of oligopoly cannot begin with assumptions concerning the way in which each firm views its interdependence with its rivals. If we adhere to the traditional theory of profit maximizing enterprises, then behaviour is no longer something to be assumed but rather something to be deduced."

4 Consistent conjectural variations

We continue to explore the properties of the conjectural variations oligopoly on the assumption -for which we see no alternative- that the solution to the oligopoly problem is a member of this class of models. Thus, with $\omega_i = w_i(a_1, \dots, a_I)$ the pay-off function of player i, the first order conditions for a pay-off maximum are

$$(4.1) \qquad \omega^i_{ii} = \sum_j w_{ij}\alpha^i_{ji} = 0, \qquad i = 1, \dots, I,$$

where $\alpha^i_{ji} = (da_j/da_i)^i$. This corresponds to (3.8) when firms use (the natural logarithm of) quantity as instrument; more generally, a_i may be either $\ln(p_i)$ or $\ln(x_i)$. We shall also consider the cross derivatives of one player's pay-off function with respect to the action of some other player, as conjectured by a third player. In general we have

$$(4.2) \qquad \omega^\ell_{ik} = \sum_j w_{ij}\alpha^\ell_{jk}, \qquad i, k, \ell, = 1, \dots, I,$$

where α^ℓ_{jk} is the conjecture of firm ℓ on the reaction of firm j to an action of firm k. There are I^3 such conjectures, I^2 of which (the α^ℓ_{jj}) are equal to one.

As we assume that players behave rationally, it is natural to impose the restrictions that *conjectured* total derivatives equal *actual* total derivatives:

$$(4.3) \qquad \alpha^\ell_{jk} = \alpha^j_{jk} = \alpha_{jk}, \qquad j, k, \ell = 1, \dots, I,$$

implying $\omega^\ell_{ik} = \omega_{ik}$ for all i, k and ℓ. This reduces the number of (as yet) unknown action-action derivatives to $I^2 - I$. *But then, if the* α_{jk} *are total derivatives they have all properties of total derivatives.* In particular, there holds

$$(4.4) \quad \alpha_{ik}\alpha_{kj} = \alpha_{ij}, \qquad i, j, k = 1, \ldots, I,$$

which simply is the chain rule for differentiation. These relations reduce the number of independent interesting action-action derivatives to I - 1.

The observation that in equilibrium the (actual) reaction coefficients α_{ij} obey the relations (4.4) is a simple but crucial step in our analysis. The rest is a matter of elementary mathematics (mainly linear algebra).

We start by noting that (4.1, 4.3-4) imply that in equilibrium all (cross) derivatives ω_{ik} are equal to zero. For $\omega_{ik} = \omega_{ii}\alpha_{ik}$, and $\omega_{ik} \neq 0$ would imply (both $\alpha_{ik} \neq 0$ and) $\omega_{ii} \neq 0$, contradicting (4.1). Defining $W = [w_{ij}]$ and $A = [\alpha_{ij}]$ we can compactly write all these equations as

$$(4.5) \quad WA = 0.$$

But since (4.4) implies that the matrix A has rank one, it suffices to consider anyone of the following sets of I equations:

$$(4.6) \quad W\alpha_i = 0, \qquad i = 1, \ldots, I,$$

where α_i is the i-th column of A. Multiplying $\omega_{jj} = 0$ by α_{ji} yields the j-th equation of the i-th set. Thus (4.6) is nothing but a set of I equivalent representations of the first order conditions (4.1).

Now observe that for given W, $W\alpha_i = 0$ is a set of I linear equations in I - 1 unknowns ($\alpha_{ii} = 1$). The existence of a solution requires that the coefficient matrix is singular, implying that its determinant equals zero:

$$(4.7) \quad |W| = 0.$$

In fact, we shall assume that in equilibrium the rank of W is I - 1, and that each of its submatrices W_{ii}, obtained by deleting the i-th row and column, has full rank I-1. This guarantees the uniqueness of the solution for α_i, given W.

In order to see more clearly the meaning of (4.7), delete the i-th equation from $W\alpha_i = 0$ and solve the reduced system for α_{ji}, $j \neq i$. Let $w_{i(-i)}$ be the i-th column of W without element i, and let $\alpha_{i(-i)}$ be defined analogously. The solution is

$$(4.8) \quad \alpha_{i(-i)} = -(W_{ii})^{-1} w_{i(-i)}, \qquad i = 1,\ldots, I.$$

Writing out the right hand side of (4.8) we see that (with $|W_{ij}|^*$ the cofactor of w_{ij})

$$(4.9) \quad \alpha_{ji} = |W_{ij}|^* / |W_{ii}|^*, \qquad i, j = 1,\ldots, I.$$

Substitution of (4.9) for α_{ji} into the equation deleted at the outset and multiplication by $|W_{ii}|^*$ yields

$$(4.10) \quad \sum_j w_{ij} |W_{ij}|^* = 0, \qquad i = 1,\ldots, I.$$

The left hand side of (4.10) simply is the determinant of the matrix W expanded in terms of the elements of row i and their cofactors, and so (4.10) represents I equivalent versions of the condition $|W| = 0$.

This derivation shows that the restrictions (4.3-4) imply that in equilibrium each of the first order conditions (4.6) collapses to the same relation. All points on the surface defined by $|W| = 0$ satisfy the first order conditions. In general, there is no unique solution to the equilibrium conditions considered until now.

A geometric interpretation helps to further clarify what we have derived sofar. Let us consider an isopay-off surface in action space for player i, defined by ω_i = c(onstant). Along such a surface there holds

$$(4.11) \quad \sum_j w_{ij}\, da_j^i = 0, \qquad i = 1,\ldots, I,$$

which is equivalent to "the" first order conditions (4.1) for a pay-off maximum. Through a point on the isopay-off surface there is a hyperplane tangent to the surface at that point. The condition (4.11) stipulates that the vector da^i lies in a tangent hyperplane to some isopay-off surface of player i. Our consistency conditions (4.3-4) simply amount to the requirements $da^i = da$, $i = 1,\ldots, I$; (4.11) then states that for a point to be an equilibrium, there must exist at that point some joint infinitesimal variation of actions that leaves each player's pay-off unchanged. In general, the I tangent hyperplanes through a point (one for each isopay-off surface) have only the point concerned in common. *The geometric interpretation of the restrictions (4.3-4) is that we confine attention to points where the tangent hyperplanes have (at least) a straight line in common.* In the case of two players this implies that the two tangent lines must coincide, which occurs at points of tangency between the isopay-off curves of the two players. These points form a curve which is analogous to Edgeworth's contract curve. Therefore we shall call the (hyper)surface defined by the equation $|W| = 0$ the *contract surface*.

The case of two players can be illustrated diagrammatically with a figure well-known from textbooks on microeconomic theory, *e.g.* Koutsoyiannis (1975, p. 221 or 234). Figure 1 is adopted from this source. It shows isopay-off curves for both players, the curves of the pay-off for player i being concave to the a_i-axis. The closer the curve is to the axis, the higher the pay-off. The curve M_1M_2 traces out the points of tangency between the isopay-off curves of both players: it is the contract curve. Also shown are the reaction curves for the case that the players are naive in the sense that player i assumes that α_{ji} $(j \neq i)$ equals zero, for i, j = 1, 2. The reaction curve for such a player is given by $w_{ii} = 0$ (for quantity setting (price setting) oligopolists this is the Cournot (Bertrand) assumption). The intersection R of the reaction curves is the "equilibrium" point, given that the players behave in this suboptimal way. The points S_1 and S_2 are the Stackelberg "equilibria" of the game: the leader or sophisticated player has determined his optimal action by finding the point of tangency between one of his isopay-off curves and the reaction curve of the follower or naive player. Again, these points represent equilibria only under the particular behavioural assumptions mentioned, which impose suboptimal behaviour on one of the players.

For the sake of concreteness, assume that the isopay-off curves for player i form a set of concentric circles with center M_i, i = 1, 2. All points on the straight line through M_1 and M_2 satisfy the first order conditions (4.6). However, at points on the line but outside the segment M_1M_2 the second order conditions are not fulfilled: at those points the pay-off for one player is at a minimum given the pay-off for the other player. Henceforward, we shall reserve the term contract surface for the set of points where both the first order and the second order conditions are met with. In the present case the contract curve is the line segment M_1M_2.

The generalization to cases with more than two players is straightforward. Let there be three players, where the isopay-off surfaces for each player i form a set of concentric balls with center M_i. The contract surface now is the triangle $M_1M_2M_3$. With four players, the contract (hyper)surface is a tetrahedron (something we can still imagine), and so on (beyond imagination).

We now return to the concept of consistency defined in Section 3.4, equation (3.11). For simplicity's sake we consider a duopoly. Note that (3.11) applied to one player (the first, say) separately is a tangency condition: Firm 1 equates the marginal rate of substitution da_2/da_1 along some isopay-off curve with the marginal rate of substitution along the reaction curve of Firm

2. The solution may be called a generalized Stackelberg "equilibrium", the generalization being that the second firm's conjectural variation need not equal zero (but is known to the other firm). Of course, for a given pair of conjectural variations there are two generalized Stackelberg "equilibria" that do not in general coincide. The exception occurs when each firm's reaction curve coincides with the contract curve. Then both firms search in vain for a point of tangency with one of their isopay-off curves, and the whole contract curve remains as the collection of candidate equilibrium points.

From the analysis in this section we conclude that rational behaviour in our game with full information and instantaneous adjustment is cooperative behaviour. The same result has been obtained for infinitely repeated games, but we have derived it for the one-shot oligopoly game that is being played only once, by simply imposing the almost self-evident requirement that the solution be invariant to the arbitrary choice of instrument. On closer scrutiny, the result is not as unexpected as it may seem at first sight. In Section 3 we have reviewed a number of noncooperative "solution" concepts, uncritically accepting that they are applicable to the problem at hand. Now, the distinguishing features of a noncooperative game are that the players 1. may not communicate, and 2. cannot make binding contracts. But in a world with perfect information there is no need for communication, and if moreover adjustment is instantaneous, then there is no need for contracts either. So these two restrictions on the players' behaviour are not binding.

5 Joint profit maximization and economic forces

As the contract curve M_1M_2 in Figure 1 is the locus of points where the isopay-off curves of the players are tangent and where the second order conditions are fulfilled, a movement along the curve necessarily means that while the pay-off to one player is increased, the pay-off to the other player is lowered. But consider any point off the contract curve, for example R. By following the isopay-off curves through R we see that the pay-off to one player may be increased while holding constant the pay-off to the other player. Thus at all points off the contract curve there are untapped possibilities for pay-off increases.

This discussion implies that the point (or points) where the joint pay-off $\omega = \omega_1 + \omega_2$ reaches a (or its) maximum, is a point on the contract curve. It is quite easy to prove this mathematically for the general case of I players. The first order conditions for a joint pay-off maximum are

$$(5.1) \qquad \partial\omega/\partial a_i = \sum_j w_{ji} = 0, \qquad i = 1,\ldots, I.$$

With ι' a row vector of ones we can state these conditions as

$$(5.2) \qquad \iota'W = 0.$$

At a point a^* in action space where these conditions are satisfied, the matrix W is singular. Hence a^* is a point on the contract surface. This result also holds true if the joint pay-off is not the simple sum but some other function of the individual pay-offs, in which case we must replace ι' by the row vector of first partial derivatives of this function with respect to its arguments.

Now, why is it that we do not find just one "equilibrium" point or only a few but a whole continuum of them? Let us return to Figure 1 and give it another interpretation. Point M_1 represents the earth and point M_2 represents the moon. These celestial bodies are surrounded by a gravity field. The curves centered at M_1 connect the points where the potential function of the field of the earth takes the same value. At each point in the field, the gradient of the potential function in some direction is equal to the force of the field in that direction. Analogously for the field of the moon. For a massive body somewhere on the curve M_1M_2 there is no force pulling it off the curve. That does not mean that the system is in equilibrium, for the attractive forces of the earth and the moon need not cancel at the particular point. To be sure, there is a point on the curve where they do cancel, and this is an equilibrium point; the equilibrium is unstable, because even a small displacement from the point will cause a massive body to start falling either to the earth or to the moon.

By now it will be clear what is missing from game theory: a theory of forces exerted by players. Game theorists, and social scientists making use of game theory (economists for example), have only been willing to state that players do not exert forces in some directions (at some point, all directions in the tangent hyperplane to their isopay-off surface at that point) but have remained silent on the intensity of effort in other directions. The result is a literally (and figuratively?) forceless theory. Still, in many applications the introduction of the concepts of force and power seems quite natural. In fact, economists do speak of market forces and economic power. And even the carrier of the force has been identified, as is apparent from the title of a song in the film Cabaret: Money Makes The World Go Round.

It is not the purpose of the present paper to elaborate a theory of force in game theory or, more specifically, in economic theory. But we do want to draw attention to some consequences of formally introducing a force. How the introduction can be achieved is clear from the analogy with the theory of gravity: we identify the potential function of a player with his pay-off function; the force exerted by a player at some point in action space in a particular direction is equal to the gradient of his pay-off function in that direction, evaluated at the point concerned. Now, in order for the force to have an objective meaning and to be amenable to empirical investigation, we must be able to compare forces across players, or in other words to express the forces exerted by the different players in the same units. So, cardinal properties of the pay-off functions are invoked. *In economics, the introduction of an objective force would imply the resurrection of cardinal utility, even without uncertainty.*

Accepting the existence of an (objective) force, we note that *the equilibrium points of a (noncooperative) game are those points where the joint pay-off is stationary;* for at such points the forces exerted by all players exactly cancel. This is the analogon of the result that an equilibrium in physical systems is characterized by some aggregate entity, for example potential energy, being at an extreme value.

In economics, a usual assumption is that marginal pay-off (profit, utility) is decreasing in directions of increasing pay-off. As a corollary, the strength of the economic force increases with increasing distance (in action space) from the point of maximum pay-off (in this respect the analogy with the force of gravity fails; the economic force is better likened to the strong force that binds together quarks in composite particles like protons, electrons and so on). *The assumption of diminishing marginal pay-offs implies that on the curve M_1M_2 in Figure 1 there is a unique equilibrium point and the equilibrium is stable.*

Much more can be said on the similarity between game theory and models of the natural forces. It seems that the analogy can be stretched in many respects. For example, we may confer an economic charge upon the players, stipulating that they are sensitive to the economic force. But it is time that we return to the subject matter of the paper: the theory of oligopoly.

6 Oligopoly and joint profit maximization

6.1 Introduction

In this section we maintain the hypothesis that firms -or, more generally, agents- behave in such a way that the joint profit is being maximized, and we study its consequences within the context of the model sketched in Section 2. Of course, there are many reasons why in reality the joint profit maximum will not be reached (on this subject, see Stigler (1964)), but that is not our concern here.

The joint profit of firms is given by

$$(6.1) \quad \Pi := \sum_i \Pi_i = \sum_i [p_i x_i - \gamma_i(x_i)] = Y - \sum_i \gamma_i(x_i),$$

where Y, the aggregate consumer income, is a datum. This means that, with cost functions that are strictly increasing, the joint profit of firms is maximal if $x_i = 0$, $i = 1,\ldots, I$.

The explanation for this result is, of course, that no weight has been given to the interests of consumers or, stated differently, that the economic forces exerted by consumers have been neglected. This raises the question how these interests can be taken into account.

Usually, consumers are described as budget-constrained utility maximizers. We now face the problem that the objectives of consumers and producers are not being expressed in the same units. Describing firms as utility maximizers is no way out because "utils" need not be comparable across agents.

The solution we propose is to treat consumers in a way that is completely analogous to the way producers are being treated: as profit maximizing agents. This is the subject of the next subsection. Thereafter we return to the problem of joint profit maximization.

6.2 Consumer behaviour

We feel free to present some results from the economic theory of consumer behaviour without proof. For a good treatment of the duality concepts involved, see Anderson (1980) and Weymark (1980).

We assume there are J identical consumers. We omit the subscript j (= 1,..., J). Each consumer has a given budget Y, which we take as the unit of value (Y = 1).

Traditionally, consumers are described as maximizing their utility U,

$$(6.2) \quad U = u(x_1,..., x_I),$$

subject to the budget constraint $\sum_i p_i x_i = 1$. The first order conditions are (next to the budget constraint)

$$(6.3) \quad \partial u/\partial x_i =: u_i = \lambda p_i, \qquad i = 1,..., I.$$

The solution consists of the consumer's demand functions, which give the optimal quantities as a function of the normalized prices p_i (= p^*_i/Y):

$$(6.4) \quad x_i = g_i(p_1,..., p_I), \qquad i = 1,..., I.$$

(By summing these demand functions over all consumers we obtain the market demand functions (2.1).) The economic interpretation of the Lagrange multiplier λ associated with the budget constraint is that of the *marginal utility of income*.

By substituting the functions $g_i(.)$ for x_i into the (*direct*) utility function u(.) we obtain the *indirect* utility function v(.),

$$(6.5) \quad U = v(p_1,..., p_I).$$

The dual problem of minimizing v(.) subject to the budget constraint with given quantities leads to the first order conditions

$$(6.6)\quad \partial v/\partial p_i =: v_i = -\mu x_i, \qquad i = 1,\ldots, I.$$

The optimal prices are a function of the quantities:

$$(6.7)\quad p_i = f_i(x_1,\ldots, x_I), \qquad i = 1,\ldots, I.$$

These are the consumer's inverse demand functions. Again, the Lagrange multiplier associated with the budget constraint, μ, stands for the consumer's marginal utility of income.

In fact, if we know the indirect utility function we can easily obtain the (ordinary) demand functions from the first order conditions of the dual problem:

$$(6.8)\quad x_i = v_i/\sum_j p_j v_j, \qquad i = 1,\ldots, I.$$

This is the Identity of Ville and Roy. The Lagrange multiplier μ is given by

$$(6.9)\quad \mu = -\sum_i p_i v_i.$$

Analogously, we obtain from the primal problem

$$(6.10)\quad p_i = u_i/\sum_j x_j u_j, \qquad i = 1,\ldots, I,$$

which is the Identity of Hotelling and Wold, and

$$(6.11)\quad \lambda = \sum_i x_i u_i.$$

In an equilibrium λ equals μ.

We next turn to the description of consumers as profit maximizing agents. Just like firms, consumers maximize profits in two stages. In the first stage, each consumer minimizes the cost of "producing" a given but arbitrary level of utility U according to the utility function U = u(.). This yields the *cost* or *expenditure function*,

$$(6.12)\quad Y = c(p_1,\ldots, p_I, U) = \min_{x_i} \left(\sum_j p_j x_j \mid u(x_1,\ldots, x_I) \geq U\right).$$

Solving the equation c(.) = 1 for U yields the indirect utility function v(.).

In order to define the profit of a consumer we introduce the price of utility, p_U. The consumer's profit then is

$$(6.13)\quad \Pi := p_U U - c(p_1,\ldots, p_I, U).$$

The first order condition for a profit maximum is

$$(6.14)\quad p_U = \partial c/\partial U =: c_U(.),$$

because p_U is taken as given by the consumer. In equilibrium, the price of utility equals the marginal cost of producing utility.

By putting the price of utility equal to the reciprocal of the marginal utility of income we can guarantee that the solution to the problem of profit maximization coincides with the solution to the traditional (formulation of the) problem.

The dual of the expenditure function is the *distance function*, defined by

$$(6.15)\quad d(x_1,\ldots, x_I, U) := \min_{p_i} \left(\sum_j p_j x_j \mid v(p_1,\ldots, p_I) \leq U\right).$$

Solving the equation d(.) = 1 yields the direct utility function U = u(.). There holds

(6.16) $c_U = -d_U$.

A well-known result is Shephard's Lemma, which states that

(6.17) $x_i = \partial c/\partial p_i =: c_i(p_1, \ldots, p_I, U)$.

The functions $c_i(.)$ are the compensated demand functions. The dual result is the Shephard-Hanoch Lemma, stating that

(6.18) $p_i = \partial d/\partial x_i =: d_i(x_1, \ldots, x_I, U)$.

The functions $d_i(.)$ are the compensated inverse demand functions. In Section 3.3, where we described the theory of monopolistic competition, we made use of the concept of compensated (inverse) demand functions.

The results (6.16-18) are useful in converting the "price version" of the problem of joint profit maximization into the "quantity version", and *vice versa*.

6.3 Joint profit maximization

In this section, by "joint profit" we mean the sum of the profit of consumers and producers. For the sake of ease we consider the case that there is but one consumer.

The expenditures of the consumer equal the revenues of the firms. The joint profit thus is

(6.19) $\Pi := p_U U - \sum_i \gamma_i(x_i)$.

In the case of price taking behaviour by firms (and the consumer, of course), we may treat p_U as a parameter of the problem. By substituting u(.) for U and taking partial derivatives with respect to x_i we obtain the first order conditions

$$(6.20)\quad MC_i = p_U . u_i, \qquad i = 1,\ldots, I;$$

by setting p_U equal to the reciprocal of the marginal utility of income we can guarantee that the consumer exhausts his budget. Obviously, these conditions for a cooperative equilibrium are equivalent to the conditions for a (partial) competitive equilibrium, which state that the marginal cost of producing good i equals the price of good i.

In the oligopoly case we must take the dependence of p_U on the decision variables of the agents into account. There are several equivalent formulations of the problem. One is

$$(6.21)\quad \underset{x_i,\, U}{\text{maximize}} \quad -d_U(x_1,\ldots, x_I, U).U - \sum_i \gamma_i(x_i)$$

$$\text{subject to } d(x_1,\ldots, x_I, U) = 1.$$

This is the pure "quantity version" as it involves only quantities as decision variables. (Of course, we may use the constraint to eliminate U from the objective function; this amounts to replacing $-d_U$ by $1/\sum_i x_i u_i$ and U by u(.).) From the first order conditions we derive

$$(6.22)\quad MC_i = d_i(1 - d_{iU}.U/d_i + d_{UU}.U/d_U), \qquad i = 1,\ldots, I,$$

where d_{UU} is the derivative of d_U with respect to U and d_{iU} the derivative of d_U with respect to x_i. In equilibrium, the marginal cost of producing good i equals the marginal revenue to the consumer of good i. Under competitive circumstances the latter equals the price of good i, d_i. In the case of oligopoly account must be taken of the effect of the marginal change in utility on the price of the good and on the price of utility. The former effect is measured by minus the utility elasticity of the price offered for good i and the latter effect by the utility elasticity of the marginal cost of utility, both multiplied by the price of the good.

Another formulation is

$$(6.23)\quad \underset{p_i,\ U}{\text{maximize}}\ c_U(p_1,\ldots, p_I, U).U - \sum_i \gamma_i[c_i(p_1,\ldots, p_I, U)]$$

$$\text{subject to } c(p_1,\ldots, p_I, U) = 1,$$

which is an example of a "mixed version" of the problem. From the first order conditions we now derive

$$(6.24)\quad \sum_j MC_j(c_{ji} - c_{jU}.c_i/c_U) = -c_i(1 - c_{iU}.U/c_i + c_{UU}.U/c_U),$$

$$i = 1,\ldots, I,$$

in obvious notation. These conditions express the equality of the marginal cost for firms to the marginal revenue for the consumer of a marginal change in the price of good i. Under competitive circumstances the contribution of an infinitesimal change in the price of good i to the consumer's revenue equals minus the quantity of the good, c_i. In the case of oligopoly account must be taken of the effect of the marginal change in utility on the quantity of the good and on the price of utility. The former effect is measured by minus the utility elasticity of the quantity demanded of good i and the latter by the utility elasticity of the marginal cost of utility, both multiplied by minus the quantity of the good. The marginal cost for firms of a small change in the price of good i is a weighted sum of the marginal cost of producing the goods. It is easily verified that the weights are the derivatives with respect to p_i of the consumer's uncompensated demand functions.

Using the constraint to eliminate U from the objective function (6.23) we obtain the unconstrained problem

$$(6.25)\quad \underset{p_i}{\text{maximize}}\quad - v(p_1,\ldots, p_I)/\sum_i p_i v_i(p_1,\ldots, p_I)$$
$$- \sum_i \gamma_i[g_i(p_1,\ldots, p_I)].$$

We refrain from presenting the first order conditions for this problem.

It is clear from the first order conditions (6.22) and (6.24) that the equilibrium depends on cardinal properties of the consumer's utility function. We conclude this section by giving two examples, each with one and the same preference ordering but with a different cardinal representation. In both examples there is a quantity aggregator x that is a linearly homogeneous function of the quantities of the individual goods. Dual to x there is a price aggregator p that is a linearly homogeneous function of the prices of the individual goods. The product p.x equals the consumer's budget.

Example 1

Let the consumer's utility function be

$$(6.26)\quad U = x^{\nu}.$$

The distance function then is

$$(6.27)\quad d(x_1,\ldots, x_I, U) = x.U^{-1/\nu}.$$

We find that $d_U = -d/\nu U$, $d_{UU} = (1 + \nu)d/\nu^2U^2$, and $d_{iU} = -d_i/\nu U$. By substituting these results into (6.22) we obtain

$$(6.28)\quad MC_i = d_i[1 - (1 + \nu)/\nu + 1/\nu] = 0, \qquad i = 1,\ldots, I.$$

The explanation for this result is the following. If the utility function is homogeneous, a proportionate change in all quantities can be compensated by an equal proportionate change of all prices in the opposite direction so as to leave the consumer's revenue unchanged: along each ray in quantity (price) space through the origin the consumer's revenue is constant. Therefore the joint profit is maximal if total production costs are minimal.

Example 2

Let the consumer's utility function be

$$(6.29) \quad U = \ln(x).$$

The distance function then is

$$(6.30) \quad d(x_1, \ldots, x_I, U) = x.e^{-U}.$$

We find that $d_U = -d$, $d_{UU} = d$, $d_{iU} = -d_i$. By substituting these results into (6.22) we obtain

$$(6.31) \quad MC_i = d_i, \qquad i = 1, \ldots, I.$$

This is equal to the competitive outcome.

These examples show that there is a wide range of equilibria for a given preference ordering, corresponding to different cardinal representations.

7 Conclusion

We have reconsidered the oligopoly problem in one of its most elementary forms: static, with perfect information and no uncertainty, and with differentiated products. There are several equivalent ways to represent the demand side: by means of ordinary, inverse, or mixed demand functions. They enable firms to travel from quantity space to price space or any "mixture" space, and *vice versa*. Hence the choice of instrument is essentially arbitrary. A minimal requirement of rationality on the part of firms is that the solution does not depend on the arbitrary choice of instrument. This simple observation, that surprisingly has not been made before, has a far reaching implication: it discards the Cournot equilibrium and the Bertrand equilibrium and, more generally, any Nash noncooperative equilibrium as a valid solution concept based on rational behaviour. This is the first, negative result of our study.

The requirement that the solution be invariant to the arbitrary choice of instrument, price or quantity, by firms suggests that the solution belongs to the class of conjectural variations oligopolies. Conjectural variations have long confused students of the oligopoly problem. We have given their correct mathematical interpretation and shown that the solution to all conjectural variations oligopolies is the contract curve (or rather, hypersurface). This is the second, positive result of our study.

The contract curve is much too weak a solution concept to be of any use. A special point on the contract curve is the point where the joint pay-off is maximal. We have suggested a rationale for choosing this point as *the* solution by noting an analogy between game theory and the theory of gravity. A player's pay-off function may be regarded his potential function. The gradient of this function in some direction represents the (economic) force exerted by the player in that direction. An equilibrium is a point where the forces exerted by all players exactly cancel. Such is the case at the joint pay-off maximizing point. This is the third, suggestive result of our study.

The proof of the pudding is in the eating. Application of the idea of joint pay-off maximization to the oligopoly problem requires that the pay-off to consumers and producers be expressed in the same units. It appears convenient to describe consumers as profit maximizing agents. We have defined the revenues to a consumer as the product of the quantity and price of utility; the price of utility is just the reciprocal of the marginal utility of the consumer's income. This way of formulating the problem has the attractive feature that the (partial) competitive equilibrium results as the solution to the problem of joint profit maximization in the special case that firms take prices as given. In general, however, the solution depends on cardinal properties of the consumers' utility functions.

Does the introduction into economic theory of a formal concept of economic force have other benefits besides resolving the indeterminacy of the solution to the oligopoly problem? It might be helpfull in formalizing a theory of economical dynamics. This is a suggested field for future research.

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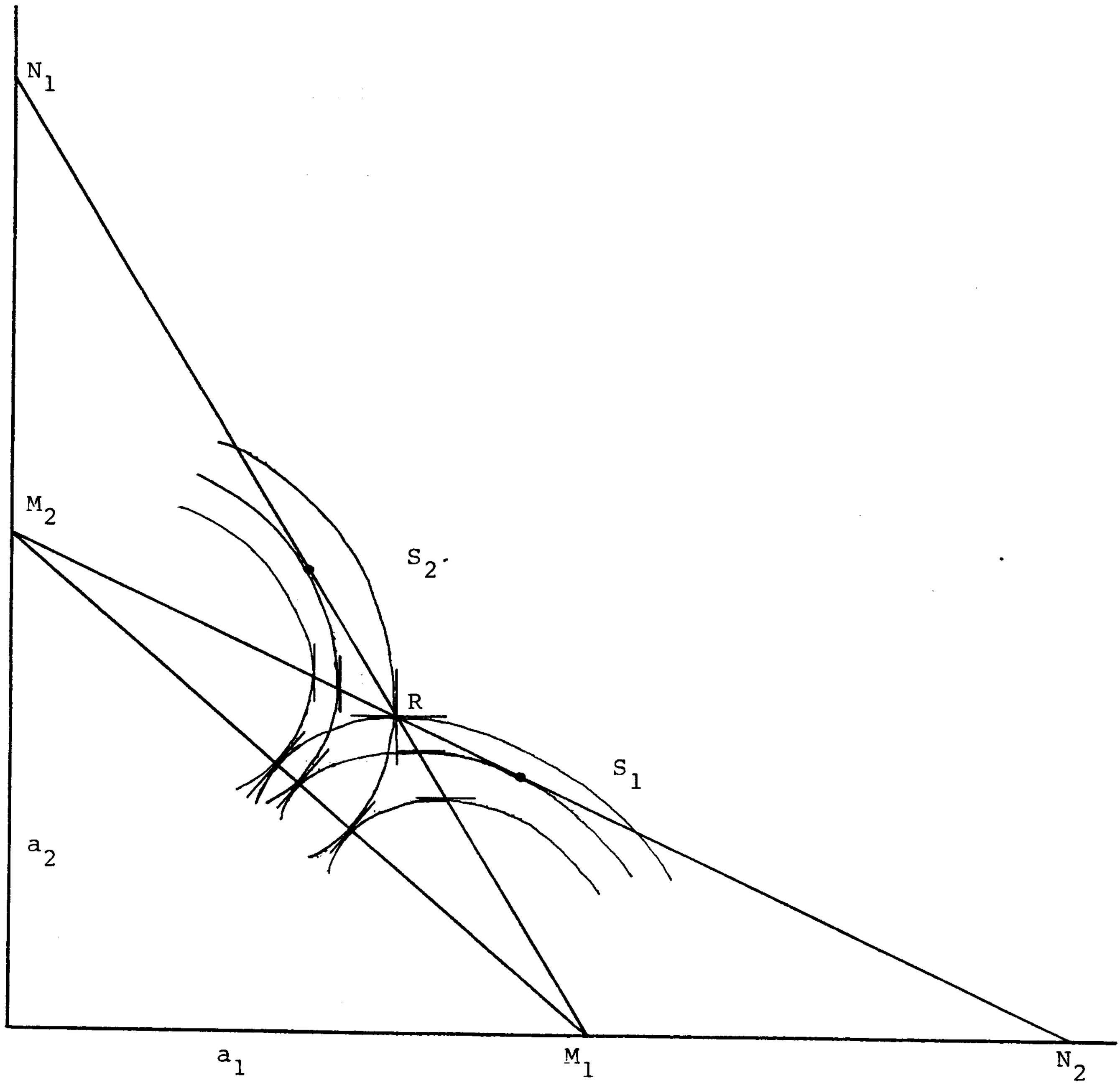
FIGURE 1



M_iN_i Reaction curve of player i (defined by $w_{ii} = 0$, i = 1, 2)
M_1M_2 Contract curve ($\alpha_{12}\alpha_{21} = 1$)
R Nash noncoooperative equilibrium
S_i Stackelberg equilibrium with player i as leader