RIEMANN ZETA FUNCTION

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Abstract. In this article, known complex analysis techniques will be used to prove by reduction to absurdity that the zeta function must admit at least one non-trivial zero outside the critical line.

The central idea of the article is to assume the Riemann hypothesis as true and obtain a accretion for the function $\frac{\xi(s)'}{\xi(s)(s-1)^4}$ in the region $Re(s) > 0$ using Hadamard's factorization theorem and known facts about the zeta function.

Using Cauchy's theorem for a closed rectangular path with side 2N and center (a+N,0) and the aforementioned accretion, it is possible to prove that $\int_{-N}^{N} \frac{\xi(a+it)'}{\xi(a+it)(a-1+it)^4} dt \approx 0,1854 + \frac{2}{3\sqrt{2}N^3 \left|(a-1)\left(a-\frac{1}{2}\right)\right|}$ for every $\frac{1}{2} < a < 1$ and $N \in \mathbb{R}$. However, from other techniques, it appears that such a result cannot be true, thus showing the absurdity.

Keywords: Riemann hypothesis, Critical line, Zeta function, Non-trivial zeros

1. Introduction

When Bernhard Riemann was invited in 1859 to become a corresponding member of the Berlin Academy, as was customary on such occasions, he submitted an article with the research he was developing. Riemann's famous article in 1859 was entitled *"On the number of primes less than a given quantity"*, in which Riemann showed that the non-trivial zeros of the function are intrinsically connected with the prime numbers, which is one of the most important results in mathematics.

In the same article, Riemann formulated the conjecture that became known as the Riemann hypothesis and that proved to be a difficult problem to be solved. One that is extremely important and is related to several mathematical results, including the best estimate of the error in the Theorem of Prime Numbers .

The Riemann Hypothesis is the only problem that simultaneously belongs to the list of Hilbert problems (biggest mathematical challenges of the century), the list of millennium problems (7 biggest mathematical challenges for this millennium) and Smale's problems (biggest mathematical challenges for the century), which contextualizes its eminent importance. Indeed, the Riemann Hypothesis is a celebrated conjecture within mathematics.

In what follows, we will enunciate general results related to complex analysis and some well-established theorems about the Riemann zeta function, later, we will assume the Riemann Hypothesis as true and use these results to arrive at a contradictory result, thus proving that the Riemann Hypothesis can't be true.

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2. General Theorems

Theorem 2.1 (Cauchy). *Let f be a holomorphic function defined on a simply connected set Ω, if γ is a closed path contained in Ω, then:*

$$\oint f = 0$$

Proof. [1] □

Theorem 2.2 (Cauchy). *Let f be a holomorphic function defined on a simply connected set Ω, if γ is a closed path contained in Ω, and s_0 is an interior point on the contour of the path γ, then:*

$$\oint \frac{f(s)}{(s-s_0)^n} = 2\pi i \frac{f^{(n-1)}(s_0)}{(n-1)!}$$

Proof. [1] □

Theorem 2.3 (Hadamard). *Suppose f is an integer function such as $|f(s)| < Ae^{B|s|^p}$ where p is the smallest real number in which the inequality is satisfied for every s. Let k be an integer such thatk $\leq p < k+1$ and $\{a_n\}_{n\in\mathbb{N}}$ the set of zeros of f, then:*

$$f(s) = s^m e^{P(s)} \prod E_k\left(\frac{s}{a_n}\right),\ E_k(s) = (1-s)\, e^{s+\frac{s^2}{2}+\frac{s^3}{3}+...+\frac{s^k}{k}}$$

where $P(s)$ is a polynomial of degree k and m the multiplicity of the root s=0.

Proof. [2] □

Theorem 2.4 (Cauchy). *Suppose f is an integer function such as $|f(s)| < Ae^{B|s|^p}$ and $\{a_n\}_{n\in\mathbb{N}}$ the set of zeros of f, then:*

$$\sum \frac{1}{|a_n|^s} < +\infty,\ if\ s > p$$

Proof. [3] □

3. Facts about the Zeta Function

Proposition 1. *The zeta function $\xi : \mathbb{C}/\{1\} \to \mathbb{C}$ is a holomorphic function with a simple pole at $s = 1$. Besides the trivial zeros $\{-2, -4, ..., -2n...\}$ ξ can only have possible roots in the critical range $H = \{s \in \mathbb{C}/0 < Re(s) < 1\}$.*

Proof. [1] □

Proposition 2. *The function $G(s) = \xi(s)(s-1)$ is holomorphic in the region $Re(s) > 0$, and the roots of G in this region are the same roots of ζ, in particular, G does not vanish in the region $Re(s) \geq 1$*

Proof. Trivial. □

Proposition 3. *Let $\xi(s)$ be the Riemann zeta function, for every $\varepsilon > 0$ exists A_ε e B_ε such that $|\xi(s)(s-1)| < A_\varepsilon e^{B_\varepsilon\ |s|^{1+\varepsilon}}$.*

Proof. [2] □

Corollary 1. *Let $\{a_n\}_{n\in\mathbb{N}}$ be the set of zeros in the zeta function, then:*

$$\sum \frac{1}{|a_n|^s} < +\infty,\ if\ s > 1$$

Proof. Immediate consequence of proposition 3. $\square$

Proposition 4. *If $\xi(s) = 0$, then $\xi(s*) = 0$.*

Proof. [3] $\square$

4. Proof

Hypothesis 1. *All non-trivial zeros of the zeta function $\xi : \mathbb{C}/1 \to \mathbb{C}$ are on the critical line $Re(s) = 1/2$.*

Proposition 5.

$$\lim_{|s|\to\infty} \frac{\xi(s)'}{\xi(s)(s-1)^4} = 0.$$

Proof. Let $G(s) = \xi(s)(s-1)$, as $G(s)$ is an integer function whose zeros are the same zeros from $\xi(s)$. By theorem 3 we have:

$$\xi(s)(s-1) = e^{As+B}\prod (1 - \frac{s}{a_n})e^{\frac{s}{a_n}}$$

$$\frac{\xi(s)'}{\xi(s)} = -s\sum \frac{1}{a_n(a_n - s)} + A - \frac{1}{s-1}$$

$$\frac{\xi(s)'}{\xi(s)(s-1)^4} = -\frac{s}{(s-1)^4}\sum \frac{1}{a_n(a_n - s)} + \frac{A}{(s-1)^4} - \frac{1}{(s-1)^5}$$

Where $a_n = -2k_n, k_n \in \mathbb{N}$ or $\frac{1}{2} + it_n, t_n \in \mathbb{R}$, according to hypothesis 1. $\square$

Note that:

$$|s-1|\,|s-a_n| \geq |a_n|\left|(Re(s) - 1)\left(Re(s) - \frac{1}{2}\right)\right|$$

In fact, be $s = a + it$ and $a_n = \frac{1}{2} + it_n$, then:

$$|s-1|^2\,|s-a_n|^2 = ((a-1)^2 + t^2)\left(\left(a - \frac{1}{2}\right)^2 + (t - t_n)^2\right) \geq (a-1)^2(t-t_n)^2 + \left(a - \frac{1}{2}\right)^2 t^2$$

Defining:

$$f(t) = (a-1)^2(t-t_n)^2 + \left(a - \frac{1}{2}\right)^2 t^2$$

As $f(t) \to \infty$ if $|t| \to \infty$, it follows that $f(t)$ admits a global minimum at the point where $f(t*)' = 0$.

$$f(t*)' = 0 \ \Rightarrow\ t* = \frac{(a-1)^2 t_n}{(a-1)^2 + \left(a - \frac{1}{2}\right)^2}$$

Thus:

$$f(t) \geq f(t*) = (a-1)^2 \left(a - \frac{1}{2}\right)^2 t_n^2$$

That is:

$$|s-1|\,|s-a_n| \geq |a_n| \left|(Re(s)-1)\left(Re(s)-\frac{1}{2}\right)\right|$$

If $s = a + it$ and $a_n = -2k_n$, then:

$$|s-1|^2\,|s-a_n|^2 = \left((a-1)^2 + t^2\right)\left((a+2k_n)^2 + (t-t_n)^2\right)$$

Applying the same procedure it is found that:

$$|s-1|\,|s-a_n| \geq |a_n|\,|(Re(s)-1)(Re(s)+2k_n)|$$

As $|a_n|\,|(Re(s)-1)(Re(s)+2k_n)| > |a_n| \left|(Re(s)-1)\left(Re(s)-\frac{1}{2}\right)\right|$, we have

$$|s-1|\,|s-a_n| \geq |a_n| \left|(Re(s)-1)\left(Re(s)-\frac{1}{2}\right)\right|$$

in every case.

Thus, we have:

$$\left|\sum \frac{1}{a_n\left(a_n-s\right)(s-1)}\right| \leq \sum\left|\frac{1}{a_n\left(a_n-s\right)(s-1)}\right| \leq \frac{1}{\left|\left(Re\left(s\right)-1\right)\left(Re\left(s\right)-\frac{1}{2}\right)\right|} \sum\left|\frac{1}{a_n^2}\right|$$

Therefore:

$$\left|\frac{\xi\left(s\right)'}{\xi(s)(s-1)^4}\right| \leq \left|\frac{s}{\left(s-1\right)^3}\right| \frac{1}{\left|\left(Re\left(s\right)-1\right)\left(Re\left(s\right)-\frac{1}{2}\right)\right|} \sum\left|\frac{1}{a_n^2}\right| + \left|\frac{A}{\left(s-1\right)^4}\right| + \left|\frac{1}{\left(s-1\right)^5}\right|$$

And:

$$\lim_{|s| \to \infty} \frac{\xi(s)'}{\xi(s)(s-1)^4} = 0.$$

Given that $\sum\left|\frac{1}{a_n^2}\right| < \infty$ by corollary 1.

Proposition 6.

$$\lim_{N \to \infty}\left|\int_{-N}^{N} \frac{\xi\left(a+it\right)'}{\xi\left(a+it\right)\left(a-1+it\right)^4} dt\right| < \infty, \ \frac{1}{2} < a < 1$$

Proof. Let $G\left(s\right) = \xi\left(s\right)\left(s-1\right)$,

$$\frac{G\left(s\right)'}{G\left(s\right)\left(s-1\right)^4} = \frac{\xi\left(s\right)'}{\xi\left(s\right)\left(s-1\right)^4} + \frac{1}{\left(s-1\right)^5}$$

Integrating the left side of the equation along a square path of side 2N and center $(a+N, 0)$, $\frac{1}{2} < a < 1$, we have, according to Theorem 2:

$$\oint \frac{G\left(s\right)'}{G\left(s\right)\left(s-1\right)^4} = \frac{2\pi i}{3} \frac{d^3}{d^3} \frac{G\left(s\right)'}{G\left(s\right)}, s = 1$$

□

Note that:

$$\oint \frac{G\left(s\right)'}{G\left(s\right)\left(s-1\right)^4} = \oint \frac{\xi\left(s\right)'}{\xi\left(s\right)\left(s-1\right)^4} + \frac{1}{\left(s-1\right)^5} = \oint \frac{\xi\left(s\right)'}{\xi\left(s\right)\left(s-1\right)^4}$$

as $\oint \frac{1}{(s-1)^5} = 0$.

If $N \to \infty$, according to proposition 5, the only remaining contribution of the closed square path of side 2N and center $(a+N,0)$ is:

$$\lim_{N\to\infty}\int_{-N}^{N}\frac{\xi\left(a+it\right)'}{\xi\left(a+it\right)\left(a-1+it\right)^{4}}idt=\oint\frac{\xi\left(s\right)'}{\xi\left(s\right)\left(s-1\right)^{4}}$$
$$=\oint\frac{G\left(s\right)'}{G\left(s\right)\left(s-1\right)^{4}}i=\frac{2\pi i}{3}\frac{d^{3}}{d^{3}}\frac{G\left(s\right)'}{G\left(s\right)},s=1$$

Corollary 2.

$$\lim_{N\to\infty}\int_{-N}^{N}\frac{\xi\left(a+it\right)'}{\xi\left(a+it\right)\left(a-1+it\right)^{4}}\ dt=\frac{2\pi}{3}\frac{d^{3}}{d^{3}}\frac{G\left(s\right)'}{G\left(s\right)},s=1\approx 0,1854$$

Proof. According to proposition 6:

$$\lim_{N\to\infty}\int_{-N}^{N}\frac{\xi\left(a+it\right)'}{\xi\left(a+it\right)\left(a-1+it\right)^{4}}idt=\oint\frac{\xi\left(s\right)'}{\xi\left(s\right)\left(s-1\right)^{4}}$$
$$=\oint\frac{G\left(s\right)'}{G\left(s\right)\left(s-1\right)^{4}}=\frac{2\pi i}{3}\frac{d^{3}}{d^{3}}\frac{G\left(s\right)'}{G\left(s\right)},s=1$$

And utilizing Wolfram Mathematica:

$$\frac{d^{3}}{d^{3}}\frac{G\left(s\right)'}{G\left(s\right)},s=1\ \approx 0,0851 \tag{1}$$

Thus:

$$\frac{2\pi}{3}\frac{d^{3}}{d^{3}}\frac{G\left(s\right)'}{G\left(s\right)},s=1\approx 0,1854$$

□

Proposition 7. *Let $\{a_n\}_{n\in N}$ be the set of zeros in the zeta function, then:*

$$\sum\frac{1}{a_n^2}\in\mathbb{R}$$

Proof. According to hypothesis 1:

$$\sum\frac{1}{a_n^2}=\sum\frac{1}{\left(2n\right)^2}+\sum\frac{1}{\left(\frac{1}{2}+it_n\right)^2}+\frac{1}{\left(\frac{1}{2}-it_n\right)^2}$$

$$\sum\frac{1}{\left(\frac{1}{2}+it_n\right)^2}+\frac{1}{\left(\frac{1}{2}-it_n\right)^2}=-\frac{1}{2}\sum\frac{\left(t_n^2-1\right)}{\left(\frac{1}{4}+t_n^2\right)^2}$$

And:

$$\sum\frac{1}{a_n^2}=\frac{1}{4}\sum\frac{1}{n^2}-\frac{1}{2}\sum\frac{\left(t_n^2-1\right)}{\left(\frac{1}{4}+t_n^2\right)^2}\in\mathbb{R}$$

□

Corollary 3.

$$\sum |\frac{1}{a_n^2}| = \frac{1}{2}\sum \frac{1}{n^2} - \sum \frac{1}{a_n^2}$$

Proof. Trivial. □

As:

$$\frac{\xi(s)'}{\xi(s)} = -s\sum \frac{1}{a_n\left(a_n - s\right)} + A - \frac{1}{s-1}$$

We have:

$$\frac{\xi(0)'}{\xi(0)} + 1 = A$$

Utilizing Wolfram Mathematica:

$$A = \ln\left(2\pi\right) - 1$$

and

$$\lim_{s\to 0}\frac{1}{s}\left(\frac{\xi(s)'}{\xi(s)} + \frac{1}{s-1} - \ln\left(2\pi\right) + 1\right) = -\sum \frac{1}{a_n^2} \approx -0,3650$$

According to corollaries 3 and 2, respectively:

$$\sum \left|\frac{1}{a_n^2}\right| = \frac{\pi^2}{12} - 0,3650 \approx 0,475 < \frac{1}{2}$$

$$\oint \frac{\xi(s)'}{\xi(s)(s-1)^4} = \int_{-\infty}^{\infty} \frac{\xi\left(a+it\right)'}{\xi\left(a+it\right)\left(a-1+it\right)^4}\,dt = 0,1854$$

While also utilizing proposition 5:

$$\frac{\xi\left(s\right)'}{\xi(s)(s-1)^4} = -\frac{s}{\left(s-1\right)^4}\sum \frac{1}{a_n\left(a_n - s\right)} + \frac{A}{\left(s-1\right)^4} - \frac{1}{\left(s-1\right)^5}$$

We have:

$$\int_N^{\infty}\left|\frac{\xi\left(a+it\right)'}{\xi\left(a+it\right)\left(a-1+it\right)^4}\right|dt \le \int_N^{\infty}\left|\frac{a+it}{\left(a+it-1\right)^4}\sum \frac{1}{a_n\left(a_n - a + it\right)}\right| +$$

$$+\left|\frac{A}{\left(a+it-1\right)^4}\right| + \left|\frac{1}{\left(a+it-1\right)^5}\right|dt$$

Which follows:

$$\leq \int_N^\infty \left|\frac{a+it}{(a+it-1)^3}\right| \left|\frac{1}{(a-1)\left(a-\frac{1}{2}\right)}\right| \sum \left|\frac{1}{a_n^2}\right| + \left|\frac{A}{(a+it-1)^4}\right| + \left|\frac{1}{(a+it-1)^5}\right| dt$$

$$< \int_N^\infty \left|\frac{a+it}{(a+it-1)^3}\right| \left|\frac{\frac{1}{2}}{(a-1)\left(a-\frac{1}{2}\right)}\right| + \left|\frac{\ln(2\pi)-1}{(a+it-1)^4}\right| + \left|\frac{1}{(a+it-1)^5}\right| dt$$

$$\leq \frac{1}{3\sqrt{2}N^3\left|(a-1)\left(a-\frac{1}{2}\right)\right|} + \frac{\ln(2\pi)-1}{N^5} + \frac{1}{N^6}$$

Likewise:

$$\int_{-\infty}^{-N} \left|\frac{\xi(a+it)'}{\xi(a+it)(a-1+it)^4}\right| dt \leq \frac{1}{3\sqrt{2}N^3\left|(a-1)\left(a-\frac{1}{2}\right)\right|} + \frac{\ln(2\pi)-1}{N^5} + \frac{1}{N^6}$$

Thus:

$$\left|\int_{-N}^{N} \frac{\xi(a+it)'}{\xi(a+it)(a-1+it)^4} dt - 0,1854\right| \leq \frac{2}{3\sqrt{2}N^3\left|(a-1)\left(a-\frac{1}{2}\right)\right|} + \frac{2\ln(2\pi)-2}{N^5} + \frac{2}{N^6}$$

For all $\frac{1}{2} < a < 1$, N.

Computing the Integral with $a = \frac{5}{6}$ and $N = 10$ in Wolfram Mathematica we have:

$$\left|\int_{-10}^{10} \frac{\xi\left(\frac{5}{6}+it\right)'}{\xi\left(\frac{5}{6}+it\right)\left(-\frac{1}{6}+it\right)^4} dt - 0,1854\right| \leq \frac{6}{1000\sqrt{2}} + 2\left(\frac{\ln(2\pi)-1}{10^5}\right) + \frac{2}{10^6}$$

However,

$$\int_{-10}^{10} \frac{\xi\left(\frac{5}{6}+it\right)'}{\xi\left(\frac{5}{6}+it\right)\left(-\frac{1}{6}+it\right)^4} dt = 3.7007478695173663$$

Thus:

$$\left|\int_{-10}^{10} \frac{\xi\left(\frac{5}{6}+it\right)'}{\xi\left(\frac{5}{6}+it\right)\left(-\frac{1}{6}+it\right)^4} dt - 0,1854\right| > 3$$

Which is an absurd.

5. Acknowledgements

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