Nonrelativistic Quantum Electrodynamics of extended spinning charges

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Abstract

A non-relativistic QED equation is developed for extended spinning charges in the Schrodinger picture. It is based on the corresponding classical electromagnetic - particle system, and the Hamiltonian associated with it.

1. Introduction

Most studies of non-relativistic QED consider a system of non-spinning point charges interacting with the electromagnetic field, for example see Healy[1]. Spin can be included by adding a Pauli spin term to the Hamiltonian, see for example Messiah[2]. Moniz and Sharp[3,4], Grotch and Kazes[5], and Roa-Neri and Jimenez[6] consider non-relativistic QED with extended charges, but do not include spin. They quantize the system in the Heisenberg picture, and look at the effect of the self-fields. Spohn[7] considers non-relativistic QED with extended charges and includes spin by introducing the Pauli spin matrices into the non-spinning QED equations. He using a method developed by Feynman and shown by Sakurai[8] which automatically yields the value g = 2. Our method develops an equation for any g factor starting with a Hamiltonian with the spin included and then quantizing the whole system.

Nodvik[9], Appel and Kiessling[10] and Spohn[7] look at relativistic rigid extended spinning charges, and derive equations of motion from a Lagrangian, but do not quantize the system.

Kiessling[11] looks at the conservation of total charge, energy, linear and angular momentum for the combined system of the electromagnetic field and a set of non-relativistic extended spinning charges, and shows that spin must be included for the conservation laws to hold.

In this paper we use a non-relativistic model of Lorentz electrodynamics, that is a set of classical non-relativistic extended spinning particles interacting with the electromagnetic field, and find the corresponding Lagrangian and Hamiltonian for the field and particle system. This system, including the spin, is then quantized by a canonical quantization in the Schrodinger picture. The charges are taken to be rigid and only interact by the electromagnetic field, that is their charge distributions are allowed to overlap.

Non-relativistic QED can be derived from relativistic QED, but only for point particles, and because they are point particles there are problems with infinities and thus renormalization. Using an extended particle with spin may lead to a better understanding of these infinities. While papers [3-6] look at the effect of the self-field, in this paper the self-field is included in the general formulation but is not separated out specifically.

2. Classical Equations

Consider a system of N rigid particles of charge q_i, charge density $\rho = \sum_{i=1}^{N} \rho_i$ and current density $\mathbf{j} = \sum_{i=1}^{n} \mathbf{j}_i$ with $\rho_i(\mathbf{r},t) = q_i f(|\mathbf{r}-\mathbf{r}_i(t)|)$ and $\mathbf{j}_i = \rho_i(\mathbf{r},t)(\mathbf{v}_i(t) + \boldsymbol{\omega}_i(t) \times (\mathbf{r}-\mathbf{r}_i(t)))$ where $\mathbf{r}_i(t)$ represents the position of particle i, $\mathbf{v}_i(t)$ its velocity, and $\boldsymbol{\omega}_i(t)$ its angular velocity. $f(r')$ is a smooth function concentrated around $r' = 0$ with $r' = |\mathbf{r}'|$, and $\mathbf{r}' = \mathbf{r} - \mathbf{r}_i(t)$. Bold face will be used to indicate a vector, and * indicates the complex conjugate.

Using the transverse gauge, $\nabla \bullet \mathbf{A} = 0$, Maxwell's equations take the form, for example see Jackson [12]

$$\nabla^2 \phi = -4\pi\rho \tag{1}$$

$$\nabla^2 \mathbf{A} - \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla \frac{\partial \phi}{\partial t} = -4\pi \mathbf{j} \tag{2}$$

where $\mathbf{A}$ is the vector potential, ϕ the scalar potential, and the speed of light is taken to be one.

Since $\nabla \bullet \mathbf{A} = 0$, we can expand the vector potential as a Fourier expansion

$$\mathbf{A} = (2L^3)^{-1/2} \sum_{\mathbf{k}} \sum_{n=1}^{2} a_{n\mathbf{k}}(t) \boldsymbol{\varepsilon}^n{}_{\mathbf{k}} \exp(i\mathbf{k} \bullet \mathbf{r}) \tag{3}$$

where the $\boldsymbol{\varepsilon}^n_{\mathbf{k}}$ are unit polarization vectors which obey the relations $\boldsymbol{\varepsilon}^n_{\mathbf{k}} \bullet \boldsymbol{\varepsilon}^{n'}_{\mathbf{k}} = \delta^{nn'}$, $\boldsymbol{\varepsilon}^n_{\mathbf{k}} \bullet \mathbf{k} = 0$ with n = 1, 2. To keep $\boldsymbol{\varepsilon}^1_{\mathbf{k}}$, $\boldsymbol{\varepsilon}^2_{\mathbf{k}}$ and $\mathbf{k}$ as a right handed basis, we set $\boldsymbol{\varepsilon}^n_{-\mathbf{k}} = (\delta_{n1} - \delta_{n2})\boldsymbol{\varepsilon}^n_{\mathbf{k}}$. $\sum_{\mathbf{k}}$ represents a sum over $\mathbf{k}$ space and the system is confined to a box of size L^3. We use $(2L^3)^{-1/2}$ instead of $L^{-3/2}$ so that, in terms of the degrees of freedom, the basis functions form an orthogonal set which are normalized to one. Using this relation for $\mathbf{A}$ in eq. (2), and taking the inner product with $\int dv \exp(-i\mathbf{k}'\bullet\mathbf{r})\boldsymbol{\varepsilon}^{n'}_{\mathbf{k}'}$, eq. (2) becomes

$$L^{3/2}(-k^2 a_{n'\mathbf{k}'} - \frac{d^2 a_{n'\mathbf{k}'}}{dt^2}) = \sqrt{2}\, \boldsymbol{\varepsilon}^{n'}_{\mathbf{k}'} \bullet \int dv \exp(-i\mathbf{k}'\bullet\mathbf{r})\{-4\pi\mathbf{j} + \frac{\partial\nabla\phi}{\partial t}\} \qquad (4)$$

using $\int dv \exp(i(\mathbf{k} - \mathbf{k}')\bullet\mathbf{r}) = L^3\delta_{\mathbf{k},\mathbf{k}'}$ and $\boldsymbol{\varepsilon}^n_{\mathbf{k}} \bullet \boldsymbol{\varepsilon}^{n'}_{\mathbf{k}} = \delta^{nn'}$ with $\int dv$ representing a volume integral over the box of size L^3. Note that

$$\boldsymbol{\varepsilon}^{n'}_{\mathbf{k}'}\bullet \int dv \exp(-i\mathbf{k}'\bullet\mathbf{r}) \frac{\partial\nabla\phi}{\partial t} = \int dv \exp(-i\mathbf{k}'\bullet\mathbf{r})\nabla\bullet\{\boldsymbol{\varepsilon}^{n'}_{\mathbf{k}'} \frac{\partial\phi}{\partial t}\}$$

$$= -i\int dv \frac{\partial\phi}{\partial t} \boldsymbol{\varepsilon}^{n'}_{\mathbf{k}'}\bullet\mathbf{k}' \exp(-i\mathbf{k}'\bullet\mathbf{r}) = 0 \qquad (5)$$

using a partial integration, and taking the scalar potential ϕ to be constant at spatial infinity and the fact that $\boldsymbol{\varepsilon}^n_{\mathbf{k}}\bullet\mathbf{k} = 0$. Thus eq. (4) becomes, dropping the prime on n' and $\mathbf{k}'$

$$L^{3/2}(k^2 a_{n\mathbf{k}} + \frac{d^2 a_{n\mathbf{k}}}{dt^2}) = 4\pi\sqrt{2}\int dv\, \exp(-i\mathbf{k}\bullet\mathbf{r})\boldsymbol{\varepsilon}^n_{\mathbf{k}}\bullet\mathbf{j} = 4\pi\sqrt{2}\int dv\, \exp(-i\mathbf{k}\bullet\mathbf{r})\boldsymbol{\varepsilon}^n_{\mathbf{k}}\bullet\sum_{i=1}^{N}\mathbf{j}_i$$

$$= 4\pi\sqrt{2}\sum_{i=1}^{N}\exp(-i\mathbf{k}\bullet\mathbf{r}_i)\int dv\, \exp(-i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_i))\boldsymbol{\varepsilon}^n_{\mathbf{k}}\bullet\mathbf{j}_i$$

$$= 4\pi\sqrt{2}\sum_{i=1}^{N}q_i\, \exp(-i\mathbf{k}\bullet\mathbf{r}_i)\int dv'\, \exp(-i\mathbf{k}\bullet\mathbf{r}')\boldsymbol{\varepsilon}^n_{\mathbf{k}}\bullet(\mathbf{v}_i + \boldsymbol{\omega}_i x\mathbf{r}')f(r') \qquad (6)$$

with $\mathbf{r}' = \mathbf{r} - \mathbf{r}_i$.

The particle equations of motion for particle i are taken to be

$$m\frac{d\mathbf{v}_i}{dt} = \int dv\, (\rho_i\mathbf{E} + \mathbf{j}_i x\mathbf{B}) = \int dv\, \rho_i\{\mathbf{E} + (\mathbf{v}_i + \boldsymbol{\omega}_i x(\mathbf{r} - \mathbf{r}_i))x\mathbf{B}\} \qquad (7)$$

$$I\frac{d\boldsymbol{\omega}_i}{dt} = \int dv\, (\mathbf{r} - \mathbf{r}_i)x(\rho_i\mathbf{E} + \mathbf{j}_i x\mathbf{B}) = \int dv\, \rho_i(\mathbf{r} - \mathbf{r}_i)x\{\mathbf{E} + (\mathbf{v}_i + \boldsymbol{\omega}_i x(\mathbf{r} - \mathbf{r}_i))x\mathbf{B}\} \qquad (8)$$

where the **E** and **B** fields are due to all the particles, including particle i, so the particle self field is included. In this way the radiation reaction is included in the equations of motion. m is the bare mass of particle i, and I is its mechanical moment of inertia.

We can pick a particular form for the mass and charge distribution, and will take them to be a solid sphere of constant mass density and a shell of charge. In this case the mechanical moment of inertia is $I = \frac{2}{5}mr_0^2$ where r_0 is the radius of the solid sphere, and $\rho_i(\mathbf{r} - \mathbf{r}_i) = \frac{q_i}{4\pi r_0^2}\delta(|\mathbf{r} - \mathbf{r}_i| - r_0)$.

Then using $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla x \mathbf{A}$, and a Fourier expansion of the vector potential,

the spatial equation (7) takes the form

$$m\frac{d\mathbf{v}_i}{dt} = \int dv\, \rho_i[-\nabla\phi + (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2}\{-\frac{da_{n\mathbf{k}}}{dt}\boldsymbol{\varepsilon}^n{}_{\mathbf{k}} + (\mathbf{v}_i + \boldsymbol{\omega}_i x(\mathbf{r} - \mathbf{r}_i))x(i\mathbf{k}x\boldsymbol{\varepsilon}^n{}_{\mathbf{k}})a_{n\mathbf{k}}\}\exp(i\mathbf{k}\bullet\mathbf{r})]$$

$$= -\int dv\, \rho_i\nabla\phi + (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2}\exp(i\mathbf{k}\bullet\mathbf{r}_i)\int dv'\, q_i f(r')\{-\frac{da_{n\mathbf{k}}}{dt}\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}$$

$$+ i\{(\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}\bullet(\mathbf{v}_i + \boldsymbol{\omega}_i x\mathbf{r}'))\mathbf{k} - (\mathbf{v}_i\bullet\mathbf{k})\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}\}a_{n\mathbf{k}}\}\exp(i\mathbf{k}\bullet\mathbf{r}') \qquad (9)$$

Since $\nabla^2\phi = -4\pi\rho$,

$$\phi(\mathbf{r}) = \int dv' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \int dv' \sum_{j=1}^{N} q_j \frac{f(|\mathbf{r}' - \mathbf{r}_j|)}{|\mathbf{r} - \mathbf{r}'|} \qquad (10)$$

so that

$$\int dv\, \rho_i\nabla\phi = \int dv\, q_i f(|\mathbf{r} - \mathbf{r}_i|)\nabla\{\int dv' \sum_{j=1}^{N} q_j \frac{f(|\mathbf{r}' - \mathbf{r}_j|)}{|\mathbf{r} - \mathbf{r}'|}\}$$

$$= \int dv''\, q_i f(r'')\nabla''\int dv''' \sum_{j\neq i} q_j \frac{f(r''')}{|\mathbf{r}_i - \mathbf{r}_j + \mathbf{r}'' - \mathbf{r}'''|}$$

$$= \int dv'' q_i f(r'')\nabla_i \int dv''' \sum_{j\neq i} q_j \frac{f(r''')}{|\mathbf{r}_i - \mathbf{r}_j + \mathbf{r}'' - \mathbf{r}'''|} \tag{11}$$

where $\mathbf{r}'' = \mathbf{r} - \mathbf{r}_i$, $\mathbf{r}''' = \mathbf{r}' - \mathbf{r}_j$, and the j = i term has been left out since it makes no contribution. ∇'' is the gradient with respect to $\mathbf{r}''$, and ∇_i is the gradient with respect to $\mathbf{r}_i$. Thus the spatial equation (9) takes the form

$$m\frac{d\mathbf{v}_i}{dt} = -\int dv'' q_i f(r'')\nabla_i \int dv''' \sum_{j\neq i} q_j \frac{f(r''')}{|\mathbf{r}_i - \mathbf{r}_j + \mathbf{r}'' - \mathbf{r}'''|}$$

$$+ (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2}\exp(i\mathbf{k}\bullet\mathbf{r}_i)\int dv' q_i f(r')\{-\frac{da_{n\mathbf{k}}}{dt}\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}$$

$$+ i\{(\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}\bullet(\mathbf{v}_i + \boldsymbol{\omega}_i x\mathbf{r}'))\mathbf{k} - (\mathbf{v}_i\bullet\mathbf{k})\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}\}a_{n\mathbf{k}}\}\exp(i\mathbf{k}\bullet\mathbf{r}') \tag{12}$$

Now use $\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla x\mathbf{A}$, and the Fourier expansion of the vector potential in the rotational equation (8) to obtain

$$I\frac{d\boldsymbol{\omega}_i}{dt} = (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2}\exp(i\mathbf{k}\bullet\mathbf{r}_i)\int dv' q_i f(r')\{-(\mathbf{r}' x\boldsymbol{\varepsilon}^n{}_{\mathbf{k}})(\frac{da_{n\mathbf{k}}}{dt} + i(\mathbf{v}_i\bullet\mathbf{k})a_{n\mathbf{k}})$$

$$+ i\{(\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}\bullet(\boldsymbol{\omega}_i x\mathbf{r}'))\mathbf{r}' x\mathbf{k} - (\mathbf{k}\bullet(\boldsymbol{\omega}_i x\mathbf{r}'))\mathbf{r}' x\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}\}a_{n\mathbf{k}}\}\exp(i\mathbf{k}\bullet\mathbf{r}') \tag{13}$$

using the relation $\mathbf{k}x\int dv' f(r')\mathbf{r}'\exp(i\mathbf{k}\bullet\mathbf{r}') = 0$ and $\int dv\, \rho_i(\mathbf{r} - \mathbf{r}_i)x\nabla\phi = 0$ for a spherically symmetric charge distribution.

The first condition $\mathbf{k} \mathrm{x} \int dv' f(r')\mathbf{r}' \exp(i\mathbf{k} \bullet \mathbf{r}') = 0$ can be seen to be true by choosing the coordinate system such that only k_x is non-zero, and noting that

$$\int dv' f(r') y' \exp(ikx') = \int dv' f(r') z' \exp(ikx') = 0 \quad (14)$$

by spherical symmetry. The second relation $\int dv\, \rho_i(\mathbf{r} - \mathbf{r}_i) \mathrm{x} \nabla\phi = 0$ then follows by making a Fourier expansion of ϕ, and using the first relation.

Now look at the degrees of freedom of the system. For the vector potential, we will use a method similar to that of Kroll[13]. Setting $a_{n\mathbf{k}} = a^R_{n\mathbf{k}} + i a^I_{n\mathbf{k}}$, take $a^R_{n\mathbf{k}}$ and $a^I_{n\mathbf{k}}$ as independent real variables only for $\mathbf{k} > 0$, that is for $\mathbf{k}$ vectors with $k_z > 0$. Then require that $a_{n-\mathbf{k}} = (\delta_{n1} - \delta_{n2}) a_{n\mathbf{k}}^*$ so as to satisfy the condition that the vector potential be real and that $\boldsymbol{\varepsilon}^n_{-\mathbf{k}} = (\delta_{n1} - \delta_{n2}) \boldsymbol{\varepsilon}^n_{\mathbf{k}}$.

For spin degrees of freedom we will use the Euler angles of each particle, ϕ_i , θ_i and ψ_i so, following Goldstein [14],

$$\boldsymbol{\omega}_i = \left(\cos(\phi_i)\frac{d\theta_i}{dt} + \sin(\phi_i)\sin(\theta_i)\frac{d\psi_i}{dt}\right)\mathbf{x} + \left(\sin(\phi_i)\frac{d\theta_i}{dt} - \cos(\phi_i)\sin(\theta_i)\frac{d\psi_i}{dt}\right)\mathbf{y}$$

$$+ \left(\frac{d\phi_i}{dt} + \cos(\theta_i)\frac{d\psi_i}{dt}\right)\mathbf{z} \quad (15)$$

where **x**, **y**, and **z** are the unit vectors in the x, y and z directions.

3. Variational Principles

For a Lagrangian of the combined system try

$$L = \frac{1}{2}\sum_{i=1}^{N} m\, v_i^2 + \frac{1}{2}\sum_{i=1}^{N} I\, \omega_i^2 - \frac{1}{2}\int dv \int dv' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \int dv\, \mathbf{j} \bullet \mathbf{A}$$

$$+ \frac{1}{8\pi}\int dv \left(\frac{\partial \mathbf{A}}{\partial t} \bullet \frac{\partial \mathbf{A}}{\partial t} - \mathbf{B}\bullet\mathbf{B}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{N} m\, v_i^2 + \frac{1}{2}\sum_{i=1}^{N} I\, \omega_i^2 - \frac{1}{2}\int dv \sum_{i=1}^{N} q_i f(|\mathbf{r} - \mathbf{r}_i|) \int dv' \sum_{j=1}^{N} q_j \frac{f(|\mathbf{r}' - \mathbf{r}_j|)}{|\mathbf{r} - \mathbf{r}'|}$$

$$+ (2L^3)^{-1/2} \sum_{\mathbf{k}} \sum_{n=1}^{2} a_{n\mathbf{k}} \int dv \sum_{i=1}^{N} q_i f(|\mathbf{r} - \mathbf{r}_i|)\, (\mathbf{v}_i + \boldsymbol{\omega}_i x(\mathbf{r} - \mathbf{r}_i)) \bullet \boldsymbol{\varepsilon}^n{}_{\mathbf{k}} \exp(i\mathbf{k}\bullet\mathbf{r})$$

$$+ \frac{1}{8\pi} \sum_{+\mathbf{k}} \sum_{n=1}^{2} \left\{ \frac{da_{n\mathbf{k}}}{dt} \frac{da_{n\mathbf{k}}}{dt}^{*} - k^2 a_{n\mathbf{k}} a_{n\mathbf{k}}^{*} \right\}$$

$$= \frac{1}{2}\sum_{i=1}^{N} m\, v_i^2 + \frac{1}{2}\sum_{i=1}^{N} I\, \omega_i^2 - \frac{1}{2}\int dv'' \sum_{i=1}^{N} q_i f(r'') \int dv''' \sum_{j=1}^{N} q_j \frac{f(r''')}{|\mathbf{r}_i - \mathbf{r}_j + \mathbf{r}'' - \mathbf{r}'''|}$$

$$+ (2L^3)^{-1/2} \sum_{\mathbf{k}} \sum_{n=1}^{2} a_{n\mathbf{k}} \sum_{i=1}^{N} q_i\, e^{i\mathbf{k}\bullet\mathbf{r}_i} \int dv' f(r')\, (\mathbf{v}_i + \boldsymbol{\omega}_i x\mathbf{r}') \bullet \boldsymbol{\varepsilon}^n{}_{\mathbf{k}} \exp(i\mathbf{k}\bullet\mathbf{r}')$$

$$+ \frac{1}{8\pi} \sum_{+\mathbf{k}} \sum_{n=1}^{2} \left\{ \frac{da_{n\mathbf{k}}}{dt} \frac{da_{n\mathbf{k}}}{dt}^{*} - k^2 a_{n\mathbf{k}} a_{n\mathbf{k}}^{*} \right\} \tag{16}$$

where $\sum_{+\mathbf{k}}$ represents the sum over the **k** vectors for $k_z > 0$, and $k = |\mathbf{k}|$. Richoz[15] writes down a similar Lagrangian but for a single charge and then derives equations of motion from it.

Variation of L with respect to the spatial particle coordinates $\mathbf{r}_i$ yields the equations

$$m\frac{d\mathbf{v}_i}{dt} + (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2} q_i \exp(i\mathbf{k}\bullet\mathbf{r}_i)(i\mathbf{k}\bullet\mathbf{v}_i a_{n\mathbf{k}} + \frac{da_{n\mathbf{k}}}{dt})\int dv' f(r')\, \boldsymbol{\varepsilon}^n_{\mathbf{k}} \exp(i\mathbf{k}\bullet\mathbf{r}')$$

$$= -\frac{1}{2}\int dv''\sum_{j=1}^{N} q_j f(r'')\int dv'''\sum_{k=1}^{N} q_k f(r''')(\delta_{ik} + \delta_{ij})\,\nabla_i(|\mathbf{r}_j - \mathbf{r}_k + \mathbf{r}'' - \mathbf{r}'''|^{-1})$$

$$+ (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2} a_{n\mathbf{k}}\, q_i \exp(i\mathbf{k}\bullet\mathbf{r}_i) i\mathbf{k}\int dv' f(r')\,(\mathbf{v}_i + \boldsymbol{\omega}_i x \mathbf{r}')\bullet\boldsymbol{\varepsilon}^n_{\mathbf{k}} \exp(i\mathbf{k}\bullet\mathbf{r}')$$

$$= -\int dv'' q_i f(r'')\int dv'''\sum_{j=1}^{N} q_j f(r''')\,\nabla_i(|\mathbf{r}_i - \mathbf{r}_j + \mathbf{r}'' - \mathbf{r}'''|^{-1})$$

$$+ (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2} a_{n\mathbf{k}}\, q_i \exp(i\mathbf{k}\bullet\mathbf{r}_i) i\mathbf{k}\int dv' f(r')\,(\mathbf{v}_i + \boldsymbol{\omega}_i x \mathbf{r}')\bullet\boldsymbol{\varepsilon}^n_{\mathbf{k}} \exp(i\mathbf{k}\bullet\mathbf{r}') \qquad (17)$$

switching the j and k indices, and the $\mathbf{r}''$ and $\mathbf{r}'''$ integrations. This agrees with eq. (12).

Variation of L with respect to the individual particle Euler angles yields the equations $\frac{d(\nabla_{\omega_i} L)}{dt} = \boldsymbol{\omega}_i \, x \, \nabla_{\omega_i} L$ as can be shown by combining the Euler-Lagrange equations for the individual Euler angles, ϕ_i , θ_i and ψ_i as long as $\sin(\theta_i)$ is not zero. ∇_{ω_i} is the gradient with respect to $\boldsymbol{\omega}_i$. This yields the relations

$$I\frac{d\boldsymbol{\omega}_i}{dt} - (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2} q_i \exp(i\mathbf{k}\bullet\mathbf{r}_i)(\frac{da_{n\mathbf{k}}}{dt} + i(\mathbf{v}_i\bullet\mathbf{k})a_{n\mathbf{k}})\int dv' f(r')(\boldsymbol{\varepsilon}^n_{\mathbf{k}} x \mathbf{r}') \exp(i\mathbf{k}\bullet\mathbf{r}')$$

$$= -(2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2} a_{n\mathbf{k}} q_i \exp(i\mathbf{k}\bullet\mathbf{r}_i)\int dv' f(r')\boldsymbol{\omega}_i x(\boldsymbol{\varepsilon}^n{}_{\mathbf{k}} x\mathbf{r}')\exp(i\mathbf{k}\bullet\mathbf{r}') \qquad (18)$$

which will equal eq. (13) if

$$\int dv' f(r')\, \boldsymbol{\omega}_i x(\boldsymbol{\varepsilon}^n{}_{\mathbf{k}} x\mathbf{r}')\exp(i\mathbf{k}\bullet\mathbf{r}')$$

$$= -i\int dv' f(r')\, \{(\boldsymbol{\varepsilon}^n{}_{\mathbf{k}} \bullet(\boldsymbol{\omega}_i x\mathbf{r}'))\mathbf{r}' x\mathbf{k} - (\mathbf{k}\bullet(\boldsymbol{\omega}_i x\mathbf{r}'))\mathbf{r}' x\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}\}\exp(i\mathbf{k}\bullet\mathbf{r}') \qquad (19)$$

This can be shown to be true by choosing the coordinate system such that **k** is only in the x direction, and noting that by spherical symmetry,

$\int dv' f(r')\, x' \exp(ikx') = ik\int dv' f(r')\, y'^2 \exp(ikx')$.

Variation of L with respect to the Fourier components of the vector potential, that is the real and imaginary parts of $a_{n\mathbf{k}}$ for $\mathbf{k} > 0$, yields eq. (6).

The conjugate momentum take the form

$$\mathbf{p}^i = \nabla_{\mathbf{v}_i} L = m\mathbf{v}_i + (2L^3)^{-1/2}\sum_{\mathbf{k}}\sum_{n=1}^{2} a_{n\mathbf{k}} q_i \exp(i\mathbf{k}\bullet\mathbf{r}_i)\int dv' f(r')\, \boldsymbol{\varepsilon}^n{}_{\mathbf{k}}\exp(i\mathbf{k}\bullet\mathbf{r}')$$

$$= m\mathbf{v}_i + \int dv\, \rho_i \mathbf{A} \qquad (20)$$

$$p^R{}_{n\mathbf{k}} = \frac{\partial L}{\partial \frac{da^R{}_{n\mathbf{k}}}{dt}} = \frac{1}{4\pi}\frac{da^R{}_{n\mathbf{k}}}{dt} \qquad (21a)$$

$$p^I_{n\mathbf{k}} = \frac{\partial L}{\partial \frac{da^I_{n\mathbf{k}}}{dt}} = \frac{1}{4\pi}\frac{da^I_{n\mathbf{k}}}{dt} \tag{21b}$$

where $\nabla_{\mathbf{v}_i}$ is the gradient with respect to $\mathbf{v}_i$. Using the notation

$$\mathbf{p}_{\boldsymbol{\omega}_i} = \nabla_{\omega_i} L = I\boldsymbol{\omega}_i + \int dv\, \rho_i(\mathbf{r} - \mathbf{r}_i)x\mathbf{A} \tag{22}$$

we have

$$p_{\phi_i} = \frac{\partial L}{\partial \frac{d\phi_i}{dt}} = \mathbf{p}_{\boldsymbol{\omega}_i} \bullet \frac{\partial \boldsymbol{\omega}_i}{\partial \frac{d\phi_i}{dt}} \tag{23a}$$

$$p_{\theta_i} = \frac{\partial L}{\partial \frac{d\theta_i}{dt}} = \mathbf{p}_{\boldsymbol{\omega}_i} \bullet \frac{\partial \boldsymbol{\omega}_i}{\partial \frac{d\theta_i}{dt}} \tag{23b}$$

$$p_{\psi_i} = \frac{\partial L}{\partial \frac{d\psi_i}{dt}} = \mathbf{p}_{\boldsymbol{\omega}_i} \bullet \frac{\partial \boldsymbol{\omega}_i}{\partial \frac{d\psi_i}{dt}} \tag{23c}$$

The Hamiltonian is then

$$H = \sum_{i=1}^{N} \mathbf{v}_i \bullet \mathbf{p}^i + \sum_{i=1}^{N} (p_{\phi_i} \frac{d\phi_i}{dt} + p_{\theta_i} \frac{d\theta_i}{dt} + p_{\psi_i} \frac{d\psi_i}{dt}) + \sum_{+\mathbf{k}} \sum_{n=1}^{2} (p^R{}_{n\mathbf{k}} \frac{da^R{}_{n\mathbf{k}}}{dt} + p^I{}_{n\mathbf{k}} \frac{da^I{}_{n\mathbf{k}}}{dt}) - L$$

$$= \sum_{i=1}^{N} \mathbf{v}_i \bullet \mathbf{p}^i + \sum_{i=1}^{N} \mathbf{p}_{\boldsymbol{\omega}_i} \bullet \boldsymbol{\omega}_i + \sum_{+\mathbf{k}} \sum_{n=1}^{2} (p^R{}_{n\mathbf{k}} \frac{da^R{}_{n\mathbf{k}}}{dt} + p^I{}_{n\mathbf{k}} \frac{da^I{}_{n\mathbf{k}}}{dt}) - L \qquad (24)$$

using the relation $p_{\phi_i} \frac{d\phi_i}{dt} + p_{\theta_i} \frac{d\theta_i}{dt} + p_{\psi_i} \frac{d\psi_i}{dt} = \mathbf{p}_{\boldsymbol{\omega}_i} \bullet \boldsymbol{\omega}_i$.

Using the Lagrangian and conjugate momentum, we then have

$$H = \frac{1}{2} \sum_{i=1}^{N} \{ m v_i^2 + I\omega_i^2 \} + \frac{1}{2} \int dv \int dv' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$+ \frac{1}{8\pi} \sum_{+\mathbf{k}} \sum_{n=1}^{2} \{ \frac{da_{n\mathbf{k}}}{dt} \frac{da_{n\mathbf{k}}{}^*}{dt} + k^2 a_{n\mathbf{k}} a_{n\mathbf{k}}{}^* \} \qquad (25)$$

which in terms of the conjugate momentum becomes

$$H = \sum_{i=1}^{N} \{ \frac{1}{2m} (\mathbf{p}^i - \int dv\, \rho_i \mathbf{A})^2 + \frac{1}{2I} (\mathbf{p}_{\boldsymbol{\omega}_i} - \int dv\, \rho_i (\mathbf{r} - \mathbf{r}_i) x \mathbf{A})^2 \}$$

$$+ \frac{1}{2} \int dv \int dv' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{8\pi} \sum_{+\mathbf{k}} \sum_{n=1}^{2} \{ (4\pi)^2 p_{n\mathbf{k}} p_{n\mathbf{k}}{}^* + k^2 a_{n\mathbf{k}} a_{n\mathbf{k}}{}^* \} \qquad (26)$$

using $p_{n\mathbf{k}} = p^R{}_{n\mathbf{k}} + i p^I{}_{n\mathbf{k}}$.

4. Quantization

A canonical quantization of the system is carried out in the Schrodinger picture by replacing $\mathbf{p}^i$ by $-i\hbar\nabla_i$, $p_{n\mathbf{k}}$ by $-i\hbar\left(\frac{\partial}{\partial a^R_{n\mathbf{k}}} + i\frac{\partial}{\partial a^I_{n\mathbf{k}}}\right)$, and, following Bopp and Haag[16], $\mathbf{p}_{\omega_i}$ by $-i\hbar\hat{\mathbf{D}}_i$ where

$$\hat{\mathbf{D}}_i = \mathbf{x}\{\cos(\phi_i)\frac{\partial}{\partial\theta_i} + \frac{\sin(\phi_i)}{\sin(\theta_i)}(\frac{\partial}{\partial\psi_i} - \cos(\theta_i)\frac{\partial}{\partial\phi_i})\}$$

$$+ \mathbf{y}\{\sin(\phi_i)\frac{\partial}{\partial\theta_i} - \frac{\cos(\phi_i)}{\sin(\theta_i)}(\frac{\partial}{\partial\psi_i} - \cos(\theta_i)\frac{\partial}{\partial\phi_i})\} + \mathbf{z}\frac{\partial}{\partial\phi_i} \qquad (27)$$

and a ^ indicates an operator. Using eq. (26), the corresponding Schrodinger equation takes the form

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi = [\sum_{i=1}^{N}\{\frac{1}{2m}(-i\hbar\nabla_i - \int dv\,\rho_i\mathbf{A})^2 + \frac{1}{2I}(-i\hbar\hat{\mathbf{D}}_i - \int dv\,\rho_i(\mathbf{r} - \mathbf{r}_i)x\mathbf{A})^2\}$$

$$+ \frac{1}{2}\int dv\int dv'\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{8\pi}\sum_{+\mathbf{k}}\sum_{n=1}^{2}\{-(4\pi\hbar)^2(\frac{\partial^2}{\partial a^{R\,2}_{n\mathbf{k}}} + \frac{\partial^2}{\partial a^{I\,2}_{n\mathbf{k}}}) + k^2 a_{n\mathbf{k}}{}^* a_{n\mathbf{k}}\}]\psi \qquad (28)$$

where $\psi = \psi(\mathbf{r}_i,\phi_i,\theta_i,\psi_i,a^R_{n\mathbf{k}},a^I_{n\mathbf{k}},t)$. Now define the following creation and annihilation operators for $k_z > 0$

$$\hat{b}^R_{n\mathbf{k}}{}^* = \frac{1}{\sqrt{2}}\{\sqrt{\frac{k}{2h}}\,a^R_{n\mathbf{k}} - \sqrt{\frac{2h}{k}}\,\frac{\partial}{\partial a^R_{n\mathbf{k}}}\} \qquad (29a)$$

$$\hat{b}^R_{n\mathbf{k}} = \frac{1}{\sqrt{2}} \{ \sqrt{\frac{k}{2h}}\, a^R_{n\mathbf{k}} + \sqrt{\frac{2h}{k}}\, \frac{\partial}{\partial a^R_{n\mathbf{k}}} \} \qquad (29b)$$

$$\hat{b}^I_{n\mathbf{k}}{}^* = \frac{1}{\sqrt{2}} \{ \sqrt{\frac{k}{2h}}\, a^I_{n\mathbf{k}} - \sqrt{\frac{2h}{k}}\, \frac{\partial}{\partial a^I_{n\mathbf{k}}} \} \qquad (29c)$$

$$\hat{b}^I_{n\mathbf{k}} = \frac{1}{\sqrt{2}} \{ \sqrt{\frac{k}{2h}}\, a^I_{n\mathbf{k}} + \sqrt{\frac{2h}{k}}\, \frac{\partial}{\partial a^I_{n\mathbf{k}}} \} \qquad (29e)$$

using $h = 2\pi\hbar$. To extend these operators over all of **k** space, introduce the new operators

$$\hat{b}_{n\mathbf{k}}{}^* = \frac{1}{\sqrt{2}} (\hat{b}^R_{n\mathbf{k}}{}^* - i\, \hat{b}^I_{n\mathbf{k}}{}^*) \text{ for } k_z > 0$$

$$= \frac{1}{\sqrt{2}} (\hat{b}^R_{n\text{-}\mathbf{k}}{}^* + i\, \hat{b}^I_{n\text{-}\mathbf{k}}{}^*)(\delta_{n,1} - \delta_{n,2}) \text{ for } k_z < 0 \qquad (30a)$$

$$\hat{b}_{n\mathbf{k}} = \frac{1}{\sqrt{2}} (\hat{b}^R_{n\mathbf{k}} + i\, \hat{b}^I_{n\mathbf{k}}) \text{ for } k_z > 0$$

$$= \frac{1}{\sqrt{2}} (\hat{b}^R_{n\text{-}\mathbf{k}} - i\, \hat{b}^I_{n\text{-}\mathbf{k}})(\delta_{n,1} - \delta_{n,2}) \text{ for } k_z < 0 \qquad (30b)$$

Kroll[13] defines the same type of operators, but leaves out the $(\delta_{n,1} - \delta_{n,2})$ terms. These terms are needed because $\boldsymbol{\varepsilon}^n_{-\mathbf{k}} = (\delta_{n,1} - \delta_{n,2})\boldsymbol{\varepsilon}^n_{\mathbf{k}}$.

In terms of these operators we have

$$\mathbf{A} = (2L^3)^{-1/2}\sum_{+\mathbf{k}}\sum_{n=1}^{2}(a_{n\mathbf{k}}\exp(i\mathbf{k}\bullet\mathbf{r}) + a_{n\mathbf{k}}^*\exp(-i\mathbf{k}\bullet\mathbf{r}))\boldsymbol{\varepsilon}^n{}_{\mathbf{k}}$$

$$= L^{-3/2}\sum_{\mathbf{k}}\sum_{n=1}^{2}\sqrt{\frac{h}{k}}\,(\hat{b}_{n\mathbf{k}}\exp(i\mathbf{k}\bullet\mathbf{r}) + \hat{b}_{n\mathbf{k}}{}^*\exp(-i\mathbf{k}\bullet\mathbf{r}))\boldsymbol{\varepsilon}^n{}_{\mathbf{k}} = \hat{\mathbf{A}} \tag{31}$$

and

$$\frac{1}{8\pi}\sum_{+\mathbf{k}}\sum_{n=1}^{2}\{-(4\pi\hbar)^2(\frac{\partial^2}{\partial {a^R{}_{n\mathbf{k}}}^2} + \frac{\partial^2}{\partial {a^I{}_{n\mathbf{k}}}^2}) + k^2 a_{n\mathbf{k}} a_{n\mathbf{k}}^*\} = \sum_{\mathbf{k}}\sum_{n=1}^{2}\hbar k(\hat{b}_{n\mathbf{k}}{}^*\hat{b}_{n\mathbf{k}} + \frac{1}{2}) \tag{32}$$

so that eq. (28) becomes

$$i\hbar\frac{\partial\psi}{\partial t} = [\sum_{i=1}^{N}\{ \frac{1}{2m}(-i\hbar\nabla_i - \int dv\,\rho_i\hat{\mathbf{A}})^2 + \frac{1}{2I}(-i\hbar\hat{\mathbf{D}}_i - \int dv\,\rho_i(\mathbf{r} - \mathbf{r}_i)\mathrm{x}\hat{\mathbf{A}})^2\}$$

$$+ \frac{1}{2}\int dv\int dv'\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \sum_{\mathbf{k}}\sum_{n=1}^{2}\hbar k(\hat{b}_{n\mathbf{k}}{}^*\hat{b}_{n\mathbf{k}} + \frac{1}{2})]\psi \tag{33}$$

If we restrict the system to spin $\frac{1}{2}$ particles, then try a solution of the form

$$\psi = \sum_{m_1=1}^{2}\sum_{m_2=1}^{2}\ldots\sum_{n_1=1}^{2}\sum_{n_2=1}^{2}\ldots\,\psi_{m_1,m_2,\ldots n_1,n_2,\ldots}(\mathbf{r}_i,a^R{}_{n\mathbf{k}},a^I{}_{n\mathbf{k}},t)\,\chi_{m_1,n_1}(\phi_1,\psi_1,\theta_1)\,\chi_{m_2,n_2}(\phi_2,\psi_2,\theta_2).. \tag{34}$$

where the matrix χ is given by

$$\chi(\phi_i,\psi_i,\theta_i) = \frac{1}{2\pi}\begin{vmatrix} \exp(\frac{i}{2}(\phi_i+\psi_i))\cos(\frac{\theta_i}{2}) & i\exp(\frac{i}{2}(-\phi_i+\psi_i))\sin(\frac{\theta_i}{2}) \\ i\exp(\frac{i}{2}(\phi_i-\psi_i))\sin(\frac{\theta_i}{2}) & \exp(-\frac{i}{2}(\phi_i+\psi_i))\cos(\frac{\theta_i}{2}) \end{vmatrix} \qquad (35)$$

The $\frac{1}{2\pi}$ factor is for normalization. χ does not appear to be single valued in the orientation coordinate space, but this is because the space is doubly connected, for example see Merzbacher[17]. Then using the fact that

$$\hat{\mathbf{D}}_i \bullet \hat{\mathbf{D}}_i\, \chi(\phi_i,\psi_i,\theta_i) = -\frac{3}{4}\chi(\phi_i,\psi_i,\theta_i) \quad \text{and} \quad \hat{\mathbf{D}}_i\, \chi(\phi_i,\psi_i,\theta_i) = \frac{i}{2}\chi(\phi_i,\psi_i,\theta_i)\boldsymbol{\sigma} \qquad (36)$$

and the fact that different $\chi_{m_1,n_1}\chi_{m_2,n_2}$ .... are independent, eq. (33) yields 2^N identical sets of the 2^N equations

$$i\hbar\frac{\partial\psi}{\partial t} = [\sum_{i=1}^{N}\{ \frac{1}{2m}(-i\hbar\boldsymbol{\nabla}_i - \int dv\,\rho_i\,\hat{\mathbf{A}})^2 + \frac{1}{2I}\{\frac{3}{4}\hbar^2 - \hbar\boldsymbol{\sigma}\bullet\int dv\,\rho_i((\mathbf{r}-\mathbf{r}_i)\mathrm{x}\,\hat{\mathbf{A}})$$

$$+ (\int dv\,\rho_i(\mathbf{r}-\mathbf{r}_i)\mathrm{x}\,\hat{\mathbf{A}})^2\}\} + \frac{1}{2}\int dv\int dv'\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \sum_{\mathbf{k}}\sum_{n=1}^{2}\hbar k(\hat{b}_{n\mathbf{k}}{}^*\hat{b}_{n\mathbf{k}} + \frac{1}{2})]\psi \qquad (37)$$

where now ψ stands for $\psi_{m_1,m_2,...n_1,n_2,...}$ and $\sum_{i=1}^{N}\boldsymbol{\sigma}\bullet\int dv\,\rho_i((\mathbf{r}-\mathbf{r}_i)\mathrm{x}\,\hat{\mathbf{A}})\psi$ stands for

$$\sum_{i=1}^{N}\sum_{n'_i=1}^{2}\boldsymbol{\sigma}_{n_i}{}^{n'_i} \bullet \int dv\ \rho_i((\mathbf{r} - \mathbf{r}_i) x\, \hat{\mathbf{A}})\, \psi_{m_1, m_2, .. m_i ... n_1, n_2, .. n'_i, ...} \tag{38}$$

where the $\boldsymbol{\sigma}$ are the Pauli spin matrices.

Bohm and Hiley[18] also look into the many body Pauli equation but conclude that the above method is not viable because the number of equations does not agree with the number of degrees of freedom of the system, that is there are 2^n equations for 3n Euler angles. They are considering a hidden variable theory while here we are not and the total wavefunction, eq. (34), is just a function of all the position and Euler angle coordinates. We never try to specify the values of the particular Euler angles, but just the probability of obtaining a particular value of them.

Now set $\psi = \psi' \exp(-i\hbar(\frac{3N}{8I})t)$ so that, dropping the prime on ψ, eq. (37) becomes

$$i\hbar \frac{\partial \psi}{\partial t} = [\sum_{i=1}^{N}\{ \frac{1}{2m}(-i\hbar\boldsymbol{\nabla}_i - \int dv\ \rho_i\, \hat{\mathbf{A}})^2 + \frac{1}{2I}\{-\hbar\boldsymbol{\sigma}\bullet\int dv\ \rho_i((\mathbf{r} - \mathbf{r}_i) x\, \hat{\mathbf{A}})$$

$$+ (\int dv\ \rho_i(\mathbf{r} - \mathbf{r}_i) x\, \hat{\mathbf{A}})^2\}\} + \frac{1}{2}\int dv \int dv' \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \sum_{\mathbf{k}}\sum_{n=1}^{2}\hbar k(\hat{b}_{n\mathbf{k}} * \hat{b}_{n\mathbf{k}} + \frac{1}{2})]\psi \tag{39}$$

To see how this equation changes as the size of the particles is reduced and the external fields vary little over the size of the particle, expand the vector potential about the center of particle i so that

$$\hat{\mathbf{A}}(\mathbf{r}) = \hat{\mathbf{A}}(\mathbf{r}_i) + \sum_{j=1}^{3}(r^j - r^j{}_i)\hat{\mathbf{A}}_{,j}(\mathbf{r}_i) + \frac{1}{2}\sum_{j=1}^{3}\sum_{k=1}^{3}(r^j - r^j{}_i)(r^k - r^k{}_i)\hat{\mathbf{A}}_{,jk}(\mathbf{r}_i) \tag{40}$$

where a comma represents a partial derivative, and we have dropped terms which involve third and higher order derivatives of the vector potential. Then using the spherical symmetry of the charge distribution, eq. (39) becomes

$$i\hbar\frac{\partial\psi}{\partial t} = [\sum_{i=1}^{N}\{ \frac{1}{2m}(-i\hbar\nabla_i - q_i\hat{\mathbf{A}}(\mathbf{r}_i) - \frac{1}{6}Q_i\nabla^2\hat{\mathbf{A}}(\mathbf{r}_i))^2 - \frac{gq_i\hbar}{4m}\boldsymbol{\sigma}\bullet\hat{\mathbf{B}}(\mathbf{r}_i)$$

$$+ g\frac{q_i}{12m}Q_i\hat{\mathbf{B}}(\mathbf{r}_i)\bullet\hat{\mathbf{B}}(\mathbf{r}_i)\} + \frac{1}{2}\int dv\int dv'\frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \sum_{\mathbf{k}}\sum_{n=1}^{2}\hbar k(\hat{b}_{n\mathbf{k}}{}^*\hat{b}_{n\mathbf{k}} + \frac{1}{2})]\psi \qquad (41)$$

where $Q_i = \int dv\, \rho_i r^2$. The g factor is defined using $g\mathbf{L}_i = 2\frac{m}{q_i}\boldsymbol{\mu}_i$ with $\boldsymbol{\mu}_i = \frac{1}{3}\boldsymbol{\omega}_i Q_i$, the magnetic moment, and $\mathbf{L}_i = I\boldsymbol{\omega}_i$, so that $g = \frac{2}{3}\frac{mQ_i}{q_i I}$.

The m and I that go into the g definition here are the mechanical values only, that is they do not include the self-field electromagnetic contribution. The reason that the self-field contribution is not included is because we have ignored higher order derivatives of the vector potential which will have a large contribution from the self-fields of the particles.

Now look at equation (41) using our value of $I = \frac{2}{5}mr_0{}^2$ and our charge distribution $\rho_i(\mathbf{r} - \mathbf{r}_i) = \frac{q_i}{4\pi r_0{}^2}\delta(|\mathbf{r} - \mathbf{r}_i| - r_0)$. Using our charge distribution, Q_i takes the form

$$Q_i = \int dv\, \rho_i r^2 = \frac{q_i}{4\pi r_0^{\ 2}} 4\pi \int_0^\infty dr r^2 \delta(r - r_0) r^2 = q_i r_0^{\ 2} \tag{42}$$

so that using $I = \frac{2}{5} m r_0^{\ 2}$ we have $g = \frac{5}{3}$ and equation (41) becomes

$$i\hbar \frac{\partial \psi}{\partial t} = [\sum_{i=1}^{N} \{ \frac{1}{2m} (-i\hbar \nabla_i - q_i \hat{\mathbf{A}}(\mathbf{r}_i) - \frac{1}{6} q_i r_0^{\ 2} \nabla^2 \hat{\mathbf{A}}(\mathbf{r}_i))^2 - \frac{5 q_i \hbar}{12m} \boldsymbol{\sigma} \bullet \hat{\mathbf{B}}(\mathbf{r}_i)$$

$$+ \frac{5 q_i^{\ 2} r_0^{\ 2}}{36m} \hat{\mathbf{B}}(\mathbf{r}_i) \bullet \hat{\mathbf{B}}(\mathbf{r}_i) \} + \frac{1}{32\pi^2} \sum_{i=1}^{N} q_i \sum_{j=1}^{N} q_j \int d\Omega'' \int d\Omega''' \frac{1}{|\mathbf{r}'' - \mathbf{r}''' + \mathbf{r}_i - \mathbf{r}_j|}$$

$$+ \sum_{\mathbf{k}} \sum_{n=1}^{2} \hbar k (\hat{b}_{n\mathbf{k}}{}^* \hat{b}_{n\mathbf{k}} + \frac{1}{2})]\psi \tag{43}$$

where we have set $\mathbf{r}'' = \mathbf{r} - \mathbf{r}_i$, $\mathbf{r}''' = \mathbf{r}' - \mathbf{r}_j$ and used spherical coordinates about the $\mathbf{r}_i$ and $\mathbf{r}_j$ centers in the next to last term. The length of r’’ and r’’’ is restricted to r_0 and $\int d\Omega'' \int d\Omega'''$ represents the integrals over the angular parts of the r’’ and r’’’ coordinates.

In the limit of r_0 going to zero, r’’ and r’’’ go to zero and the angular dependence of $\frac{1}{|\mathbf{r}'' - \mathbf{r}''' + \mathbf{r}_i - \mathbf{r}_j|}$ disappears. So in this limit

$$\int d\Omega'' \int d\Omega''' \frac{1}{|\mathbf{r}'' - \mathbf{r}''' + \mathbf{r}_i - \mathbf{r}_j|} = \frac{(4\pi)^2}{|\mathbf{r}_i - \mathbf{r}_j|} \tag{44}$$

Thus in the limit of a point particle equation (43) reduces to

$$i\hbar \frac{\partial \psi}{\partial t} = [\sum_{i=1}^{N}\{ \frac{1}{2m}(-i\hbar\nabla_i - q_i \hat{\mathbf{A}}(\mathbf{r}_i))^2 - \frac{5q_i\hbar}{12m}\boldsymbol{\sigma}\bullet\hat{\mathbf{B}}(\mathbf{r}_i)\} + \frac{1}{2}\sum_{i=1}^{N} q_i \sum_{j=1}^{N} q_j \frac{1}{|\mathbf{r}_i - \mathbf{r}_j|}$$

$$+ \sum_{\mathbf{k}}\sum_{n=1}^{2}\hbar k(\hat{b}_{n\mathbf{k}}{}^{*}\,\hat{b}_{n\mathbf{k}} + \frac{1}{2})]\psi \qquad (45)$$

Here $g = \frac{5}{3}$ which just includes the mechanical mass and moment of inertia.

5. Conclusions

This paper investigated non-relativistic QED of extended spinning particles with an arbitrary g factor included and came up with a general equation which includes the self-field terms. In the limit of a point particle the self-field terms have not been ignored. If these can be included in the limit of a point particle then using the result of this calculation may lead to a better understanding of the infinities associated with a point particle with spin. No definite predictions can be made from these equations until a way to include the self-field terms in the limit of a point particle can be achieved. This inclusion of the self-field terms probably has to be done in the Heisenberg picture using methods similar to those used in papers 3-7.

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