A Reflection on the Planck Scale with Countable Measure for all Quantities of Length and Time in the Universe

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Abstract

We argue that the Planck scale for length and time is the only countable scale for measurements of lengths and times in the universe. Accordingly, any measurement of length which seeks a precision below the Planck length is unmeasurable and meaningless; so, every length in the universe has a countable number of Planck lengths. We see that there is a distinction between the physical *quantities* of length and time in the universe, and *number*, which is distinct from the physical universe.

1. Background

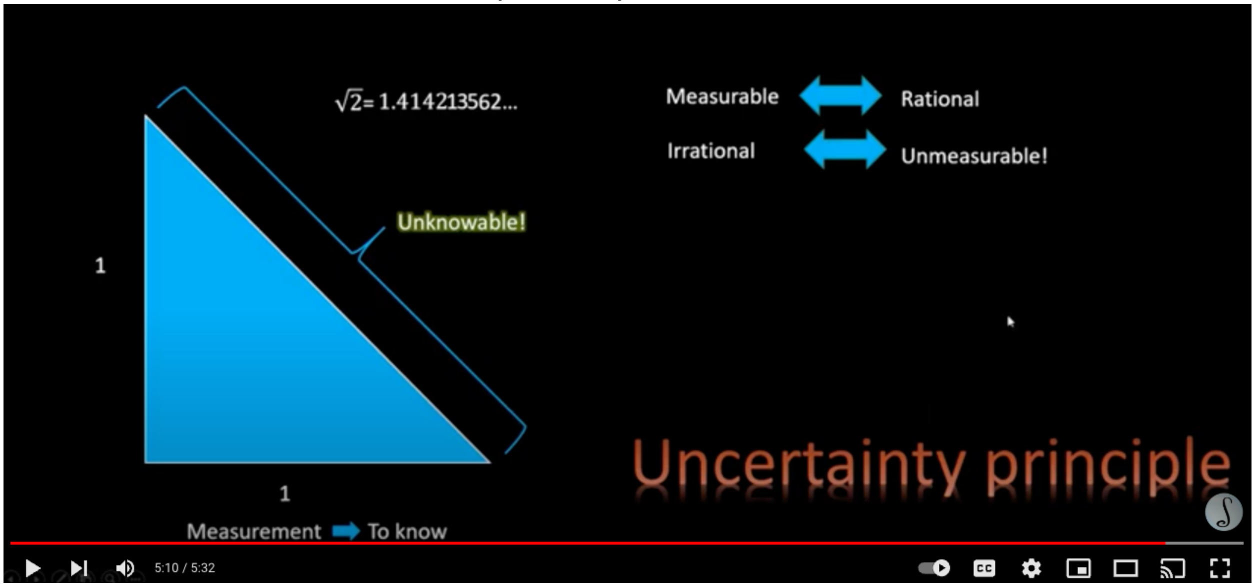
There is a very interesting and short (5:32) YouTube video, called "Why Irrational Numbers Don't Make Sense"[1] by Shirley Zhu from Learning0to1. For background purposes and later discussion, we mention some observations made in the video:

- Measurements are made using some standard unit of measure (*i.e.*, for 1 assigned unit).
- Some measurements can be made using ratios of a countable number of these units, and these are rational measurements.
- Some measurements cannot be made using ratios of a countable number of these units, and these are irrational.

The video concludes with the following text, starting at 4:02. The subsequent screen shot is taken at 5:10.

> Being measurable is related to being a rational number. So, if it's irrational, then it means it can't be measured. So, if it's irrational, then it means it's unmeasurable - because how can it be measurable if you can't even quote how many units it occupies. It has nothing to do with how you do the measurement. Hence eventually, irrational numbers mean something unmeasurable. But measurement is our critical tool to know the world. So, if the length is unmeasurable, that means it's unknowable.

A length of square root of 2 can be approximated with greater and greater accuracy. But you never know exactly what it is. Each decimal place is unpredictable, until you actually compute it. But, how come we see the length exists in front of us, yet there is no way of knowing.
It reminds me of the uncertainty principle in quantum mechanics - which says that the essential property of an electron, it's position and momentum (or simply velocity) can't be known precisely. It's pretty interesting that both math reasoning and physics seem to suggest that the world can't be known precisely.



2. Introduction

Zhu's YouTube video, discussed in the previous section, captures some very important observations about length measurement and what is knowable. In this paper, we will expand upon those observations.

In the following sections we show that, 1) measurements and the quantities used for length are only relatable to each other by a unit scale. 2) anything quantifiable is finite, 3) the Planck-length scale and the Planck-time scale have the ultimate precisions and the only countable (whole number) measurement scales for these quantities, 4) there is a distinction between all real-world physical *quantities* of length and time, and *numbers*, which need not be correlated with anything physical in the universe.

Some of our arguments are similar to those found in the literature and YouTube, which is a popular source for the layperson. We include YouTube sources that we believe are trustworthy and which explain concepts clearly and correctly. These YouTube sources may also be more accessible and engaging to a broader audience.

The arguments presented in this paper could be associated with a limited view of finitism. A passage about finitism from Calmet and Hsu[2] state that “[s]ome mathematicians, called *finitists*, accept only finite mathematical objects and procedures.”

However, Incurvati[3] discusses that there are different conceptualizations of finitism, which might not agree or be consistent with each another. In any case, we seek to avoid any controversy with the usage of finitism; we will not attempt to take a specific position on its usage. Instead, we will let our arguments stand (or fall) on their own.

We begin our discussion in the next section by considering quantities of lengths.

3. Measurements and Finite Length Unit Scales

Zhu’s YouTube video shows an isosceles right triangle where sides can be expressed as ratios. These ratios can only provide for the length of one of the sides when one length is known. In order to have a quantity for a length, we must have a measurement unit. In other words, measurements of length can only be made with the adoption of a unit scale, such as meter, inch, etc., which is a key point in this paper.

Key Point 1: Measurements and the quantities used for length are only relatable to each other by a unit scale.

The units used for a measurement determine what can be measured rationally. A rational number can be expressed as the ratio (fraction) of two whole numbers. An irrational number is a real number that is not expressible as a ratio. Examples of irrational numbers are $\sqrt{2}$ and π. If a length could be determined to be irrational, then its length would not be commensurate with the unit scale used.

Courant and Robbins[4] make an important point about irrational numbers used for lengths:

> From a physical point of view, the definition of an irrational number by a sequence of nested intervals corresponds to the determination of the value of some observable quantity by a sequence of measurements of greater and greater accuracy. Any given operation for determining, say, a length, will have a practical meaning only within the limits of a certain possible error which measures the precision of the operation. Since the rational numbers are dense on the line, it is impossible to determine by any physical operation, however precise, whether a given length is rational or irrational. Thus it might seem that the irrational numbers are unnecessary for the adequate description of physical phenomena.

The text goes on to point out the advantage of irrational numbers for the limit concept and the number continuum. However, the passage recognizes a limited precision for a measurement, even when a greater precision is possible. This limited measurement precision is a result of a human endeavor/activity. This is different than the ultimate precision possible for a length, which will be discussed later. The passage also indicates a distinction between *number*, such as an irrational number, and a *physical quantity* for length. This distinction will also be discussed later.

Our main purpose here is to recognize that, although an irrational number may have an infinite decimal expansion of digits that never repeat, a quantifiable length is always finite. This is true in general.

Key Point 2: Anything quantifiable is finite.

Although a unit scale is required to quantify a length, the scale can almost be arbitrarily designated. For example, drawing a line on a piece of paper and designating 0 and 1 at the end points will assign an implied unit length against which things can be measured; it is a “ruler” with its own assigned unit measure. This arbitrary nature of a unit scale is key in determining whether one specified length is identified with a rational or irrational measure.

Consider a right triangle, similar to the one shown by Zhu, with 1 meter (*m*) sides and hypotenuse $\sqrt{2}\ m$. Here the sides are rational, and the hypotenuse is irrational. Suppose a different measurement scale is adopted where the length of the hypotenuse ($\sqrt{2}\ m$) is assigned a unit called "sqrt2". So, the hypotenuse has a length of 1 *sqrt2*. In this measurement scale, it is the side lengths of the right triangle that are now irrational; they do not have a rational unit of measure relative to 1 *sqrt2*. In fact, in the scale of sqrt2, all rational lengths in meters, besides 0 *m*, are irrational (and some irrational numbers are now rational). So, it might seem that a specific length may always be considered to be either rational or irrational depending on the unit scale used.

However, math and science have consilience when it comes to quantities of length.

4. What we can know about a physical length: The Planck length precision

We now consider what we can possibly know about a length in terms of its ultimate precision in the universe, which we will identify as the Planck length. This section will quote heavily from interesting YouTube sources since they provide for excellent descriptions and explanations. It is hoped that readers will watch these videos.

For background purposes, we mention that the Planck length has, somewhat controversially, been referred to as the smallest length. An interesting YouTube video by Don Lincoln[5] discusses how Alden Mead's work (after Max Planck's death) resulted in other people saying that the Planck length was the smallest. Don summarizes this as follows:

> The bottom line is that the Planck length is *not* necessarily the shortest length, but it is a length at which existing physics *has* to fail and needs to be replaced with something better. So, it's an important size, but it may not be the smallest size.

Matt O'Dowd[6] also discusses the smallest length, the limits of physics, and the Planck length:

> So, is there a smallest length? Well, there's a smallest meaningful length, at least for any intuitive conception of space. Quantum uncertainty thwarts our attempts to understand the universe by simply splitting it into smaller parts…

At another point in the video, O'Dowd states:

> There's no limit to the number of times you can half a number, but the same might not be true of space. The Planck length is thought to represent the minimum length for which the concept of length is even meaningful. Here, the illusion that space is smooth and continuous breaks down.

Whether or not the Planck length is the smallest length, physics says it is the ultimate measure before things become unmeasurable. This is the key for measurability – and is a main argument of this paper.

O'Dowd uses an example where a light wave is envisioned to measure a distance with increasing accurately by using a beam with higher frequencies, and shorter wavelengths. However, the uncertainty principle becomes a limiting factor, and "… the Planck constant represents the limit to which we can measure the universe."

O'Dowd reviews the physics for how this happens, including the warping of space itself:

> So, this is one way of thinking about it – the Planck length represents the best possible resolution that any distance can be measured. It also represents the minimum size that you meaningfully ascribe to anything.

In any attempt to measure a length below the Planck length, the uncertainty becomes 100%. The universe is described as seemingly anthropomorphic to thwart measurements below the Planck length:

> So, the universe seems to be conspiring to stop us measuring distances or sizes smaller than the Planck length.

We agree with O'Dowd statement that the Planck length "represents the fundamental limit of measurability of space". So, we take the Planck-length scale to have the ultimate precision possible in the universe. If a different unit scale, say one using the meter, is used in an attempt to specify a number (irrational or rational) for a measurement, then this measurement eventually becomes meaningless below the Planck length scale.

Key Point 3: Lengths can only have a precision down to the Planck length. The Planck length has the ultimate precision possible for length.

A unit length is inextricably linked with what can be measured. And every possible measurement length is measurable by a whole number of Plank lengths. This shows that:

Key Point 4: The Planck-length scale provides for the only rational number line. Ultimately, all lengths have countable measure in the Planck-length scale.

5. What we can know about physical time: The Planck time precision

So far, this paper has discussed the physical quantity for length measurement and the Planck length. However, everything discussed for length also applies to time and the Planck time. That is, the physical quantity of time in the universe has its ultimate precision with the Planck time. Both the Planck length and Planck time are shown by Don Lincoln and Arvin Ash[7] in SI units:

Planck Length $l_P = \sqrt{\frac{\hbar G}{c^3}}$	Planck Time $t_P = \sqrt{\frac{\hbar G}{c^5}}$

Arvin Ash elaborates further on the Planck time:[8]

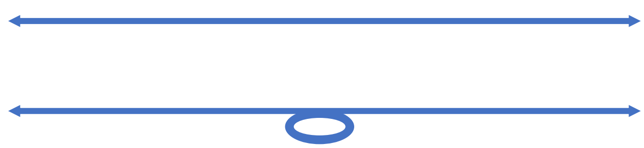
> Planck time is the time it takes for light to travel one Planck length. It is the smallest measurement of time that has any meaning in quantum mechanics. Planck time is the closest time to the beginning of the universe, occurring right after the Big Bang, that we could theoretically model.

Both time and length, which are key concepts of physics, have a corresponding Planck scale. In the next section we ask about the infinite, using length as an example.

6. Infinite?

Up to this point, we have discussed how anything quantifiable is finite. But what about the infinite, such as a hypothesized infinite length in the universe?

The observable universe is finite. If such a thing as an infinite length did exist in the entire (observable and unobservable) universe, then it could not be quantified by any unit scale. Consider that there is an infinite line in the universe. Since no practical ruler can provide for a countable measure of this infinite line, then we might try to assign the infinite line itself with a unit length (say 1 "inf") itself. Now suppose that there is a parallel line which also has the same 1 *inf* measure. Consider that you pinch this line and twist it to have a small loop.



This line still has the same 1 *inf* measure, but it can be matched up to the first line, with a loop left over. The parallel line has 1 *inf* measure and a measure that is also larger than 1 *inf*. So, we have a contradiction showing that there is no infinite quantity which is countably measurable.

This demonstration is different from the Hilbert's Hotel, which is well-known. The hotel is envisioned to have an infinite number of rooms, each with a guest. The hotel can accommodate any number of guests, even infinitely many, by moving guests to other rooms such that new guests can be accommodated. Hilbert's hotel does not specify a completed infinity, as 1 *inf* attempts to do. The idea of a complete infinite magnitude has been discussed before. In an oft-quoted statement by Gauss:[9]

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which

> certain ratios approach indefinitely close, while others are permitted to increase without restriction.

We agree that a complete infinite magnitude does not exist; it is non-quantifiable. For example, there is no meaning in a sequence of an infinite string of 9s to represent an infinite magnitude or an infinite strings of 1s.

7. *Quantities* of length and time vs *numbers*

In this paper, we adopt the position that there is a distinction between *numbers* and the physical *quantities* of length and time. We have shown that all such quantities are countable with the Planck scale, whereas real numbers themselves include irrational numbers that are not ascribed to anything physical.

Courant and Robbins[10] reference a definition of number as "…the continuum of numbers, or real number system … is the totality of infinite decimals." A real number may be rational or irrational, but it need not be associated with anything quantifiable in order to have mathematical existence.

Our position about this distinction between *number* and *quantity* is similar to those expressed by others, including Calmet and Hsu:[2]

> Our intuitions about the existence and nature of a continuum arise from perceptions of space and time […]. But the existence of a fundamental Planck length suggests that space-time may not be a continuum. In that case, our intuitions originate from something (an idealization) that is not actually realized in Nature.

An example of this distinction is Zeno's well-known paradox, for which discrete Planck-lengths have already been proposed as a solution.[11] of the tortoise and the hare. The tortoise and hare can only traverse a length (*i.e.*, a countable number of Planck lengths) per unit time (*i.e.*, a countable number of Planck times). Eventually, the faster moving tortoise will overtake the hare since only discrete measures from the Planck scale are involved. So, it is not necessary to discuss a continuum of infinitely divisible lengths in order to resolve the paradox. In other words, physical quantities in the universe resolve the paradox without requiring the notion of a continuum of numbers.

Another example of the distinction between physical quantities and numbers emerges from calculus. In calculus, a continuous real number line is used in order to perform integration, etc. In the physical universe, the discrete Planck length unit is so small that it is practically infinitesimal (although it is really finite). We can imagine that this is the infinitesimal that is actually being used in calculus; any measurement below the Planck length is unknowable anyway. A sequence of these discrete infinitesimals “appears” to be continuous for any human-scale measurement.

8. Conclusion

The abstract concept of *numbers* is distinct from physical *quantities* for length and time in the universe. In this paper, we show that measurement requires the use of finite unit scales. The ultimate measurement scales for length and time in our universe are their corresponding Planck scales. Any other measurement scale which attempts to define a precision below its Planck length or Planck time becomes meaningless. These Planck scales are the only countable scales for length and time quantities. Ultimately, these quantities can only be known to a certain precision in the universe – the Planck scale - and it makes no sense to pretend otherwise.

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