Constructive QFT : a condensed math point of view I

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Abstract

In this series, we show a road map to prove the existence of ϕ^4 quantum field theory over 4-d Euclidean spacetime. We suggest a new axiomatic approach on constructive quantum field theory via condensed mathematics. The goal is to extend this new approach to cover the mathematical viability of field theories of standard model and beyond.

I. Introduction

1 A journey to achieve legitimacy of field theories

Field theories govern the current understanding of the universe. Quantized field theory is a fundamental language of standard model. And Einstein's tensor field theory is a language of gravity. In spite of it's huge success, the field theory was suffering from infinity issue already at the beginning. For example in electromagnetism, a charged particle should have mass

$$m_{EM} = \frac{1}{2}\int E^2\, dV = \frac{1}{2}\int_{r_e}^{\infty} \left(\frac{q}{4\pi r}\right)^2 4\pi r^2\, dr = \frac{q^2}{8\pi r_e},$$

So by this equation one knows that the mass depends on the electron radius. And if $r_e = 0$, then the mass should be infinity. At the time, elementary particle was treated as a 0-dimensional point particle, which can't have radius. But by this equation, in order to have viable theory one has to make the electron to have radius. Physically, with infinite mass the particle cannot be accelerated. Mathematically, with solution being in infinity, one cannot have the existence of the theory. In this example, physical and mathematical viability both claim the new physical realization. And the electron now is considered to have the radius of $\approx 10^{-15}m$ rather than radius 0. So in this process one encounters new physical understanding by resolving both the mathematical inconsistency and physical nonsense.

However, in modern day physics, where physical viability is hardly presumed, mathematical viability of any theory is an important guide to have an insight on uncharted realm of physics. Especially in the theory of quantized field, where mathematical viability is still required to be constructed in many cases, it is worthwhile to have various tools to search the viability.

In this series of notes, we suggest new tool kits to construct general QFT. The foundation is from the standard analytic approaches such as Wightman, Osterwalder-Schrader and Glimm-Jaffe. We use the condened mathematics to reinterpret the analytic approach of current building blocks of constructive QFT. So one would see some unfamiliar treatment on the subject. In order to do that, we will suggest algebraic axiom sets to cover the analytic axioms of (OS). As an application, we heuristically show that ϕ^4 interating theory exists at any coupling constant λ.

2 Notes before departing

2.1 Notational disasters

Physicists sometimes call functions, when it is distributions at best. And They treat some function space as manifold, when it is not even a *space* with suitable axioms. Karma has stacked, now some mathematicians named certain mathematical ideas *condensed, solid, liquid* and called whole subject as *condensed mathematics*. Even though this is not the first time that physical nomenclature is used in mathematics without direct connections, such as nuclear modules, crystalline cohomology and etc, this terminology is not adequate when it's application can really meet the field of statistical mechanics from which these nomenclatures originate.

However, we stick to the terminology as the inventors coined in these notes. Maybe there will require the need for renaming these technology apt to physicists' taste. Until then, we use *condensed mathematics* terminology for now. So readers should keep in mind that these *condensed* notations are merely from mathematical purpose. Since in these notes, applications on statistical mechanics is not pursued, we hope there is less confusion on the terminology. For convenience, we will use emphatic font for these technological tool kits, to remind notational bug.[1] Also other than *condensed* terminology, we introduce new notations to call some of mathematical definitions only for physical use. And we use loose notations in physical standards, too.

2.2 From pure condensed mathematical point of view

Condensed math is a recently developed idea for rebuilding the world of topology. Constructors are bold enough to claim that they want to cover large part of analytic geometry by this method, as if algebraic geometry is successfully overarching so many

[1]Loosely speaking, one can appreciate the terms, *solid* or *liquid*, and *condensed* for both, as a kind of characteristics of certain QFT. It doesn't have to do anything with particles' status and so on.

parts of mathematics. However, still the examples look elementary and the application is sometimes more expensive to take than taking old route. We want to add a belief in this direction by constructing a theory which is not covered by algebraic geometry yet. Hopefully, constructing quantized field theory via condensed math may provide more validity of why this point of view is worth while. So as much as having a legitimate field theory of elementary particles is significant to mathematical physicists, having an interesting application of condensed mathematics might be important to some mathematicians. We hope this subject is developing on both sides creating a virtuous cycle.

2.3 This is not for ending the story, but initiating a dream.

This track of study has been performed by a brain. We assume many mistakes and errors are here and there, still unnoticed by the author.[2] It will be great to have feedbacks from those interested in this journey. We only hope to get a better version of this string of procedures. It will be joyous if there are more companions involved and get better and better construction day by day.

As if condensed math has grand goal, we initiated this journey aimed at more than one could expect. We hoped to cover any field theory, classical or quantized, spinors, vectors or tensors. So our approach is essentially more general with many new definitions. Some of them will appear in the series, but we decide to conceal most of them for the clarity of the message. We will focus on a specific scalar quantized field interacting theory, and show gadgets only in need to construct it.

It is focused on showing the viability rather than having nicer calculational methods by using the language of condensed mathematics. Squeezing out better calculational technics by derived homological algebra of condensed modules would be another whole business.

The end result, which is the existence of ϕ^4-theory is heuristic. There will require another hundred of pages to sort out every details. Instead we focus on advertizing the potential power of a new algebraic approach on field theory, at this stage.

3 A blueprint

We show what we have in mind in a nutshell.

3.1 from physical point of view

What we want to construct is a kind of legitimate pro-effective field theories on a *condensed* lattice spacetime. Field theory whether it is quantum or not has singularity issues on any continuous spacetime. In order to resolve singularities, the technics called regularization are required. One of prominent regularization method is to start from lattice. Cutting off some scales beyond interest is how effective field theory works.

[2] We only hope the number is not infinite.

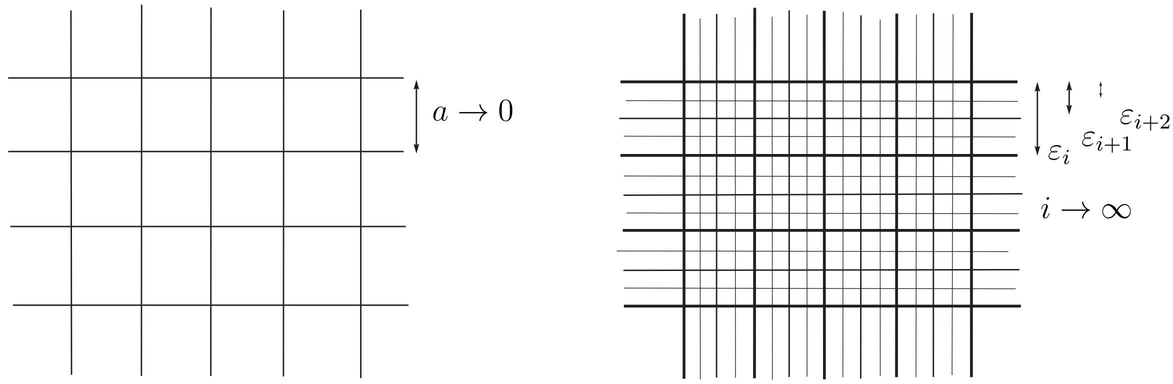

Fig. 1: In a lattice field theory, one starts from a lattice then taking limit that the gap $a \to 0$. In a *condensed* lattice, we work with a inverse profinite limit of lattices where $i \to \infty$.

But legitimacy of continuum field theory requires that cut-off should shrink to zero. This lattice approach is still a best option for studying QCD. *Condensed* QFT, or constructive QFT, we build is based on this latticed spacetime.

However, the limit process is different from regular lattice based field theory. Instead of taking both $a \to \infty$, $|\Lambda| \to \infty$ limit, we take *condensed* lattice limit. Local limit is defined via inverse limit of profinite lattices. And global limit is defined via patching the local *tissues* rather than just take the limit of the volume of the lattice to infinity. Both new approach on the local limit and global limit of lattice brings a new *texture* on the *condensed* spacetime. The discrete nature of limit process on the localization is the key point of the *condensed* regularization technics. The globalization idea is defined via pure algebraic geometrical method, and it will play some crucial role to show the existence of field theory in general.

Thinking about the fact that the *texture* of the spacetime should be continuous is doubtful sometimes. Since the success of classical mechanics defined via differential geometry on smooth manifold, the *tissue* of position or momentum observables are obviously treated as having a smooth and continuous *texture* like $\mathbb{R}$. However, after the development of quantum mechanics, we now know that the observables of any elementary particles, such as momentum, energy state or spin states are distinguished in a discrete numbers. Even for positional information, there is Planck length that any forces in nature breaks down. And by uncertainty principle, having an infinitely precise information of position makes momentum uncertain as much as possible, which blows up the amount of energy requires to pin point the particle's position. So there is no one-to-one correspondence in representing positional information by real numbers. We argue that this is a physical motivation to weave *texture* of the local spacetime by profinite procedures.

Technically, quantization scheme used in this series is Feynman's path integral method, as standard constructive QFT does. And following Osterwalder-Schrader and

Glimm-Jaffe, the study is on correlation functions on Euclidean 4-d spacetime. Especially our interest is on ϕ^4 scalar field interaction theory, where the result of Glimm-Jaffe applies to Osterwalder-Schrader.[3] We rewrite analytic axioms of (OS) defined by Glimm-Jaffe by *condensed* version of algebriac axioms ($\underline{\text{OS}}$). Our strategy is to lean on the existence of *condensed*[4] measures on the $Hom(\Lambda_{cond}, \mathbb{R})$, which is defined as a test functions, or paths space, on *condensed* lattice. We claim that the new axiom set ($\underline{\text{OS}}$) applies to (OS).[5] Then we show that ϕ^4 theory, constructed via *condensed* way, satisfies ($\underline{\text{OS}}$).

Some part[6] rely on the existence of *condensed* measures and some[7] rely on the lattice construction. Interesting part[8] is the Euclidean covariance axiom, which is not working well with lattice set-up. We argue that this works over new base field $\mathbb{K}$, which was used to show the *liquidity* theorem. There seems to exist extra symmetry group over the Euclidean covariance in this approach, but we don't assume what it is about yet.

3.2 from mathematical point of view

In a sentence, we want to axiomatize quantum field theory by algebraic geometrical methodology. In order to do that we follow *condensed philosophy.* We translate anaytic definition of quantum field theory into *condensed language.* Pros of working inside the algebraic axioms is that one can construct viable theories by assemblying viable building blocks in exact sense. If the theory supports measures satisfying decent axioms such as (Ab), the theory building is achievable from bottom to up. This is not how the viability theorem is shown in analytic axioms. In analytic world, one requires enough inequalities to show something is bounded and exists. So exact solutions deplete quickly. Instead there is numerical merit in analytic approch via using many approximation tools. We think both axiom sets can be used at it's best use, assuming the equivalence between both. But for showing existence only, algebraic axioms have advantages. Also general categorical approach provides a way to compare properties of different types of field theories. There is more to come from this method for sorting out every field theories in physics.

Coordinates of lattice is $(\mathbb{Z}a, \mathbb{Z}a, \cdots)$, then continuum limit is taken by $a \to 0$. For *condensed* lattice coordinates, we start from base field $\mathbb{K} := \mathbb{Z}((T))_{>r}$. We use this base for both *solid* field theory and *liquid* field theory.[9] Roughly saying, the coordinates vector is $(\mathbb{Z}\varepsilon_i + \mathbb{Z}\varepsilon_{i+1} + \cdots, \cdots)$, which represents fig.1. So the coordinates of *condensed* lattice is $(\mathbb{K}, \mathbb{K}, \cdots)$, just like Euclidean continuous coordinates space

[3]For general any degree of interacting theory, there is caveat to apply Glimm-Jaffe axioms to OS axioms. So for convenience we restrict a special case for simplicity.

[4]both *solid* and *liquid*

[5]Maybe, the other direction is also true.

[6]($\underline{\text{OS0}}$),($\underline{\text{OS1}}$)

[7]($\underline{\text{OS3}}$),($\underline{\text{OS4}}$)

[8]($\underline{\text{OS2}}$)

[9]Difference is from cut-offs. And each measures will be used to cover either perturbative or non-perturbative cases.

$(\mathbb{R}, \mathbb{R}, \cdots)$. However, one should note that we treat the base field $\mathbb{K}$ as a *condensed* version of base field $\mathbb{R}$.

The input of path integral quantum field theory is paths space, which is operator valued test functions on the coordinates space. As any internal *Homs* of condensed sets is again condensed set, the input is a condensed set. Axioms (E), established by Osterwalder-Shrader is about Schwinger correlators on these inputs. Schwinger functions can be built by qunatization scheme either canonical or path integral.

While axioms (OS) established by Glimm-Jaffe, can be charaterized differently. It is on the Feynman path integral measures. By showing the meausures satisfying axioms (OS), one can guarantee that the correlators out of measures through test function satisfies (E). Then with Osterwalder-Shrader theorem, which states $(E) \Rightarrow (W)$, one has the existence of QFT which satisfies Wightman axioms on Minkowski space. Establishing axioms (OS) has merits when the existence of the measure is known and the test function is specified. We try to show the (OS) for specific ϕ^4 interacting theory.[10]

The final scheme for showing the existence of quantum field theory is by[11]

$$(\underline{\text{OS}}) \Rightarrow (\text{OS}) \Rightarrow (\text{E}) \Rightarrow (\text{W})$$

4 What to expect in this series of notes

We divide the construction site into several parts.

1. Define ϕ^4_{cond}-theory.
2. Construct axiom set $(\underline{\text{OS}})$.
3. Show $(\underline{\text{OS}})_{\phi^4_{cond}} \Rightarrow (\text{OS})_{\phi^4}$.
4. Show ϕ^4_{cond}-theory satisfies $(\underline{\text{OS}})$.

One note will be at most 10 pages. Several notes will cover each step above. Motivations for short pages per each note is as follows. First, it would be easier for readers to pick and consume. Second, it is good for the author to slice out single content out of gigantically entangled forest of both *condensed* world and QFT world. One of reasons of writing these notes is to arrange things sorted out for the author oneself. Third, it is convenient to manipulate each notes separately. Because we don't think this is a completed task, we want some space for inserting whole new notes and revisions in progress. Finally, we want to attract more people to study field theories in this way. So we add short introductory notes to make it easier to approach from other sides. We hope that more pages are filled with ideas by other fine brains in the future.

[10] Sometimes we use natation $(OS)_{\phi^4}$ to distinguish from general (OS).

[11] (E) can be skipped. But there is potential route from $(\underline{\text{E}})$ theory, so we keep it.

On step 1

- Remind Feynman's approach on QFT with lattice regularization.
- Remind ϕ^4-theory. Define Schwinger functions out of path integral measures.
- Define *condensed* Schwinger functions.
- Define a measure theoretic approach on *condensed* Schwinger functions.

On step 2

- Remind (W),(E) and (E) $\Rightarrow$ (W).
- Remind (OS), and (OS) $\Rightarrow$ (E) $\Rightarrow$ (W).
- Introduce ($\underline{\text{OS}}$).

On step 3

- From ($\underline{\text{OS0}}$) & ($\underline{\text{OS1}}$) to (OS0) & (OS1).
- From ($\underline{\text{OS2}}$) to (OS2)
- From ($\underline{\text{OS3}}$) & ($\underline{\text{OS4}}$) to (OS3) & (OS4).

On step 4

- From *condensed* measures satisfying (Ab) to *condensed* measures satisfying ($\underline{\text{OS}}$).
- Free scalar field theory from *solid* measures.
- ϕ^4-theory with weak coupling $\lambda < 1$ from *solid* measures.
- ϕ^4-theory with strong coupling $\lambda \geq 1$ from *liquid* measures.

On auxiliary part

- Remind renormalization group flow[12], and it's interpretation in *condensed* measures.
- About globalization. About patching the *tissues*.
- Introduce *DO philosophy*. Introduce *Idea* and *canvas* theory.
- λ and ℓ^p-norms. The set of ℓ^p-norms for representing field theories of different couplings.

In the next note, we lay the foundations of the construction. And we define *condensed* scalar field theory with ϕ^4 interacting terms.

[12] running coupling constant

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