A Proof of Riemann Hypothesis by Strip Mapping Pattern Contradictions

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Abstract: The Riemann zeta function(RZF), $\zeta(s)$, is a function of a complex variable $s = x + iy$, which is analytic for $x > 1$. The Dirichlet Eta Function(DEF), $\eta(s)$, is also a function of a complex variable s, which is analytic for $x > 0$. The zeros of RZF and DEF are all same. The Riemann hypothesis(RH) states that the non-trivial zeros of RZF is of the form $s = 0.5 + iy$. The clue of our proof stems from the symmetry properties of RZF zeros, stating that if there exists a zero whose real part is not 0.5, such as $\zeta(\alpha + i\beta) = 0, 0 < \alpha < 0.5$, also $\zeta(1 - \alpha + i\beta) = 0$, called *critical line symmetry*. Then, the two symmetric zeros should be on the two edge lines of a strip $\alpha \le x \le 1 - \alpha$. In the strip there are infinitely many lines that are parallel to the edge lines. Our question was, when that strip is mapped by DEF, will these parallel relationships be kept? If the parallel relationships are kept, RH is true, if not, RH may be false. So, we identified four possible graphic patterns that may satisfy the critical line symmetry. We found that DEF can't satisfy any of the four patterns. So, RH is true.

1. Introduction

In this work we studied the implications of the symmetry properties of the zeros of DEF. If there exists a zero whose real part is not 0.5, i.e., $\eta(\alpha + i\beta) = 0, 0 < \alpha < 0.5$, the symmetry properties of the zeros of DEF forces $\eta(1 - \alpha + i\beta) = 0$. This implies that the two zeros should be on the two parallel edge lines of a strip $\alpha \le x \le 1 - \alpha$.

We identified four possible graphic patterns that may satisfy the critical line symmetry of DEF zeros. If there exist zeros which deviate the critical line, i.e., $\eta(\alpha - i\beta) = \eta(1 - \alpha + i\beta) = 0$, two trajectories of $\eta(\alpha - iy)$ and $\eta(1 - \alpha + iy)$ must intersect at the origin when $y = \beta$.

To verify if this can happen, we studied whether the mapping of the strip $\alpha \le x \le 1 - \alpha, 0 < \alpha < 0.5$ by $\eta(s)$ can geometrically generate $\eta(\alpha - i\beta) = \eta(1 - \alpha + i\beta) = 0$. We found that $\eta(\alpha - i\beta) = \eta(1 - \alpha + i\beta) = 0$ can't be achieved, proving that RH is true.

2. Symmetry Properties of the Zeros of RZF

RZF [1][2][3][4][5] $\zeta(s)$ and DEF [6] $\eta(s)$ are functions of a complex variable $s = x + iy$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{2.1}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \tag{2.2}$$

RH [1][7][8] states that all the non-trivial zeros of $\zeta(s)$ are of the form $s = 0.5 + iy$. The line $x = 0.5$ is called *critical line*.

It is well known that the following three equations are true, where $\xi(s)$ is the Riemann's Xi-function [8][9].

$$\xi(s) = \frac{1}{2} s(s-1) \Gamma\left(\frac{s}{2}\right) \zeta(s) \pi^{\frac{-s}{2}} \tag{2.3}$$

$$\xi(s) = \xi(1 - s) \tag{2.4}$$

$$\zeta(\bar{s}) = \overline{\zeta(s)} \tag{2.5}$$

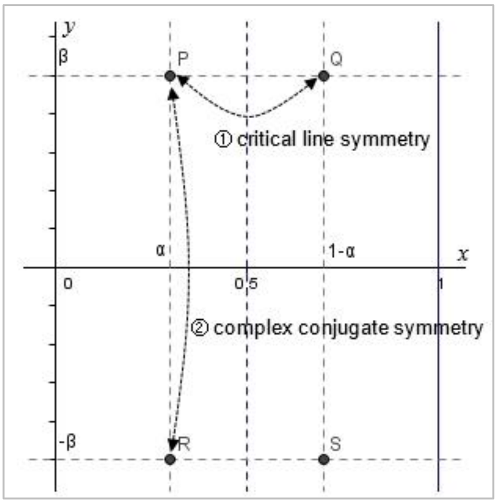
The right side of the equations (2.2) and (2.3) include $\zeta(s)$, so, the zeros of $\zeta(s)$ are also the zeros of $\eta(s)$ and $\xi(s)$.

Lemma 2.1. Equations (2.4) and (2.5) means that there exist two types of symmetries of the RZF zeros, as in Figure 1.

① **Critical line symmetry:** Symmetry of (2.4), which means that if $s = \alpha + i\beta$ is a zero, then $1 - \alpha + i\beta$ is also a zero.

② **Complex conjugate symmetry:** Symmetry of (2.5), which means that if $s = \alpha + i\beta$ is a zero, then $s = \alpha - i\beta$ is also a zero.

Figure 1. Zero symmetries of RZF.



Proof. Let $s = \alpha + i\beta$. First, in (2.5), $\zeta(\alpha - i\beta) = \overline{\zeta(\alpha + i\beta)} = 0$, in Figure 1, $\zeta(R) = \overline{\zeta(P)} = 0$, which is the complex conjugate symmetry. Second, in (2.4), $\xi(\alpha + i\beta) = \xi\{1 - (\alpha + i\beta)\} = 0$, in Figure 1, $\xi(P) = \xi(S) = 0$.By the complex conjugate symmetry, $\xi(S) = \xi(Q) = 0$. So, $\xi(P) = \xi(Q) = 0$, which is the critical line symmetry. ■

3. Strip Mapping Concept

Instead of using RZH, let's use DEF which is analytic for all complex numbers whose real part $x > 0$. As previously stated, the zeros of RZF and DEF are all same.

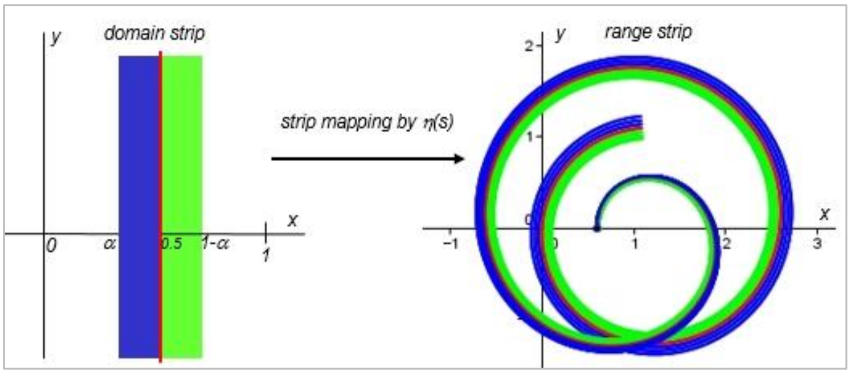
Definition 3.1. Domain strip: A strip $\alpha \le x \le 1 - \alpha, 0 < \alpha < 0.5$, $-\infty \le y \le \infty$. Let's assume for some β, $\eta(\alpha + i\beta) = \eta(1 - \alpha + i\beta) = 0$.

Definition 3.2. Strip mapping: The mapping of the domain strip by DEF.

Definition 3.3. Range strip: A strip generated by the strip mapping.

Figure 2 depicts an example strip mapping for a domain strip $0.44 \le x \le 0.54$, $0.44 \le x < 0.5$ (blue strip), $x = 0.5$ (red line), $0.5 < x < 0.54$ (green strip), and for $0 \le y \le 15$.

Figure 2. A strip mapping example.



Definition 3.4. Edge lines: Two edge lines of a strip, which are $x = \alpha$ and $x = 1 - \alpha$ or $\eta(\alpha + iy)$ and $\eta(1 - \alpha + iy)$.

Definition 3.5. Inside strip: The inside of a stripe which is bounded by two edge lines.

For later use, let's prove the monotonously increasing property of DEF with respect to $x, 0 < x < 1, y = 0$.

Lemma 3.6. $f(x) = a^{-x}, a > 1$ is a monotonously decreasing function for $0 < x < 1$.

Proof. $f'(x) = -(\ln a)a^{-x} < 0$, so, $f(x)$ is a monotonously decreasing function. ■

Lemma 3.7. $g(x) = (\ln a)a^{-x}, a > 1$ is a monotonously decreasing function for $0 < x < 1$.

Proof. $g'(x) = -(\ln a)^2 a^{-x} < 0$, so, $g(x)$ is a monotonously decreasing function. ■

Lemma 3.8. The mapping of a line segment $\alpha \le x \le 1 - \alpha, y = 0, 0 < \alpha < 0.5$, by DEF is one to one mapping.

Proof. Let's show that $\eta(x),\ 0 < x < 1,$ monotonously increases.

$$\eta(x) = \left(\frac{1}{1^x} - \frac{1}{2^x}\right) + \left(\frac{1}{3^x} - \frac{1}{4^x}\right) + \left(\frac{1}{5^x} - \frac{1}{6^x}\right) + \cdots.$$

$$\frac{d}{dx}\left(\frac{1}{n^x} - \frac{1}{(n+1)^x}\right) = -(\ln n)n^{-x} + \ln(n+1)\,(n+1)^{-x}.$$

By Lemma 3.7, $(\ln n)n^{-x}$ monotonously decreases for $n > 1$. So,

$$(\ln n)n^{-x} > \ln(n+1)\,(n+1)^{-x}.$$

$$(\ln n)n^{-x} - \ln(n+1)\,(n+1)^{-x} > 0.$$

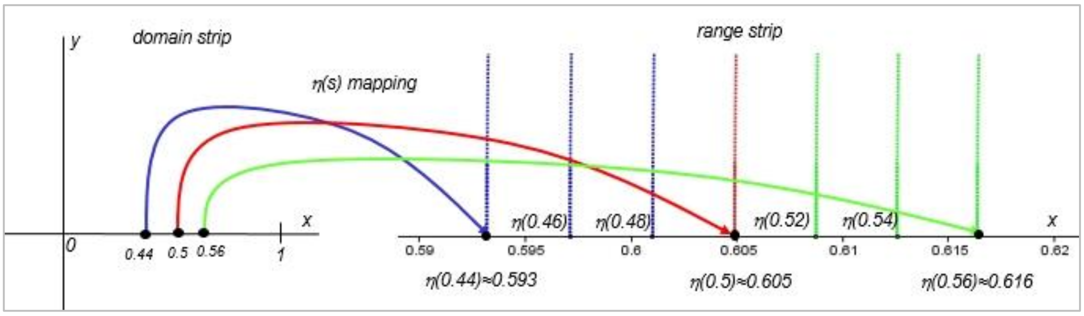
$$\eta'(x) = \ln 1(1)^{-x} + \{\ln(2)\,(2)^{-x} - (\ln 3)31^{-x}\} + \{\ln(4)\,4^{-x} - (\ln 5)5^{-x}\} + \cdots$$

$$= 0 + \{\ln(2)\,(2)^{-x} - (\ln 3)3^{-x}\} + \{\ln(4)\,4^{-x} - (\ln 5)5^{-x}\} + \cdots > 0.$$

Because $\eta'(x) > 0$, $\eta(x)$ monotonously increases, meaning that the mapping of a line segment $\alpha \le x \le 1 - \alpha, y = 0, 0 < \alpha < 0.5,$ by DEF is one to one mapping. ■

Figure 3 depicts the enlarged *x*-axis of the range strip in Figure 3. We can see the value of $\eta(x)$ monotonously increases as *x* increases.

Figure 3. An example of the monotonous increase of $\eta(x)$.



Lemma 3.9. To achieve $\eta(\alpha + i\beta) = \eta(1 - \alpha + i\beta) = 0, 0 < \alpha < 0.5$, two range strip edge lines should intersect at the origin when $y = \beta$, while y moves from $y = 0$ to $y = \beta$.

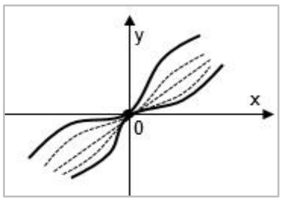
Proof. If two range strip edge lines do not intersect or do not intersect at the origin, obviously, $\eta(\alpha + i\beta) = \eta(1 - \alpha + i\beta) = 0$ can't be achieved. So, two edge lines must intersect at the origin. ■

Lemma 3.10. There are four patterns of the range strip edge line intersection, as shown in Figures from 4 to 7.

Proof. The geometries of the possible edge line intersecting cases are as follows.

1) **Shrink pattern:** The width of the range strip shrinks to zero at the origin when $y = \beta$, as in Figure 4.

Figure 4. Shrink pattern.



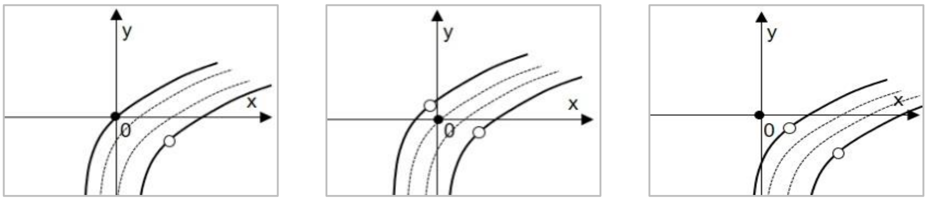
2) **Jump patterns:** One or two of the edge line(s) discontinuously jump(s) to the origin when $y = \beta$, as in Figure 5, (a) ~ (c) .

   (a) One edge line passes the origin while the other jumps to the origin, when $y = \beta$.

   (b) Two edge lines jump to the origin inside of the range strip, when $y = \beta$.

   (c) Two edge lines jump to the origin outside of the range strip, when $y = \beta$.

Figure 5. Jump patterns.



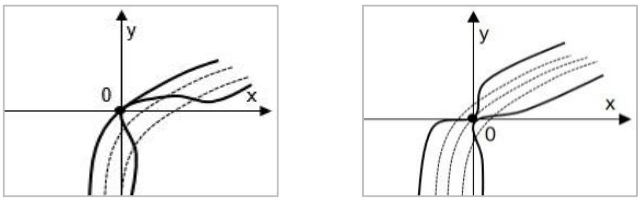
(a) Jump pattern 1 (b) Jump pattern 2 (c) Jump pattern 3

3) **Cross pattern:** One or two of the range strip edge line(s) begin(s) to step into the inside strip at some $y < \beta$, and intersect at the origin when $y = \beta$, as in Figure 6, (a) and (b) .

   (a) One edge line passes the origin while the other crosses into the range strip.

(b) Two edge lines crosses into the range strip.

Figure 6. Cross patterns.



(a) Cross pattern 1 (b) Cross pattern 2

4) **Hybrid pattern:** The mixed patterns of 1), 2) and 3) as in Figure 7, (a) ~ (d).

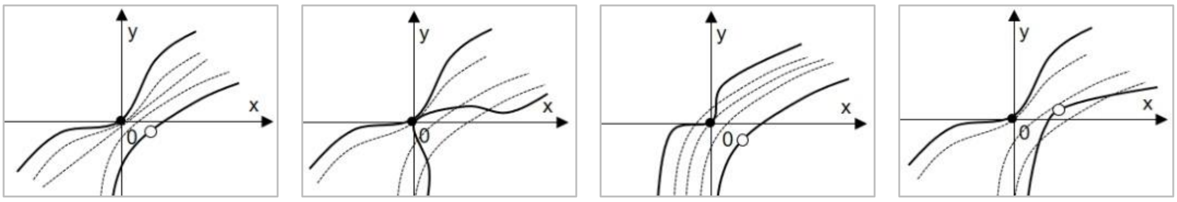
   (a) A mixed pattern of the squeeze and the jump pattern.

   (b) A mixed pattern of the squeeze and the cross pattern.

   (c) A mixed pattern of the jump and the cross pattern.

   (d) A mixed pattern of the three patterns.

Figure 7. Hybrid patterns.



(a) Squeeze-jump pattern (b) Squeeze-cross pattern (c) Jump-cross pattern (d) Squeeze-jump-cross pattern

It is obvious that there can't be other patterns. ■

4. Proof of RH

Lemma 4.1. To make two range strip edge lines intersect at the origin, at least one of the four patterns in Lemma 3.10 must be satisfied. But all four patterns can't be satisfied. So, RH is true.

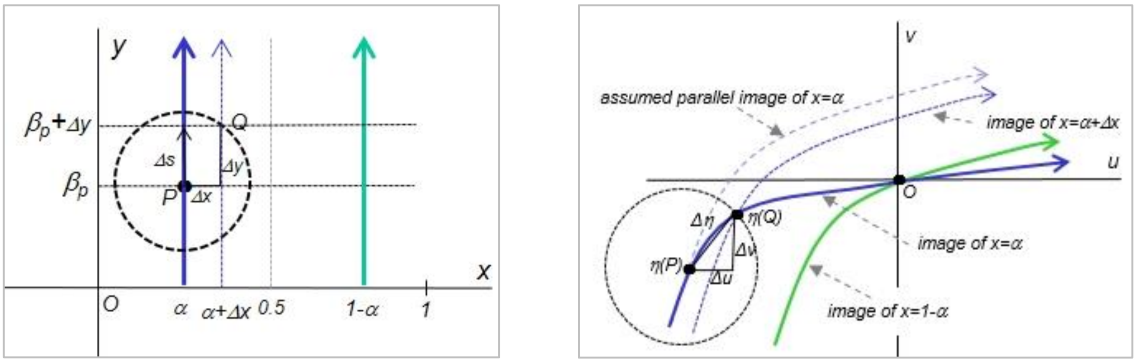
Proof. For each patterns there exists at least one contradiction.

1) **Shrink pattern:** If the width of the range strip shrinks to zero at the origin when $y = \beta$, it means that $\eta(x + i\beta) = 0,$ for all x in $\alpha \leq x \leq 1 - \alpha$, which contradicts.

2) **Jump pattern:** If one of or two of the range strip edge lines jump to the origin when $y = \beta$, it contradicts to the fact that an analytic function is continuous for all values in the domain strip, $\alpha \leq x \leq 1 - \alpha$, $-\infty \leq y \leq \infty$.

3) **Cross pattern:** As in Lemma 3.9, starting from $y = 0$, there must be a moment when the range strip edge line begins to step into the inside of the range strip at some point $y < \beta_p$, as in Fig. 8. It leads to the conclusion that $\eta(s) = constant = 0$, as follows.

Figure 8. Derivative view of cross pattern.



(a) Domain strip

(b) Range strip

The derivative [10][11] of $\eta(s)$ at a point P, $s = \alpha + i\beta_p$, is defined by

$$\eta'(s) = \lim_{\Delta s \to 0} \frac{\eta(s+\Delta s) - \eta(s)}{\Delta s} \tag{4.1}$$

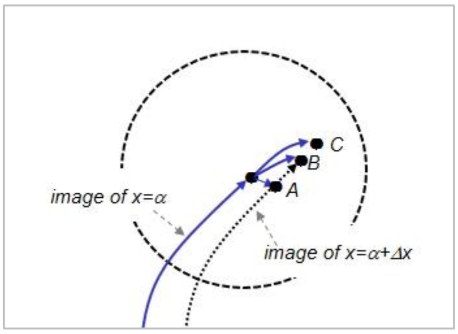
for all neighborhood of a disk centered at P.

Let's assume the image of $x = \alpha$ approaches to the image of $x = 1 - \alpha$, which crosses the origin, and the two image lines meet at the origin. When the image of $x = \alpha$ begins to step into the inside of the range strip at some point P, $s = \alpha + i\beta_p$, there should be a moment when the image of $x = \alpha$ meet the image of $x = \alpha + \Delta x$, at $y = \beta_p + \Delta y$. Here, $\Delta s = i\Delta y$, which means that there is no x movement, i.e., $\Delta x = 0$. So,

$$\eta(s + i\Delta y) = \eta(s + \Delta x + i\Delta y_1), \Delta x \neq 0. \tag{4.2}$$

The reason we used Δy_1 instead of Δy, is because there are three cases when the image of $x = \alpha$ steps into the range strip, as in Figure 9.

Figure 9. Three cases of the (4.2).



Case A: The image of $x = \alpha$ steps into the range strip and meet the past image of $x = \alpha + \Delta x$, so, $\Delta y_1 < \Delta y$.

Case B: The image of $x = \alpha$ steps into the range strip and meet the image of $x = \alpha + \Delta x$, so, $\Delta y_1 = \Delta y$.

Case C: The image of $x = \alpha$ steps into the range strip but can't meet the future image of $x = \alpha + \Delta x$ because $\Delta y_1 > \Delta y$. But as Δy reaches to Δy_1 the image of $x = \alpha + \Delta x$ will meet the image of $x = \alpha$.

In (4.2) let's $t = s + i\Delta y$, then we can rewrite (4.2) as (4.3) through following steps.

$$t = s + i\Delta y$$

$$s + \Delta x + i\Delta y_1 = s + (i\Delta y - i\Delta y) + \Delta x + i\Delta y_1$$

$$= (s + i\Delta y) + \Delta x + i(\Delta y_1 - \Delta y)$$

$$= t + \Delta x + i\Delta y_2, \Delta y_2 = \Delta y_1 - \Delta y$$

$$= t + \Delta t, \Delta t = \Delta x + i\Delta y_2 \neq 0.$$

$$\eta(t) = \eta(t + \Delta t). \tag{4.3}$$

$$\eta(t + \Delta t) - \eta(t) = 0.$$

$$\frac{\eta(t+\Delta t) - \eta(t)}{\Delta t} = 0, \ \Delta t \neq 0.$$

$$\frac{d\eta(t)}{dt} = \lim_{\Delta t \to 0} \frac{\eta(t+\Delta t) - \eta(t)}{\Delta t} = 0. \tag{4.5}$$

$$\eta(t) = constant. \tag{4.6}$$

The result (4.6) contradicts. Furthermore, if a constant function has a zero, the constant should be zero. So,

$$\eta(t) = constant = 0. \tag{4.7}$$

So, the two domain strip edge lines can't intersect when mapped by DEF, which is analytic on the domain strip. Of course, as in Figure 3, there can be such as, $\eta(\alpha + i\beta_1) = \eta(1 - \alpha + i\beta_2)$, when $\Delta\beta = \beta_1 - \beta_2$ is sufficiently large. But, when $\Delta\beta = \beta_1 - \beta_2 = 0$, i.e., $\beta = \beta_1 = \beta_2$, $\eta(\alpha + i\beta) = \eta(1 - \alpha + i\beta) = 0$ can't be achieved, except when $\eta(s) = constant = 0$.

4) **Hybrid pattern:** A hybrid pattern is the mixed patterns of the two or three of the patterns 1), 2) and 3), which can't happen. So, all hybrid patterns are also negated.

As a consequence, all four range strip edge line intersecting patterns can't be satisfied. So, RH is true. ■

5. Conclusion

in this thesis, we studied the symmetry properties of RZF zeros, stating that if there exists a zero whose real part is not 0.5, such as $\zeta(\alpha + i\beta) = 0, 0 < \alpha < 0.5$, also $\zeta(1 - \alpha + i\beta) = 0$, called critical line symmetry. We used the strip mapping concept to graphically clarify the meaning of the critical line symmetry of zeros. If there exist zeros which deviate the critical line, $\eta(\alpha - i\beta) = \eta(1 - \alpha + i\beta) = 0$ should be achieved. To verify whether it can be achieved, we identified four possible graphic patterns that may satisfy the critical line symmetry of zeros. We proved that all four graphic patterns can't achieve the critical line symmetry of zeros except when $\eta(t) = constant = 0$, which contradicts. So, RH is true.

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