Thoughts on the relationship between uncertainty, superposition and the measurement problem

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Abstract

A method is described to derive the inequality associated with the Heisenberg uncertainty principle which is valid for any object in orbit. A necessary condition for this derivation, and by implication for all such derivations, is that the position of the orbiting object as a function of time is single valued. As a consequence the existence of uncertainty is at odds with the idea of non-locality.

The presence of a harmonic series associated with the discrete energy levels of the atom must mean that there is a sampling process involved and it is this that leads to uncertainty. The position of the electron can only be related to the orbital period once per sample interval. Such a sampling process introduces ambiguity over observations of the position and velocity of the orbiting electron. In a state of superposition the electron does not exist everywhere, but is always following one of an infinite number of possible trajectories, but the precise trajectory cannot be determined since it is only accessible at intervals of a complete cycle. When it is observed the electron does not undergo any sort of physical transformation, instead such observation simply means that we are able to resolve which of the trajectories the electron is following

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The Uncertainty Principle

Werner Heisenberg formulated his uncertainty principle based on analysis of the dynamics of the hydrogen atom. Basically he arranged the terms for momentum and position in a grid or matrix, but when he came to manipulating them he discovered (or rediscovered) a quirk of matrix arithmetic: that multiplication is not commutative; in other words [A]*[B] ≠ [B]*[A]. The difference he ascribed to uncertainty; the idea that it is not possible to measure both momentum and position at the same time to an arbitrary degree of accuracy. There is always a tradeoff between these two measurements.

Eventually Niels Bohr adopted this idea as a way to circumvent the problem he had encountered with the quantum leap, arguing that uncertainty was somehow intrinsic to the electron. That it does not exist as a particle in the classical sense, but somehow is spread around in multiple places at once and that it is only when it is observed that it is transformed into a particle having both position and velocity. The electron is said to exist as a wave front in a state of quantum uncertainty or superposition, where it is deemed not to be located at a single point, but to occupy a region in space. When it is subject to an observing process, the state of uncertainty 'collapses' such that the electron manifests itself as a particle in a particular at a particular location. The measurement problem is a question of how and whether the wave front collapse occurs

To gain a complete understanding of the measurement problem it is first necessary to consider the so-called 'observer effect'. When making a measurement, it is necessary that the tools being used to make the measurement do not affect the measurement. The normal way to get around this problem is to ensure that the resolution of the measuring tool is much finer than the tolerance to which the measurement is being made. Unfortunately on the scale of the electron there are no such tools available. The only tools available are other electrons and photons and these are of the same order of magnitude as the electron being measured. The observer effect confounds any practical attempt to make measurements on this scale and it is often confused with the underlying problem of uncertainty, which is not directly related to the observer effect. While we cannot practically make measurements on this scale, we can imagine what is happening to the various particles involved as if we were able to do so.

Shortly before Heisenberg published his findings on uncertainty, Erwin Schrödinger had developed an equation which described the particle in terms of a wave. At the time there was an element of competition between Heisenberg and Schrödinger. The uncertainty principle gave Heisenberg a clear lead, however eventually Schrödinger was able to show that his wave equation could be used to derive the same expression for uncertainty as that of Heisenberg – and that in fact the two methods were equivalent. Schrödinger's method involves the use of wave theory and was therefore more easily understood by a generation of physicists brought up on waves – and so now it is the preferred method for demonstrating uncertainty.

The derivation of the uncertainty principle from Schrödinger's wave equation is rather complex and convoluted; however there is a far simpler way to derive the equation directly from the orbital motion of the electron. First as an equality and then by posing the question as to what would make this equality into an inequality.

We can describe the position of the orbiting electron in terms of two parametric equations, one in x and one in y.

$$x = R \cos(\omega t) \hspace{6cm} \text{Equation 1}$$

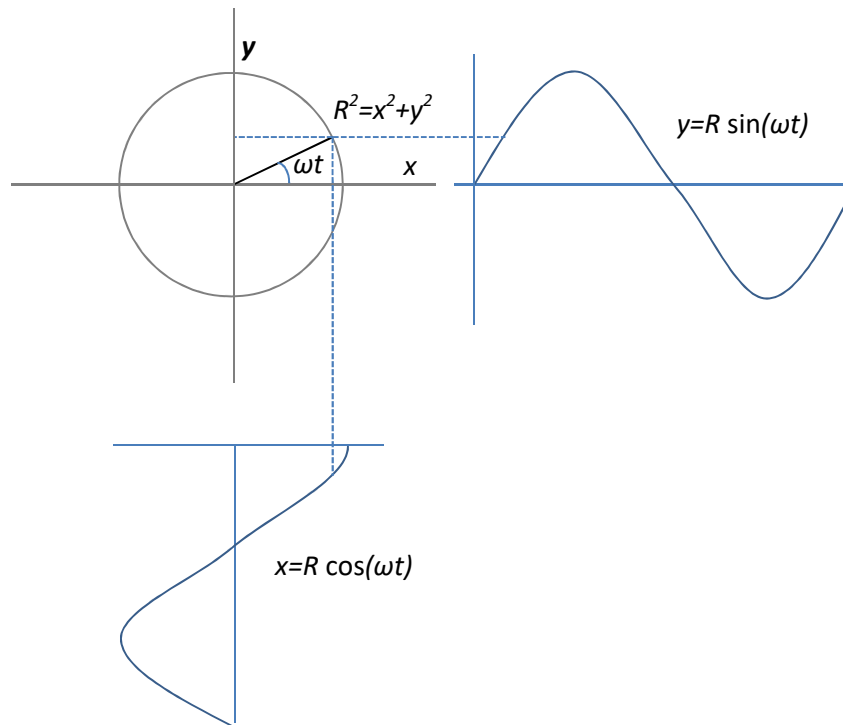$$y = R \sin(\omega t) \hspace{6cm} \text{Equation 2}$$

Figure 1 Parametric equations for the orbiting electron

The uncertainty equation is an inequality. It says that the average value of the position of the electron multiplied by its average momentum must always be greater than or equal to Planck's constant divided by 2.

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Equation 3

If we expand the terms this equation can be written as

$$\Delta x \Delta mv \geq \frac{mvr}{2}$$

Equation 4

We also note that

$$v = \frac{dx}{dt}$$

Equation 5

And so

$$v = -R\omega \sin(\omega t)$$

Equation 6

The derivation of the uncertainty makes use of the so called Cauchy Schwartz inequality[i]. This concerns the relationship between the expected values of the two variables and their product and is commonly written as

$$E(a)E(b) \geq E(ab)$$

Equation 7

Expected value, denoted by E, is a measure of the deviation of a variable from is average value and is sometimes referred to as standard deviation or in engineering terms as the Root Mean Square value of the variable.

In general the RMS value of a periodic variable which is a function of time is given by

$$\sigma f(t) = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} f(t)^2 dt}$$

The expected value of x is well understood in the world of signal processing and electrical engineering where it is referred to as the Root Mean Square (RMS) of the value.

$$\sigma(x) = \sqrt{\frac{r^2}{2\pi} \int_0^{2\pi} \cos^2(\omega t)\, d(\omega t)}$$

$$\sigma(x) = \sqrt{\frac{r^2}{2\pi} \int_0^{2\pi} \frac{1 - \sin(2\omega t)}{2} d(\omega t)}$$

$$\sigma(x) = \sqrt{\frac{r^2}{2\pi} \frac{1}{2} \left[\omega t \Big|_0^{2\pi} - \sin(2\omega t) \Big|_0^{2\pi} \right]}$$

$$\sigma(x) = \sqrt{\frac{r^2}{2\pi} \frac{1}{2} [2\pi - 0]}$$

$$\sigma(x) = \frac{r}{\sqrt{2}}$$

The calculation for momentum follows a similar argument, except for the presence of the constant m, the mass of the electron and the ω term which occurs as a result of differentiating the distance to get the velocity and gives

$$\sigma(p) = \frac{mr\omega}{\sqrt{2}}$$

The expected value for the product term follows a similar path but is slightly more complicated since it is the product of two sinusoids.

$$xmv = mr^2\omega \sin(\omega t) \cos(\omega t)$$

We can use the trigonometric identity for the product of sin and cos to give

$$xp = mr^2\omega \frac{1}{2} \sin(2\omega t)$$

And so

$$\sigma(xp) = mr^2\omega \sqrt{\frac{1}{4\pi} \int_0^{2\pi} \sin^2(2\omega t) d(\omega t)}$$ Equation 17

$$\sigma(xp) = mr^2\omega \sqrt{\frac{1}{4\pi} \int_0^{2\pi} \frac{1 - \cos(2\omega t)}{2} d(\omega t)}$$ Equation 18

$$\sigma(xp) = mr^2\omega \sqrt{\frac{1}{8\pi} \left[1 \Big|_0^{2\pi} + \sin(2\omega t) \Big|_0^{2\pi} \right]}$$ Equation 19

$$\sigma(xp) = mr^2\omega \sqrt{\frac{1}{8\pi} [2\pi + 0]}$$ Equation 20

$$\sigma(xp) = \frac{mr^2\omega}{4}$$ Equation 21

Recognizing that $\omega r = v$ and that $\hbar = mrv$ we can rewrite this as

$$\sigma(xp) = \frac{\hbar}{4}$$ Equation 22

Substituting into the Cauchy inequality gives

$$\frac{\sigma(p)}{\sqrt{2}} \frac{\sigma(x)}{\sqrt{2}} \geq \frac{\hbar}{4}$$ Equation 23

The two square root terms can be combined and cancelled to give the familiar form

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$ Equation 24

The product of the RMS value for the displacement and the RMS value for the speed multiplied by the mass of the orbiting body is always greater than or equal to half of the orbital angular momentum.

This is universally true for any orbiting body.

The inequality works equally well for all of the energy levels in the Bohr model for the atom, so if R is the Bohr radius and V the Bohr velocity we see that the n's on the LHS effectively cancel

$$\Delta R n^2 \Delta m \frac{V}{n} \geq \frac{n\hbar}{2}$$ Equation 25

From Equation 4 it is evident that the mass term is plays no part in the uncertainty. It is a constant. It is there on both sides of the inequality in order that the RHS can be expressed in terms of Planck's constant. Cancelling the mass term gives the more fundamental form which expresses the uncertainty between position and velocity for an orbiting body.

$$\Delta x \Delta v \geq \frac{vr}{2}$$ Equation 26

From this it can be seen that the real nature of uncertainty is between the velocity and position of the electron and not momentum and position or in general between a periodic variable and its derivative with respect to time.

The inequality is familiar to electrical engineers where the displacement, x, corresponds to the voltage in an AC circuit, and the velocity, v, corresponds to the current. The inequality is used to describe the power factor. The RMS value of the voltage In volts multiplied by the RMS value of the current in amps is always greater than or equal to the RMS value of the product of voltage and current; the power in watts. The ratio of these two quantities is termed the power factor and is given by

$$Power\ factor = \frac{W}{VA}$$
<div align="right">Equation 27</div>

Power factor is always less than or equal to one and is derived from the Cauchy Schwartz inequality in exactly the same way as with uncertainty,

$$E(volts)E(amps) \geq E(watts)$$
<div align="right">Equation 28[1]</div>

Since in the case of the orbiting body velocity is derived directly from the position, it begs the question: where is the uncertainty in all of this? In the electrical analogy the reduction in the power factor comes about because of the phase difference between the voltage and the current. In the terms on the LHS of the inequality, the phase is integrated out, whereas on the RHS the phase relationship is expressed by virtue of the multiplication of the two terms prior to the integration. However for an orbiting particle there is no such phase difference, the velocity is always going to be 90 degrees out of phase with the position and so we might expect the inequality to become equality and the uncertainty to evaporate.

In order to understand how such an uncertainty can come about we first need to look at the conditions which are necessary for this inequality to be true, in particular as they relate to the position or displacement variables. The inequality does not depend on the form of the periodic function, only on certain criteria that must be met. In order for the inequality to be valid it is necessary that the position variable is periodic, it must be single valued and piecewise continuous and differentiable[2][ii]. The fact that it must be single valued is of particular interest, since it is not consistent with the idea that the electron can be in more places than once at any one time. Within its orbit, the electron must be somewhere, it can be anywhere, but it cannot be everywhere. In short, if uncertainty exists then the electron must be localized to a single point.

The work of Schrödinger and Heisenberg was based on the idea of the electron as both a wave and a particle, first put forward by Louis de Broglie. One of the consequences of this idea is that there is a harmonic relationship between the various energy levels of the hydrogen atom. In de Broglie's model this is manifest as a series of standing waves., that is to say that a whole number of de wavelengths of these de Broglie waves lies with the orbital circumference of the Bohr model.

Wherever we see a harmonic series in nature there must always be a corresponding sampling process. This becomes evident if we consider the Fourier representation of a harmonic series. Such a Fourier representation comprises a series of spikes equally spaced along the frequency axis. For a real function these are disposed equally on both the positive and negative frequency axes. These spikes are referred to as Dirac or Delta functions and such a collection of equally spaced Dirac functions is referred to as a Dirac comb.

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[1] Note that the absence of the factor ½ in this equation is because voltage and current are in phase with one another whereas velocity and position are 90 degrees out of phase.
[2] Piecewise continuous means that there can be discontinuities in the velocity function but not in the position function as for example if the position were a triangle wave, then the velocity function would be a square wave.
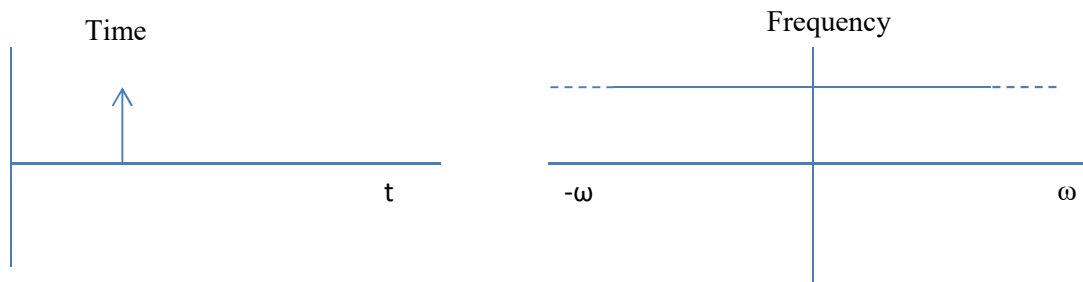
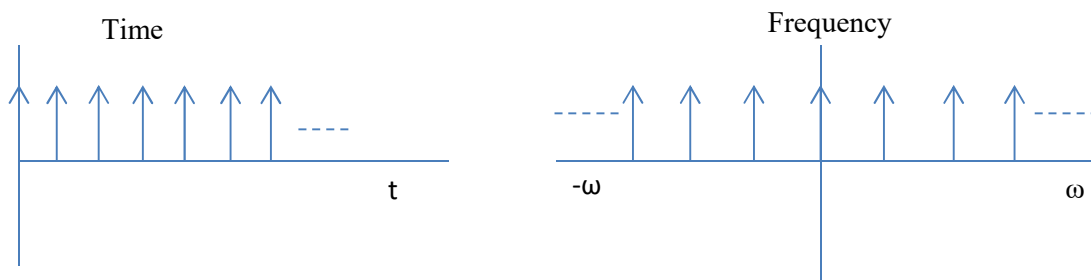Figure 2 Fourier transform of a single Dirac function
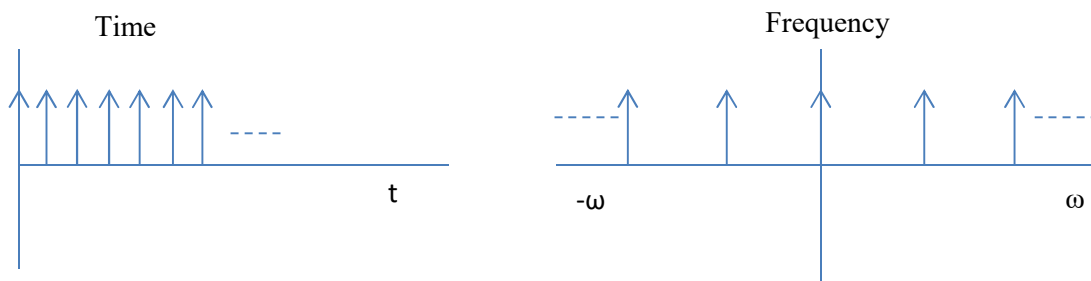

Figure 3 Fourier transform of a Dirac comb

Figure 4 Fourier transform of a higher frequency Dirac comb

The Fourier transform of a Dirac comb in the frequency domain is a function in the time domain that is itself another Dirac comb[iii]. Such a Dirac comb in the time domain can be regarded as a sampling function, since if it is multiplied by any other signal it effectively takes a sample at regular intervals in time. The sampling frequency corresponds to the lowest frequency in the harmonic series, which in the case of the atom is the frequency of the lowest energy state of the atom. It follows that the sampling rate takes place once per orbit at this frequency and it is this that provides the mechanism which brings about uncertainty. The distance travelled can only be related to the orbital period once per orbit, but the distance travelled is ambiguous. It is impossible to know how many orbits the electron has completed since the last sample was taken.

The requirement for the position of the electron to be single valued means that it always has a deterministic position, but that we cannot determine exactly what that is because of this ambiguity associated with the sampling process. The situation is shown in Figure 5 where it can be seen that the electron can be thought of as always following one of an infinite number of possible trajectories (of which only the first five are shown), sometimes referred to as aliases, appropriate to its energy level,

but we do not know which one. We cannot increase the sampling frequency because it is meaningless to try to relate the distance travelled to the time taken except over a complete orbital period.
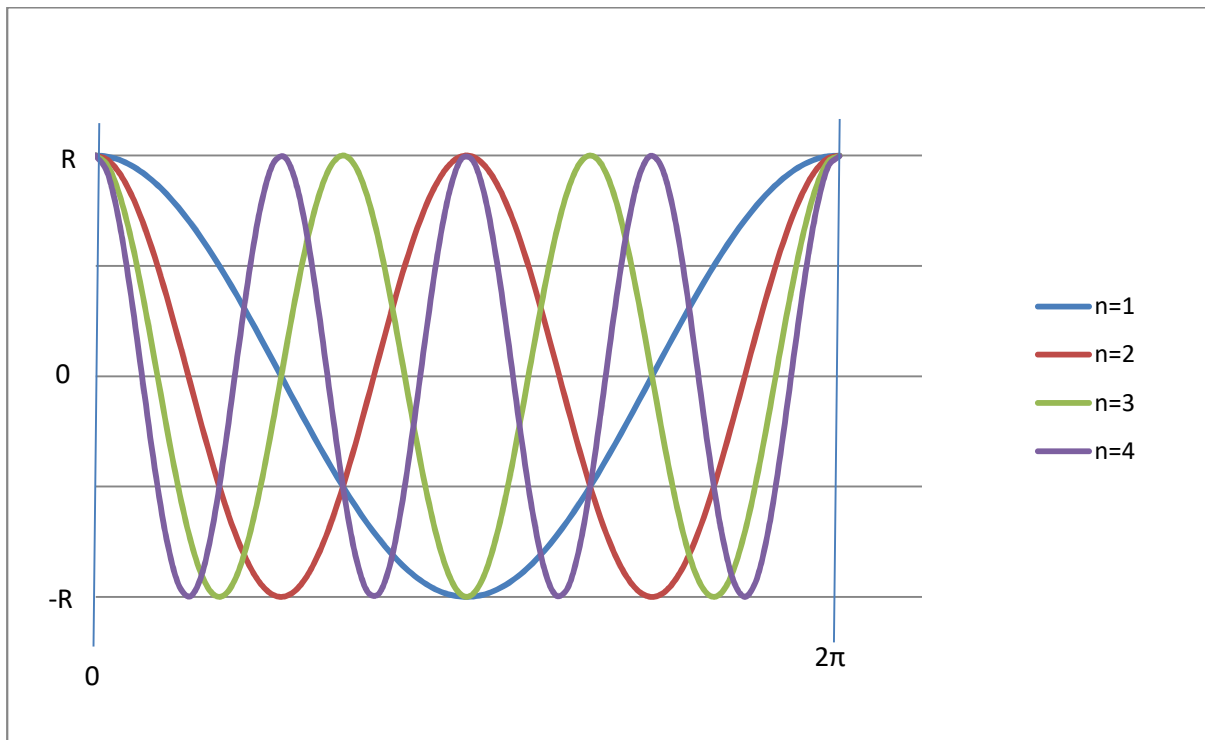

Figure 5 Electron Trajectories

The density of these infinite number of trajectories is such that the entire space is covered by at least one such trajectory and it is this that leads to the idea that the electron can be in more than one place at a the same time.

This gives us a slightly different view of the state of quantum indeterminacy or superposition. Superposition here is when the electron is in this state of it not being known which trajectory is being followed. When subject to an observing process there are no structural changes to the electron; instead the properties of the electron, such as speed, spin etc., are affected by the observing process due to the observer effect but the electron itself remains unaltered. The changes that take place are to the properties of the electron and those of the observer, not the electron itself. Hence the measurement problem does not really exist.

The observing process is inextricably linked to uncertainty, but is not the cause of it. An observing process does not necessarily mean that a human agent is involved. Consider the case of an isolated hydrogen atom in a high energy state. At some time and for whatever reason, it decays into a lower energy state and releases a photon. Such an interaction can be considered an observing process. Prior to the decay the state of the atom was indeterminate. After the decay the photon is released with a certain amount of energy and thus the photon is aware of how much energy was released by the decay. A closer examination of the Rydberg formula reveals that the amount of energy associated with any particular transition is unique and so we can identify both the energy state prior to the transition and that after the transition.   If we know the energy state of the electron we also know the corresponding alias and so we now know which trajectory the electron was following prior to the transition and which it is following after the transition. In effect, the photon has discovered the energy level of the atom prior to and after the decay event. A consequence of the observer effect is that every measurement process must involve a change in the physical properties of the object being observed, it does not mean that the observed object itself changes, just its properties such as velocity, spin, polarization etc. Uncertainty can only be resolved when an interaction takes place and that necessarily causes a change in the object being observed.

Conclusions

An examination of how the Cauchy Schwartz inequality can be applied to the problem of the uncertainty principle reveals that the position of the electron must be a single valued function of time. This in turn means that there is a fundamental incompatibility between the current view of uncertainty and non-locality. If uncertainty exists and obeys the Cauchy Schwartz inequality then the electron must be localized, that is as a particle at a single point in space and time.

The presence of a harmonic relationship between the various energy levels of the atom means that there has to be a sampling process and that the sample interval is the orbital period of the base energy state. Such a sampling process means that it is only possible to relate the distance travelled to the time taken once per orbit and that there is ambiguity over the distance travelled. The electron is seen to be following one of an infinite number of possible trajectories, each of which can be thought of as a wave. This can be regarded as a state of superposition. It is only when the electron interacts in some way with another particle that it is possible to determine which of these paths is being followed and this is equivalent to the collapsing wave front of the current theory.

Between the sampling instants the value representing the distance the electron has travelled is both unknown and unknowable to us. Such uncertainty is systematic in nature as opposed to intrinsic. That is to say that the electron always has a definite position and a definite velocity, it is just that we are not able to observe both to an arbitrary degree of accuracy because the system of sampling prevents us from doing so.

When Heisenberg first set out the uncertainty principle, he ascribed uncertainty entirely to the observer effect; in other words he felt that the process of making the one measurement necessarily affected the other and that was the complete story. Later he was persuaded that uncertainty was somehow intrinsic to the electron. That is, the electron did not exist as a discrete particle. Here we see that the two effects are inseparable. The state of the electron is unknown until it is involved in some sort of interaction. Such an interaction necessarily involves an observation process and incurs the observer effect and that the state of the electron must change in some way as a result of the interaction. The outcome, however, is that the state of the electron is revealed to the observer.

[i] Joseph K Blitzstein, Jessica Hwang, *Introduction to Probability, Second Edition,* pp 458-459, Chapman& Hall, 2019 ISBN 13: 9781138369917

[ii] Knopp, K. *Theory of Functions Parts I and II, Two Volumes Bound as One.* New York: Dover, Part I p. 103 and Part II p. 93, 1996.

[iii] Bracewell, R.N. (1986), *The Fourier Transform and Its Applications (revised ed.)*, McGraw-Hill; 1st ed. 1965, 2nd ed. 1978