Covariant and Consistent Anomalies in Supergravity

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Abstract

The scientific article reviews some properties of the low-energy effective actions for gauged supergravity models. We summarize the current state of knowledge regarding gravity theories with minimal supersymmetry. We provide an elegant extension of the theory and give a deffinitions of the anomalies in gauged supergravity. The difference between the covariant and consistent anomalies is carefully explained in terms of their different origins. The gauge structure of anomalies and the related supersymmetry currents is analyzed in detail. The results are extended to determine the structure of gravitational and mixed anomalies in supergravity. The deep relation between anomalies and inconsistency is emphasized in this research. The conditions for anomaly cancellation in these supergravity theories typically constitute determined types of equations.

1 Introduction

Supersymmetric theories are highly symmetric and magnificent beautiful. These symmetries exchange fermions with bosons, either in flat space supersymmetry or in curved space supergravity. Supergravity is a field theory, at present the most promising candidate for quantum gravity that combines the principles of supersymmetry and general relativity. Supergravity describes general relativity in the language of quantum field theory. The theory of supergravity suggested a novel approach to unification. In the quest for a unified description of gravity and matter interactions, several higher dimensional theories have been proposed. The gauged supergravity provides an interesting theoretical framework to the physics beyond the standard model. Gauged supergravities, where the global isometries of the matter lagrangian are promoted to local symmetries, have been widely explored and by now almost all allowed models for diverse spacetime dimensions. The geometry of curved superspace is shown to allow the existence of a large family of supermultiplets that can be used to describe supersymmetric matter, including vector, tensor and hypermultiplets. The main theories of interest in this publication are four and six-dimensional supergravities with a minimal amount of supersymmetry and all possible anomaly-free models. In the first part of our work we use the symplectic structure of four-dimensional minimal supergravities to study the possibility of gauged axionic shift symmetries. This leads to the introduction of generalized Chern-Simons terms, and a Green-Schwarz cancellation mechanism for gauge anomalies. Similarly, we study the possibility of adding higher order derivative corrections to the two-derivative action, leading to a cancellation of the mixed gauge-gravitational anomalies. Our models constitute the supersymmetric framework for string compactifications with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies. We discuss the presence of gauge and gravitational anomalies in theories with $N = 1$ local supersymmetry and a conventional gauging. We present a Green-Schwarz mechanism that involves Peccei-Quinn terms, generalized Chern-Simons terms, higher order derivative corrections and appropriate gauge transformations of the scalar fields. We discuss the mutual consistency conditions for all these ingredients, such that the theory is anomaly-free. In this paper we construct the complete coupling of (1,0) supergravity to all possible (1,0) multiplets, generalizing the results in order to include hypermultiplets, and extending the results to all orders in the fermi fields, while taking into account the anomalous couplings. We show that the inclusion of charged hypermultiplets gives additional terms in the supersymmetry anomaly. As was the case without hypermultiplets, the resulting theory is determined up to a quartic coupling for the gauginos, and correspondingly the supersymmetry algebra contains an extension that guarantees the consistency of the construction. We will consider the case in which the scalars in the hypermultiplets parametrize the coset, and we will describe the gauging of the full compact subgroup of the isometry group. Other cases, in which the scalars parametrize more general quaternionic symmetric spaces or are charged with respect to different subgroups of the isometry group, can be straightforwardly obtained from our results. We construct the complete coupling of (1, 0) supergravity in six dimensions to tensor multiplets, extending previous results to all orders in the fermi fields. We then add couplings to vector multiplets, as dictated by the generalized Green-Schwarz mechanism. The resulting theory embodies factorized gauge and supersymmetry anomalies, to be disposed of by fermion loops, and is determined by corresponding Wess-Zumino consistency conditions, aside from a quartic coupling for the gaugini. In addition, we show how to revert to a supersymmetric formulation in terms of covariant field equations that embody corresponding covariant anomalies. The subsequent work of some authors has developed the consistent formulation, but one can actually revert to a covariant formulation, at the price of having non-integrable field equations. The relation between the two sets of equations is one more instance of the link between covariant and consistent anomalies in field theory. This is a remarkable laboratory for current algebra, where one can play explicitly with anomalous symmetries and their consequences. The supersymmetry algebra contains a corresponding extension that plays a crucial role for the consistency of the construction. Whereas gauge and supersymmetry anomalies occur in theories with global or local supersymmetry, mixed anomalies are specific for gauged supergravities. They manifest themselves as a non-invariance of the effective action under local Lorentz transformations. Mixed anomaly usually refers to a mixture of a gauge and gravitational anomaly. These structures appear in gauge current anomalies and lead to inconsistency unless cancelled. It is common practice to attempt to cancel them by adding new fermions to the model. We have found new anomaly structures which involve scalar fields, and these can require new independent cancellation conditions. We find an intricate interplay between the gaugings and certain quantum aspects of the theory. More precisely, we obtain the general cancellation conditions for quantum anomalies, using a Green-Schwarz mechanism. We identify a number of models which obey all known low-energy consistency conditions, but which have no known string theory realization. Many of these models contain novel matter representations, suggesting possible new superstring theory constructions. We hope that the variety of new apparently consistent supergravity models identified in the advanced research will stimulate some further understanding of new superstring realizations or will help to generate new constraints on quantum theories of gravity.

2 Kinetic Action of Supergravity

In generic low energy effective field theories, gauge fields appear with non-minmal kinetic terms in which the field strengths may multiply scalar field dependent coefficients. An important example of such dependence is provided by non-minimal kinetic terms for gauge fields

$$e^{-1}\mathcal{L}_1 = -\tfrac{1}{4}\,\mathrm{Re}\, f_{AB}\mathcal{F}^A_{\mu\nu}\mathcal{F}^{\mu\nu\,B} + \tfrac{1}{4}\mathrm{i}\,\mathrm{Im}\, f_{AB}\mathcal{F}^A_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu\,B}\,, \tag{1}$$

where the gauge kinetic function $f_{AB}(z)$ is a nontrivial function of the scalar fields, z^i, which, in $\mathcal{N}=1$ supersymmetry, has to be holomorphic. The second term in (1) is often referred to as the Peccei-Quinn term. Under gauge transformation with gauge parameter $\Lambda^A(x)$, some of the z^i transform nontrivially, this may induce a corresponding gauge transformation of $f_{AB}(z)$. If this transformation is of the form of a symmetric product of two adjoint representations of the gauge group,

$$\delta(\Lambda) f_{AB} = \Lambda^C\delta_C f_{AB}\,, \qquad \delta_C f_{AB} = f_{CA}{}^D f_{BD} + f_{CB}{}^D f_{AD}\,, \tag{2}$$

with $f_{CA}{}^B$ the structure constants of the gauge group, the kinetic term (1) is obviously gauge invariant. This is what was assumed in the action of general matter-coupled supergravity. If one takes into account also other terms in the (quantum) effective action, however, a more general transformation rule for $f_{AB}(z)$ may be allowed:

$$\delta_C f_{AB} = \mathrm{i}C_{AB,C} + f_{CA}{}^D f_{BD} + f_{CB}{}^D f_{AD}\,. \tag{3}$$

Here, $C_{AB,C}$ is a constant real tensor symmetric in the first two indices, which we will recognize as a natural generalization in the context of symplectic duality transformations.

If $C_{AB,C}$ is non-zero, this leads to a non-gauge invariance of the Peccei-Quinn term in $\mathcal{L}_1$:

$$\delta(\Lambda)e^{-1}\mathcal{L}_1 = \tfrac{1}{4}\mathrm{i}C_{AB,C}\Lambda^C\mathcal{F}^A_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu\,B}\,. \tag{4}$$

For rigid parameters, $\Lambda^A=$ const., this is just a total derivative, but for local gauge parameters, $\Lambda^A(x)$, it is obviously not. If (1) is part of a supersymmetric action, the gauge non-invariance (64) also induces a non-invariance of the action under supersymmetry.

The vector multiplet in the $\mathcal{N} = 1$ superspace formulation is described by a real superfield. The latter has many more components than the physical fields describing an on-shell vector multiplet, which consists of one vector field and one fermion. The advantage of this redundancy is that one can easily construct manifestly supersymmetric actions as integrals over full or chiral superspace. As an example consider the expression

$$S_f = \int \mathrm{d}^4x \mathrm{d}^2\theta \, f_{AB}(X) W_\alpha^A W_\beta^B \varepsilon^{\alpha\beta} + \; c.c. \tag{5}$$

Here, $W_\alpha^A = \frac{1}{4}\bar{D}^2 D_\alpha V^A$, or a generalization thereof for the non-Abelian case, where V^A is the real superfield describing the vector multiplets labelled by an index A. The f_{AB} are arbitrary holomorphic functions of a set of chiral superfields denoted by X.

The integrand of (5) is itself a chiral superfield. As we integrate over a chiral superspace, the Lagrangian transforms into a total derivative under supersymmetry. Formally, this conclusion holds independently of the gauge symmetry properties of the functions $f_{AB}(X)$. For the action (5) to be gauge invariant, we should have the condition

$$\delta_C f_{AB} - f_{CA}{}^D f_{DB} - f_{AD} f_{CB}{}^D = 0\,, \tag{6}$$

where δ_C denotes the gauge transformation under the gauge symmetry related to the vector multiplet denoted by the index C.

Due to the large number of fields in the superspace formulation, the gauge parameters are not just real numbers, but are themselves full chiral superfields. To describe the physical theory, one wants to get rid of these extra gauge transformations and thereby also of many spurious components of the vector superfields. This is done by going to the so-called Wess-Zumino gauge, in which these extra gauge transformations are fixed and many spurious components of the real superfields are eliminated. Unfortunately, the Wess-Zumino gauge also breaks the manifest supersymmetry of the superspace formalism. However, a combination of this original supersymmetry and the gauge symmetries survives and becomes the preserved supersymmetry after the gauge fixing. The law that gives the preserved supersymmetry as a combination of these different symmetries is called decomposition law. Notice, however, that this preservation requires the gauge invariance of the original action (5). Thus, though (5) was invariant under the superspace supersymmetry for any choice of f_{AB}, we now need (6) for this action to be invariant under supersymmetry after the Wess-Zumino gauge.

This important consequence of the Wess-Zumino gauge can also be understood from the supersymmetry algebra. The superspace operator Q_α satisfies the anticommutation relation

$$\left\{ Q_\alpha, Q^\dagger_{\dot\alpha} \right\} = \sigma^\mu_{\alpha\dot\alpha} \partial_\mu\,. \tag{7}$$

This equation shows no mixing between supersymmetry and gauge symmetries. However, after the Wess-Zumino gauge the right-hand side is changed to

$$\left\{ Q_\alpha, Q^\dagger_{\dot\alpha} \right\} = \sigma^\mu_{\alpha\dot\alpha} \mathcal{D}_\mu = \sigma^\mu_{\alpha\dot\alpha} \left(\partial_\mu - W_\mu^A \delta_A \right)\,, \tag{8}$$

where δ_A denotes the gauge transformation. Equation (8) implies that if an action is invariant under supersymmetry, it should also be gauge invariant.

As mentioned before, the preservation of the Wess-Zumino gauges implies that the effective supersymmetry transformations are different from the ones in the original superspace formulation. The resulting supersymmetry transformations of a chiral multiplet are

$$\begin{aligned} \delta(\epsilon) z^i &= \bar\epsilon_L \chi^i_L\,, \\ \delta(\epsilon) \chi^i_L &= \tfrac{1}{2}\gamma^\mu \epsilon_R \mathcal{D}_\mu z^i + \tfrac{1}{2} h^i \epsilon_L\,, \\ \delta(\epsilon) h^i &= \bar\epsilon_R \mathcal{D}\!\!\!/\, \chi^i_L + \bar\epsilon_R \lambda^A_R \delta_A z^i\,, \end{aligned} \tag{9}$$

where we have denoted the scalar fields of the chiral multiplets as z^i, the left-chiral components of the corresponding fermions as χ_L^i and the auxiliary fields as h^i, while λ^A is the gaugino of the vector multiplet V^A. These transformations are valid for *any* chiral multiplet, in particular, they can be applied to the full integrand of (5) itself.

Compared to the standard superspace transformations, there are two modifications in (9). The first modification is that the derivatives of z^i and χ_L^i are covariantized with respect to gauge transformations. This covariant derivative acts on the chiral fermions χ_L^i as

$$\mathcal{D}_\mu \chi_L^i = \partial_\mu \chi_L^i - W_\mu^A \delta_A \chi_L^i \,. \tag{10}$$

Here, the gauge variation of the chiral fermions, $\delta_A \chi_L^i$, can be expressed in terms of the gauge variation, $\delta_A z^i$, of the scalar fields, using the fact that supersymmetry and gauge transformations commute,

$$\delta(\epsilon)\delta_A z^i = \delta_A \delta(\epsilon) z^i = \delta_A \bar{\epsilon}_L \chi_L^i = \bar{\epsilon}_L \delta_A \chi_L^i \,. \tag{11}$$

This leads to

$$\delta_A \chi^i = \frac{\partial \delta_A z^i}{\partial z^j} \chi^j \,. \tag{12}$$

The second modification is the additional last term in the transformation of the auxiliary fields h^i. The origin of this term lies in the contribution of the decomposition law for one of the gauge symmetries contained in the chiral superfield of transformations Λ, after the Wess-Zumino gauge is fixed.

To avoid the above-mentioned subtleties associated with the Wess-Zumino gauge, we will use component field expressions in the remainder of this text. Therefore, we reconsider the action (5) and in particular its integrand. The components of this composite chiral multiplet are

$$\begin{aligned} z(fW^2) &= -\tfrac{1}{2} f_{AB} \bar{\lambda}_L^A \lambda_L^B \,, \\ \chi_L(fW^2) &= \tfrac{1}{2} f_{AB} \left(\tfrac{1}{2} \gamma^{\mu\nu} \mathcal{F}_{\mu\nu}^A - \mathrm{i} D^A \right) \lambda_L^B - \tfrac{1}{2} \partial_i f_{AB} \chi_L^i \bar{\lambda}_L^A \lambda_L^B \,, \\ h(fW^2) &= f_{AB} \left(-\bar{\lambda}_L^A \mathcal{D} \lambda_R^B - \tfrac{1}{2} \mathcal{F}_{\mu\nu}^{-A} \mathcal{F}^{\mu\nu\, -B} + \tfrac{1}{2} D^A D^B \right) \\ &\quad + \partial_i f_{AB} \chi_L^i \left(-\tfrac{1}{2} \gamma^{\mu\nu} \mathcal{F}_{\mu\nu}^A + \mathrm{i} D^A \right) \lambda_L^B - \tfrac{1}{2} \partial_i f_{AB} h^i \bar{\lambda}_L^A \lambda_L^B + \tfrac{1}{2} \partial_{ij}^2 f_{AB} \bar{\chi}_L^i \chi_L^j \bar{\lambda}_L^A \lambda_L^B \,, \end{aligned} \tag{13}$$

where we used the notation $\partial_i = \frac{\partial}{\partial z^i}$. The superspace integral in (5) means that the real part of $h(fW^2)$ is (proportional to) the Lagrangian:

$$S_f = \int \mathrm{d}^4 x \ \mathrm{Re}\, h(fW^2) \,. \tag{14}$$

From (14) and (14), we read off the kinetic terms of S_f:

$$\begin{aligned} S_{f,\mathrm{kin}} &= \int \mathrm{d}^4 x \Big[-\tfrac{1}{4} \mathrm{Re}\, f_{AB} \mathcal{F}_{\mu\nu}^A \mathcal{F}^{\mu\nu B} - \tfrac{1}{2} \mathrm{Re}\, f_{AB} \bar{\lambda}^A \mathcal{D} \lambda^B \\ &\qquad + \tfrac{1}{4} \mathrm{i} \, \mathrm{Im}\, f_{AB} \mathcal{F}_{\mu\nu}^A \tilde{\mathcal{F}}^{\mu\nu B} + \tfrac{1}{4} \mathrm{i} (\mathcal{D}_\mu \mathrm{Im}\, f_{AB}) \bar{\lambda}^A \gamma^5 \gamma^\mu \lambda^B \Big] \,. \end{aligned} \tag{15}$$

We have used a partial integration to shift the derivative from the gaugini to $(\mathrm{Im}\, f_{AB})$ and rearranged the structure constants in the last term, so as to obtain a "covariant" derivative acting on $(\mathrm{Im}\, f_{AB})$. More precisely, we define

$$\mathcal{D}_\mu f_{AB} = \partial_\mu f_{AB} - 2 W_\mu^C f_{C(A}{}^D f_{B)D} \,. \tag{16}$$

In the case that the gauge kinetic matrix transforms without a shift, as in (6), the derivative defined in (16) is fully gauge covariant. The full covariant derivative has instead the new form

$$\hat{\mathcal{D}}_\mu f_{AB} \equiv \partial_\mu f_{AB} - W_\mu^C \delta_C f_{AB} = \mathcal{D}_\mu f_{AB} - \mathrm{i} W_\mu^C C_{AB,C} \,. \tag{17}$$

The last term in (15) is therefore not gauge covariant for non-vanishing $C_{AB,C}$. Hence, in presence of the new term in the transformation of f_{AB} we replace the action S_f with $\hat{S}_f$, in which we use the full covariant derivative, $\hat{\mathcal{D}}_\mu$, instead of $\mathcal{D}_\mu$. More precisely, we define

$$\hat{S}_f = S_f + S_{\text{extra}}\,, \qquad S_{\text{extra}} = \int \mathrm{d}^4x \left(-\tfrac{1}{4}\mathrm{i}W_\mu^C C_{AB,C}\bar{\lambda}^A\gamma_5\gamma^\mu\lambda^B \right)\,. \tag{18}$$

Note that we did not use any superspace expression to derive S_{extra} but simply added S_{extra} by hand in order to fully covariantize the last term of (15). As we will further discuss in the next section, S_{extra} can in fact only be partially understood from superspace expressions, which motivates our procedure to introduce it here by hand. We should also stress that the covariantization with S_{extra} does not yet mean that the entire action $\hat{S}_f$ is now fully gauge invariant. We would finally like to emphasize that, in the context of $\mathcal{N}=1$ supersymmetry, there is a priori no further restriction on the symmetry of $C_{AB,C}$ apart from its symmetry in the first two indices. This, however, is different in extended supersymmetry, as is most easily demonstrated for $\mathcal{N}=2$ supersymmetry, where the gauge kinetic matrix depends on the complex scalars X^A of the vector multiplets. These transform themselves in the adjoint representation, which implies

$$\delta(\Lambda) f_{AB}(X) = X^E \Lambda^C f_{EC}{}^D \partial_D f_{AB}(X)\,. \tag{19}$$

Hence, this gives

$$\mathrm{i}C_{AB,C} = X^E f_{EC}{}^D \partial_D f_{AB}(X) - f_{CA}{}^D f_{BD} - f_{CB}{}^D f_{AD}\,, \tag{20}$$

which leads to $C_{AB,C}X^A X^B X^C = 0$. As the scalars X^A are independent in rigid supersymmetry, this implies that $C_{(AB,C)} = 0$.

The action S_f is gauge invariant before the modification of the transformation of f_{AB}. In the presence of the $C_{AB,C}$ terms, the action $\hat{S}_f$ is not gauge invariant. However, the non-invariance comes only from one term. Indeed, terms in $\hat{S}_f$ that are proportional to derivatives of f_{AB} do not feel the constant shift $\delta_C f_{AB} = \mathrm{i}C_{AB,C} + \ldots$. They are therefore automatically gauge invariant. Also, the full covariant derivative (17) has no gauge transformation proportional to $C_{AB,C}$, and also $\mathrm{Re}\, f_{AB}$ is invariant. Hence, the gauge non-invariance originates only from the third term in (15). We are thus left with

$$\delta(\Lambda)\hat{S}_f = \tfrac{1}{4}\mathrm{i}C_{AB,C}\int \mathrm{d}^4x\, \Lambda^C \mathcal{F}_{\mu\nu}^A \tilde{\mathcal{F}}^{\mu\nu\, B}\,. \tag{21}$$

This expression vanishes for constant Λ, but it spoils the local gauge invariance.

We started to construct S_f as a superspace integral, and as such it would automatically be supersymmetric. However, we saw that when f_{AB} transforms with a shift as in (**??**), the gauge symmetry is broken, which is then communicated to the supersymmetry transformations by the Wess-Zumino gauge fixing. The $C_{AB,C}$ tensors then express the non-invariance of S_f under both gauge transformations and supersymmetry.

To determine these supersymmetry transformations, we consider the last line of (9) for $\{z^i, \chi^i, h^i\}$ replaced by $\{z(fW^2), \chi(fW^2), h(fW^2)\}$ and find

$$\delta(\epsilon)S_f = \int \mathrm{d}^4x\, \mathrm{Re}\left[\bar{\epsilon}_R \partial\!\!\!/ \chi_L(fW^2) - \bar{\epsilon}_R\gamma^\mu W_\mu^A \delta_A \chi_L(fW^2) + \bar{\epsilon}_R \lambda_R^A \delta_A z(fW^2) \right]\,. \tag{22}$$

The first term in the transformation of $h(fW^2)$ is the one that was already present in the superspace supersymmetry before going to Wess-Zumino gauge. It is a total derivative, as we would expect from the superspace rules. The other two terms are due to the mixing of supersymmetry with gauge symmetries. They vanish if $z(fW^2)$ is invariant under the gauge symmetry, as this implies by (11) that $\chi(fW^2)$ is also gauge invariant.

Using (14) and (**??**), however, one sees that $z(fW^2)$ is not gauge invariant, and (22) becomes, using also (12),

$$\delta(\epsilon)S_f = \int \mathrm{d}^4x \,\mathrm{Re}\left\{\mathrm{i}C_{AB,C}\Big[-\bar{\epsilon}_R\gamma^\mu W_\mu^C\left(\tfrac{1}{4}\gamma^{\rho\sigma}\mathcal{F}_{\rho\sigma}^A - \tfrac{1}{2}\mathrm{i}D^A\right)\lambda_L^B - \tfrac{1}{2}\bar{\epsilon}_R\lambda_R^C\bar{\lambda}_L^A\lambda_L^B\Big]\right\}. \tag{23}$$

Note that this expression contains only fields of the vector multiplets and none of the chiral multiplets. It remains to determine the contribution of S_{extra} to the supersymmetry variation, which turns out to be

$$\delta(\epsilon)S_{\mathrm{extra}} = \int \mathrm{d}^4x \,\mathrm{Re}\,\mathrm{i}C_{AB,C}\Big[-\tfrac{1}{2}W_\mu^C\bar{\lambda}_L^B\gamma^\mu\left(\tfrac{1}{2}\gamma^{\nu\rho}\mathcal{F}_{\nu\rho}^A - \mathrm{i}D^A\right)\epsilon_R - \bar{\epsilon}_R\lambda_R^B\bar{\lambda}_L^C\lambda_L^A\Big]. \tag{24}$$

By combining this with (23), we obtain, after some reordering,

$$\delta(\epsilon)\hat{S}_f = \int \mathrm{d}^4x \,\mathrm{Re}\left(\tfrac{1}{2}C_{AB,C}\varepsilon^{\mu\nu\rho\sigma}W_\mu^C\mathcal{F}_{\nu\rho}^A\bar{\epsilon}_R\gamma_\sigma\lambda_L^B - \tfrac{3}{2}\mathrm{i}C_{(AB,C)}\bar{\epsilon}_R\lambda_R^C\bar{\lambda}_L^A\lambda_L^B\right). \tag{25}$$

We describe how the addition of GCS terms and quantum anomalies can cancel the left-over gauge and supersymmetry non-invariances of equations (21) and (25).

3 Chern-Simons Action of Supergravity

Due to the gauged shift symmetry of f_{AB}, terms proportional to $C_{AB,C}$ remain in the gauge and supersymmetry variation of the action $\hat{S}_f$. To re-establish the gauge symmetry and supersymmetry invariance, we need two ingredients: GCS terms and quantum anomalies. They are of the form

$$S_{\mathrm{CS}} = \int \mathrm{d}^4x \,\tfrac{1}{2}C_{AB,C}^{(\mathrm{CS})}\varepsilon^{\mu\nu\rho\sigma}\left(\tfrac{1}{3}W_\mu^C W_\nu^A F_{\rho\sigma}^B + \tfrac{1}{4}f_{DE}{}^A W_\mu^D W_\nu^E W_\rho^C W_\sigma^B\right). \tag{26}$$

The GCS terms are proportional to a tensor $C_{AB,C}^{(\mathrm{CS})}$ that is symmetric in (A,B). Note that a completely symmetric part in $C_{AB,C}^{(\mathrm{CS})}$ would drop out of S_{CS} and we can therefore restrict $C_{AB,C}^{(\mathrm{CS})}$ to be a tensor of mixed symmetry structure

$$C_{(AB,C)}^{(\mathrm{CS})} = 0\,. \tag{27}$$

A priori, the constants $C_{AB,C}^{(\mathrm{CS})}$ need not be the same as the $C_{AB,C}$ introduced in the previous section. For $\mathcal{N}=2$ supergravity one needs them to be the same, but we will, for $\mathcal{N}=1$, establish another relation between both, which follows from supersymmetry and gauge invariance requirements.

The GCS terms can be obtained from a superfield expression

$$\begin{aligned}
S'_{\mathrm{CS}} &= C_{AB,C}^{(\mathrm{CS})}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\left[-\tfrac{2}{3}V^C\Omega^{AB}(V) + \left(f_{DE}{}^B V^C\mathcal{D}^\alpha V^A\bar{\mathcal{D}}^2\left(\mathcal{D}_\alpha V^D V^E\right) + \ c.c.\right)\right],\\
\Omega^{AB} &= \mathcal{D}^\alpha V^{(A}W_\alpha^{B)} + \bar{\mathcal{D}}_{\dot{\alpha}}V^{(A}\bar{W}^{\dot{\alpha}B)} + V^{(A}\mathcal{D}^\alpha W_\alpha^{B)}\,.
\end{aligned} \tag{28}$$

The full non-Abelian superspace expression (28) is valid only in the Wess-Zumino gauge, where it reduces to the bosonic component expression (26) plus a fermionic term

$$S'_{\mathrm{CS}} = S_{\mathrm{CS}} + (S'_{\mathrm{CS}})_{\mathrm{ferm}}\,, \qquad (S'_{\mathrm{CS}})_{\mathrm{ferm}} = \int \mathrm{d}^4x\left(-\tfrac{1}{4}\mathrm{i}C_{AB,C}^{(\mathrm{CS})}W_\mu^C\bar{\lambda}^A\gamma_5\gamma^\mu\lambda^B\right), \tag{29}$$

where we used the restriction $C^{(\mathrm{CS})}_{(AB,C)} = 0$ from (27).

Note that the fermionic term in (29) is of a form similar to S_{extra} in (18). More precisely, in (29) the fermions appear with the tensor $C^{(\mathrm{CS})}_{AB,C}$, which has a mixed symmetry, (27). S_{extra} in (18), on the other hand, is proportional to the tensor $C^{(s)}_{AB,C} + C^{(m)}_{AB,C}$. From this we see that if we identify $C^{(m)}_{AB,C} = C^{(\mathrm{CS})}_{AB,C}$, as we will do later, we can absorb the mixed part of S_{extra} into the superspace expression S'_{CS}. This is, however, not possible for the symmetric part of S_{extra} proportional to $C^{(s)}_{AB,C}$, which cannot be obtained in any obvious way from a superspace expression. As we need this symmetric part later, it is more convenient to keep the full S_{extra}, as we did in section **??**, as a part of $\hat{S}_f$, and not include $(S'_{\mathrm{CS}})_{\text{ferm}}$ here. Thus, we will further work with the purely bosonic S_{CS} and omit the fermionic term that is included in the superspace expression (28). We will show in the remainder of this subsection that for semisimple algebras the GCS terms do not bring anything new, at least in the classical theory. By this we mean they can be replaced by a redefinition of the kinetic matrix f_{AB}. This argument is not essential for the main result of this paper and the reader can thus skip this part. It shows, however, that the main application of GCS terms is for non-semisimple gauge algebras.

We start with the result

$$C^{(\mathrm{CS})}_{AB,C} = 2 f_{C(A}{}^{D} Z_{B)D}\,, \tag{30}$$

for a constant real symmetric matrix Z_{AB}, the action S_{CS} can be reabsorbed in the original action S_f using

$$f'_{AB} = f_{AB} + \mathrm{i} Z_{AB}\,. \tag{31}$$

In fact, one easily checks that with the substitution (30) in (**??**), the C-terms are absorbed by the redefinition (31). The equation (30) can be written as

$$C^{(\mathrm{CS})}_{AB,C} = T_{C,AB}{}^{DE} Z_{DE}\,, \qquad T_{C,AB}{}^{DE} \equiv 2 f_{C(A}{}^{(D}\delta^{E)}_{B)}\,. \tag{32}$$

In the case that the algebra is semisimple, one can always construct a Z_{AB} such that this equation is valid for any $C^{(\mathrm{CS})}_{AB,C}$:

$$Z_{AB} = C_2(T)^{-1}_{AB}{}^{CD} T_{E,CD}{}^{GH} g^{EF} C^{(\mathrm{CS})}_{GH,F}\,, \tag{33}$$

where g^{AB} and $C_2(T)^{-1}$ are the inverses of

$$g_{AB} = f_{AC}{}^{D} f_{BD}{}^{C}\,, \qquad C_2(T)_{CD}{}^{EF} = g^{AB} T_{A,CD}{}^{GH} T_{B,GH}{}^{EF}\,. \tag{34}$$

These inverses exist for semisimple groups. To show that (33) leads to (32) one needs (**??**), which leads to

$$g^{HD} T_H \cdot \left(\tfrac{1}{2} C^{(\mathrm{CS})}_{C} f_{DE}{}^{C} + T_{[D} \cdot C^{(\mathrm{CS})}_{E]} \right) = 0\,, \tag{35}$$

where we have dropped doublet symmetric indices using the notation $\cdot$ for contractions of such double indices. This further implies

$$g^{AB} T_E \cdot T_B \cdot C^{(\mathrm{CS})}_{A} = C_2(T) \cdot C^{(\mathrm{CS})}_{E}\,, \tag{36}$$

with which the mentioned conclusions can easily be obtained.

The GCS term S_{CS} is not gauge invariant. Even the superspace expression S'_{CS} is not gauge invariant, not even in the Abelian case. We expect that S'_{CS} is not supersymmetric in the Wess-Zumino gauge, despite the fact that it is a superspace integral. This is highlighted, in particular, by the second term in (28), which involves the structure constants. Its component expression simply gives the non-Abelian $W \wedge W \wedge W \wedge W$ correction in (26), which, as a purely bosonic object, cannot be supersymmetric by itself.

For the gauge variation of S_{CS}, one obtains

$$
\begin{aligned}
\delta(\Lambda)S_{\text{CS}} = \int \mathrm{d}^4x \Big[-\tfrac{1}{4}\mathrm{i}C^{(\text{CS})}_{AB,C}\Lambda^C F^A_{\mu\nu}\tilde{F}^{\mu\nu B} \int \mathrm{d}^4x \Big[-\tfrac{1}{8}\Lambda^C \Big(2C^{(\text{CS})}_{AB,D} f_{CE}{}^B - C^{(\text{CS})}_{DA,B} f_{CE}{}^B \\
+C^{(\text{CS})}_{BE,D} f_{CA}{}^B - C^{(\text{CS})}_{BD,C} f_{AE}{}^B \qquad (37) \\
\int \mathrm{d}^4x \Big[-\tfrac{1}{8}\mathrm{i}\Lambda^C \Big(+ C^{(\text{CS})}_{BC,D} f_{AE}{}^B + C^{(\text{CS})}_{AB,C} f_{DE}{}^B + \tfrac{1}{2}C^{(\text{CS})}_{AC,B} f_{DE}{}^B \Big) \varepsilon^{\mu\nu\rho\sigma} F^A_{\mu\nu} W^D_\rho W^E_\sigma \\
\int \mathrm{d}^4x \Big[-\tfrac{1}{8}\Lambda^C \Big(C^{(\text{CS})}_{BG,F} f_{CA}{}^B + C^{(\text{CS})}_{AG,B} f_{CF}{}^B + C^{(\text{CS})}_{AB,F} f_{CG}{}^B \Big) f_{DE}{}^A \varepsilon^{\mu\nu\rho\sigma} W^D_\mu W^E_\nu W^F_\rho W^G_\sigma \Big] ,
\end{aligned}
$$

where we used the Jacobi identity and the property $C^{(\text{CS})}_{(AB,C)} = 0$.

A careful calculation finally shows that the supersymmetry variation of S_{CS} is

$$
\delta(\epsilon)S_{\text{CS}} = -\tfrac{1}{2} \int \mathrm{d}^4x\, \varepsilon^{\mu\nu\rho\sigma}\, \mathrm{Re}\left[C^{(\text{CS})}_{AB,C} W^C_\mu F^A_{\nu\rho} + C^{(\text{CS})}_{A[B,C} f_{DE]}{}^A\, W^E_\mu W^C_\nu W^D_\rho \right] \bar{\epsilon}_L \gamma_\sigma \lambda^B_R \,. \qquad (38)
$$

4 The General Gauged Supergravity Model

The supersymmetric σ-model coupled to supergravity includes the gravitino and various coupling terms. The action is

$$
S[e^i_\mu, Aa_\mu, z^\alpha, z^{\bar{b}}, \psi^\alpha, {}^{\bar{b}}, \lambda a, \Psi_\mu] = \int d^4x \det(e^i_\mu)\mathcal{L}_{\text{SG}} \ , \qquad (39)
$$

with Lagrangian density

$$
\mathcal{L}_{\text{SG}} = \mathcal{L}_b + \mathcal{L}_f + \mathcal{L}_{\text{int}} + \text{ quartic terms} \qquad (40)
$$

with

$$
\begin{aligned}
\mathcal{L}_b &= \frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}DaDa - G_{\alpha\bar{\beta}}D_\mu z^\alpha D^\mu z^{\bar{\beta}} \ , \qquad (41)\\
\mathcal{L}_f &= \frac{1}{2}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_\nu\Psi_\rho + \frac{1}{2}\bar{\lambda}a\gamma^\mu D_\mu\lambda a + G_{\alpha\bar{b}}\bar{\psi}^{\bar{b}}\gamma^\mu D_\mu L\psi^\alpha \ , \\
\mathcal{L}_{\text{int}} &= \frac{1}{\sqrt{2}}G_{\alpha\bar{\beta}}\big[D_\mu z^{\bar{\beta}}\bar{\Psi}_\nu\gamma^\mu\gamma^\nu L\psi^\alpha + D_\mu z^\alpha\bar{\psi}^{\bar{\beta}}R\gamma^\nu\gamma^\mu\Psi_\nu\big] \\
&\quad + \frac{1}{2}Da\bar{\Psi}_\mu\gamma^\mu\gamma_5\lambda a + Fa_{\rho\sigma}\bar{\Psi}_\mu\gamma^{\rho\sigma}\gamma^\mu\lambda a + \sqrt{2}G_{\alpha\bar{\beta}}\big[Xa^{\bar{\beta}}\bar{\lambda}aL\psi^\alpha + Xa^\alpha\bar{\psi}^{\bar{\beta}}R\lambda a\big] \ .
\end{aligned}
$$

We omit the complicated set of four-fermion terms but our argument includes their effects. We assume there is no superpotential and minimal gauge kinetic functions to simplify the discussion. The gravitino covariant derivative is defined as

$$
D_\mu\Psi_\nu = \Big(\nabla_\mu + \frac{1}{2}iB_\mu\gamma_5\Big)\Psi_\nu \ = \ \Big(\partial_\mu + \frac{1}{4}\omega_{\mu ij}\gamma^{ij} + \frac{1}{2}iB_\mu\gamma_5\Big)\Psi_\nu \ , \qquad (42)
$$

in which ∇_μ includes the spin connection. Covariant derivatives of the matter fields were given previously in the current literature. One must replace $\partial_\mu \longrightarrow \nabla_\mu$ and we note that the composite Kähler connection couples to all fermions.

The model has a global $U(1)_R$ axial symmetry with transformations

$$
\delta L\psi^\alpha = i\alpha L\psi^\alpha \ , \quad \delta\lambda a = i\alpha\gamma_5\lambda a \ , \quad \delta\Psi_\mu = i\alpha\gamma_5\Psi_\mu \qquad (43)
$$

and Noether current

$$N^\mu = -\frac{i}{2}\Big[2G^{\bar b}_{\alpha\bar b}\gamma^\mu L\psi^\alpha + \bar\lambda a\gamma^\mu\gamma_5\lambda a + \bar\Psi_\rho\gamma^{\rho\mu\nu}\gamma_5\Psi_\nu\Big] \ . \tag{44}$$

It is an R-symmetry since z^α is neutral while $L\psi^\alpha$, $L\lambda^a$, and $L\Psi_\mu$ have charges $-1,+1,+1$, respectively. The $U(1)_R$ symmetry is effectively gauged by B_μ. There is also a gauge symmetry with parameters $\theta^a(x)$ with $\delta A^a_\mu = D_\mu\theta^a$. For the gauge variation of other fields we use the notation $\delta = \theta^a\delta^a$. We then have

$$\begin{aligned}
\delta a z^\alpha &= X a^\alpha \ , \quad \delta a z^{\bar b} \ = \ X a^{\bar b} \ , \\
\delta a L\psi^\alpha &= X a^\alpha_\beta L\psi^\beta - \frac{i}{2}\mathrm{Im}(Fa)L\psi^\alpha \ , \quad \delta a^{\bar b}R \ = \ X a^{\bar b}R + \frac{i}{2}\mathrm{Im}(Fa)^{\bar b}R \ , \\
\delta b\lambda a &= -f^{abc}\lambda^c - \frac{i}{2}\mathrm{Im}(Fb)\gamma_5\lambda^a \ , \quad \delta a\Psi_\mu \ = \ -\frac{i}{2}\mathrm{Im}(Fa)\gamma_5\Psi_\mu \ .
\end{aligned} \tag{45}$$

Holomorphic Killing vectors $X^{a\alpha}(z)$, $X^{a\bar b}(\bar z)$ and the holomorphic function $F^a(z)$ induced by a gauge transformation of the Kähler potential. The gauge invariance of the theory is expressed by the identity

$$\begin{aligned}
gi\delta\mathcal{L}_{\rm SG} \ = \ 0 \ = \ \theta a(x)\Big[&-D_\nu\frac{\delta\mathcal{L}_{\rm SG}}{\delta A^a_\nu} + X a^\alpha\frac{\delta\mathcal{L}_{\rm SG}}{\delta z^\alpha} + X a^{\bar b}\frac{\delta\mathcal{L}_{\rm SG}}{\delta z^{\bar b}} \\
&+\delta a^{\bar b}R\frac{\delta\mathcal{L}_{\rm SG}}{\delta^{\bar b}} + \frac{\delta\mathcal{L}_{\rm SG}}{\delta\psi^\alpha}\delta a L\psi^\alpha + \delta a\bar\lambda b\frac{\delta\mathcal{L}_{\rm SG}}{\delta\bar\lambda^b} + \delta a\bar\Psi_\rho\frac{\delta\mathcal{L}_{\rm SG}}{\delta\bar\Psi_\rho}\Big]
\end{aligned} \tag{46}$$

which is the same applied to the general supergravity Lagrangian. The gauge field equation of the model reads

$$D_\mu F a^{\mu\nu} = J a^\nu \equiv -\frac{\delta(\mathcal{L} + \frac{1}{4}F^b_{\mu\nu}F^{b\mu\nu})}{\delta A^a_\nu} = J a^\nu_b + J a^\nu_f + j a^\nu + \tfrac{1}{2}DaN^\nu + J a^\nu_{\rm int} \ . \tag{47}$$

with J^a_b and J^a_f defined in (**??**), $j^{a\nu} = \frac{1}{2}f^{abc}\bar\lambda^b\gamma^\nu\lambda^c$ and

$$J a^\nu_{\rm int} = -\frac{\delta\mathcal{L}_{\rm int}}{\delta A^a_\nu} = \frac{1}{\sqrt{2}}G_{\alpha\bar b}\Big[X a^{\bar b}\bar\Psi_\rho\gamma^\nu\gamma^\rho L\psi^\alpha + X a^{\alpha\bar b}R\gamma^\rho\gamma^\nu\Psi_\rho\Big] + 2D_\mu(\bar\Psi_\rho\gamma^{\mu\nu}\gamma^\rho\lambda a) \ . \tag{48}$$

To derive the consistency condition we now follow the same strategy as above. Assuming that the gauge variation of the action from varying bosons vanishes by the scalar equations of motion, the consistency condition arises from the fermion variations. The supergravity generalization of the expressions obtained earlier for only gauginos and for only chiral fermions turns out to be

$$0 = \langle\nabla_\nu Ja^\nu\rangle = iYa_{\alpha\bar b}\langle\nabla_\nu(^{\bar b}\gamma^\nu L\psi^\alpha)\rangle + \nabla_\mu ja^\mu \ + \tfrac{1}{2}\mathrm{Im}(Fa)\langle\nabla_\nu N^\nu\rangle \ , \tag{49}$$

in which the ∇_ν derivative carries appropriate space-time, target space and gauge connections, and $\langle ...\rangle$ again indicates just the anomalous divergences of the currents. Comparing with (47), we have dropped the divergence of $J^{a\nu}_{\rm int}$. As we argue in the appendix, this does not affect the anomaly. The condition (49) is the central result of our analysis.

The proper Kähler anomaly, proportional to $\mathrm{Im}(F^a)$, is the third term of (49). The anomaly has contributions from gauginos, from chiral fermions, and from the gravitino. The gravitino gauge anomaly is times that of a gaugino, but coupled only through the Kähler connection in (42). We obtain

$$\langle\nabla_\nu N^\nu\rangle_{\rm gauge} = \frac{1}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\Big[C_2(G)Fa_{\mu\nu}Fa_{\rho\sigma} + \frac{n_\lambda+3}{4}B_{\mu\nu}B_{\mu\nu} + C_{\mu\nu\rho\sigma}\Big] \ . \tag{50}$$

To write the contribution of the chiral fermions we first define the target space curvature tensor pulled back to spacetime. Using this we form

$$
\begin{aligned}
C_{\mu\nu\rho\sigma} &= \Sigma^{\alpha}_{\mu\nu\beta}\Sigma^{\beta}_{\rho\sigma\alpha} + F^a_{\mu\nu}F^b_{\rho\sigma}X^{a\alpha}_{\beta}X^{b\beta}_{\alpha} - \frac{1}{4}n_\psi B_{\mu\nu}B_{\rho\sigma} \\
&\quad -iF^a_{\mu\nu}B_{\rho\sigma}X^{a\alpha}_{\alpha} - 2F^a_{\mu\nu}\Sigma^{\beta}_{\rho\sigma\alpha}X^{a\alpha}_{\beta} + i\Sigma^{\alpha}_{\mu\nu\alpha}B_{\rho\sigma}\ .
\end{aligned}
\tag{51}
$$

The anomaly of the gaugino current $j^{a\mu}$ in the second term of (49) is identical to the truncated model. The contribution of the chiral fermions to the first term in (49) is

$$
\begin{aligned}
Y^a_{\alpha\bar b}\langle\nabla_\nu(^{\bar b}\gamma^\nu L\psi^\alpha)\rangle_{\text{gauge}} &= \frac{i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}G^{\beta\bar\delta}Y^a_{\alpha\bar\delta}\Big[\Sigma^{\alpha}_{\mu\nu\gamma}\Sigma^{\gamma}_{\rho\sigma\beta} + F^b_{\mu\nu}F^c_{\rho\sigma}X^{b\alpha}_{\gamma}X^{c\gamma}_{\beta} \qquad (52)\\
&-\frac{1}{4}B_{\mu\nu}B_{\rho\sigma}\delta^\alpha_\beta - iF^b_{\mu\nu}B_{\rho\sigma}X^{b\alpha}_{\beta} + i\Sigma^{\alpha}_{\mu\nu\beta}B_{\rho\sigma} - F^b_{\mu\nu}\Big(\Sigma^{\alpha}_{\rho\sigma\gamma}X^{b\gamma}_{\beta} + \Sigma^{\gamma}_{\rho\sigma\beta}X^{b\alpha}_{\gamma}\Big)\Big]\ .
\end{aligned}
$$

In the case of a flat target space $= \mathbb{C}^{n_\psi}$ and a linear realization of gauge symmetry, the Killing vector derivative reduces to constants, $X^{a\alpha}_{\beta} = X^{a\alpha}_{\beta} \to T^{a\alpha}{}_{\beta}$, a matrix generator of the gauge group G. In this case the second term of (52) reduces to the conventional cubic gauge anomaly of the chiral fermions.

The gravitational anomaly is more conventional. The contribution of the gravitino to the anomaly of the Noether current is -21 times that of a gaugino. The gaugino current $j^{a\mu}$ itself has no gravitational anomaly. The complete result is given by

$$
\begin{aligned}
\langle\nabla_\nu N^\nu\rangle_{\text{grav}} &= -\frac{1}{768\pi^2}(n_\lambda - 21 - n_\psi)\,\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\xi\tau}R_{\rho\sigma}{}^{\xi\tau}\ , \\
Y^a_{\alpha\bar b}\langle\nabla_\nu(^{\bar b}\gamma^\nu L\psi^\alpha)\rangle_{\text{grav}} &= -\frac{i}{768\pi^2}Y^a_{\alpha\bar b}G^{\alpha\bar b}\,\epsilon^{\mu\nu\rho\sigma}R_{\mu\nu\xi\tau}R_{\rho\sigma}{}^{\xi\tau}\ .
\end{aligned}
\tag{53}
$$

One can see that the gauge anomaly is very complicated. As a general observation, it is not possible to cancel the coefficient $n_\lambda + 3 - n_\psi$ of $B_{\mu\nu}B_{\rho\sigma}$ in (50) and (51) for the gauge anomaly and the gravitational anomaly in (53) at the same time by adjusting n_λ and n_ψ.

If $\mathrm{Im}(F^a) = 0$, then the $\partial_\nu N^\nu$ anomaly is absent. However, there are still several new terms involving $B_{\mu\nu}$ which can affect the consistency of the model. Anomaly cancellation will be studied with emphasis on the case of a flat target space.

The full Lagrangian of the model is a special case of (40) and rather simple. It reads

$$
\begin{aligned}
\mathcal{L}_T &= \frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - G_{S\bar S}D_\mu S D^\mu \bar S - \frac{1}{2}D^2 \\
&+\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\bar\Psi_\mu\gamma_\nu D_\rho\Psi_\sigma + \frac{1}{2}\bar\lambda\gamma^\mu D_\mu\lambda + G_{S\bar S}\bar\psi\gamma^\mu D_\mu L\psi \\
&+\frac{1}{\sqrt{2}}G_{S\bar S}\big[D_\mu\bar S\bar\Psi_\nu\gamma^\mu\gamma^\nu L\psi + D_\mu S\bar\psi R\gamma^\nu\gamma^\mu\Psi_\nu\big] \\
&+\frac{1}{2}D\bar\Psi_\mu\gamma^\mu\gamma_5\lambda + F_{\rho\sigma}\bar\Psi_\mu\gamma^{\rho\sigma}\gamma^\mu\lambda + \sqrt{2}G_{S\bar S}\big[X^{\bar S}\bar\lambda L\psi + X^S\bar\psi R\lambda\big]\ ,
\end{aligned}
\tag{54}
$$

where

$$
\begin{aligned}
D_\mu\Psi_\nu &= \Big(\nabla_\mu + \frac{1}{2}iB_\mu\gamma_5\Big)\Psi_\nu\ , \\
D_\mu\lambda &= \Big(\nabla_\mu + \frac{1}{2}iB_\mu\gamma_5\Big)\lambda\ , \\
D_\mu L\psi &= \Big(\nabla_\mu + \Gamma^S_{SS}D_\mu S + \frac{1}{2}iB_\mu\Big)L\psi\ .
\end{aligned}
\tag{55}
$$

The composite Kähler connection is

$$B_\mu = \frac{1}{2i}\Big(K_{,S}\, D_\mu S - K_{,\bar S}\, D_\mu \bar S\Big) = -\frac{1}{S+\bar S}(\partial_\mu h - eA_\mu)\ . \tag{56}$$

It is gauge invariant in this model because the Kähler potential is invariant and $\mathrm{Im}(F) = \xi = 0$. One can now directly obtain the equations of motions for A_μ and h (without going through those of S and $\bar S$ first), which are

$$\begin{aligned} \nabla_\mu F^{\mu\nu} + 2eG_{S\bar S}(\partial^\nu h - eA^\nu) &= eJ^\nu\ , \\ 2\nabla_\mu\big(G_{S\bar S}(\partial^\mu h - eA^\mu)\big) &= \nabla_\mu J^\mu\ , \end{aligned} \tag{57}$$

where

$$J^\mu = -\frac{\delta(\mathcal{L}_f + \mathcal{L}_{\rm int})}{\delta A_\mu}\ . \tag{58}$$

Applying $\frac{1}{e}\nabla_\nu$ one finds an expression which vanishes completely when the scalar equation of motion is used. Thus there is no inconsistency in this model. This agrees with the general consistency condition, because we have a gauged shift symmetry with constant Killing vectors and B_μ is gauge invariant. Therefore the fermions are invariant under the gauge transformation and all terms cancel when the scalar equations of motion are used. Hence no inconsistency can arise.

5 Generalized Chern-Simons Terms with Covariant and Consistent Anomalies in $D = 4$ Supergravity

The gauge anomalies manifest themselves as a non-invariance of the effective action under gauge transformations. We will concentrate on anomalous gauge symmetries, which are more problematic. Since gauge symmetries are needed to decouple the unphysical states of the theory, a violation of these symmetries renders the theory inconsistent. In a generic low energy effective field theory, the kinetic and the theta angle terms of vector fields, ${A_\mu}^\Lambda$, appear with scalar field dependent coefficients,

$$\mathcal{L}_{\rm g.k.} \;=\; \frac{1}{4}e\mathcal{I}_{\Lambda\Sigma}(z,\bar z){\mathcal{F}_{\mu\nu}}^\Lambda \mathcal{F}^{\mu\nu\Sigma} - \frac{1}{8}\mathcal{R}_{\Lambda\Sigma}(z,\bar z)\varepsilon^{\mu\nu\rho\sigma}{\mathcal{F}_{\mu\nu}}^\Lambda {\mathcal{F}_{\rho\sigma}}^\Sigma\,. \tag{59}$$

${\mathcal{F}_{\mu\nu}}^\Lambda \equiv 2\partial_{[\mu}A_{\nu]}{}^\Lambda + X_{\Sigma\Omega}{}^\Lambda A_\mu{}^\Sigma A_\nu{}^\Omega$ denotes the non-Abelian field strengths with $X_{\Sigma\Omega}{}^\Lambda = X_{[\Sigma\Omega]}{}^\Lambda$ being the structure constants of the gauge group. We use the metric signature $(-+++)$ and work with real $\varepsilon_{0123} = 1$. As usual, e denotes the vierbein determinant. The second term in (59) is often referred to as the Peccei-Quinn term, and the functions $\mathcal{I}_{\Lambda\Sigma}(z,\bar z)$ and $\mathcal{R}_{\Lambda\Sigma}(z,\bar z)$ depend nontrivially on the scalar fields, z^i, of the theory. One can combine these functions to a complex function

$$\mathcal{N}_{\Lambda\Sigma}(z,\bar z) = \mathcal{R}_{\Lambda\Sigma}(z,\bar z) + \mathrm{i}\mathcal{I}_{\Lambda\Sigma}(z,\bar z). \tag{60}$$

In a supersymmetric context, $\mathcal{N}_{\Lambda\Sigma}(z,\bar z)$ has to satisfy certain conditions, depending on the amount of supersymmetry. In $\mathcal{N} = 1$ global and local supersymmetry, which will be the subject of the remainder of this section, $\mathcal{N}_{\Lambda\Sigma} = \mathcal{N}_{\Lambda\Sigma}(\bar z)$ simply has to be antiholomorphic in the complex scalars of the chiral multiplets.

If, under a gauge transformation with gauge parameter $\Lambda^\Omega(x)$, acting on the field strengths as $\delta(\Lambda)\mathcal{F}^\Lambda_{\mu\nu} = \Lambda^\Xi \mathcal{F}^\Omega_{\mu\nu} X_{\Omega\Xi}{}^\Lambda$, some of the z^i transform nontrivially, this may induce a corresponding gauge transformation of $\mathcal{N}_{\Lambda\Sigma}(\bar{z})$. In case this transformation is of the form of a symmetric product of two adjoint representations of the gauge group,

$$\delta(\Lambda)\mathcal{N}_{\Lambda\Sigma} = \Lambda^{\Omega}\delta_{\Omega}\mathcal{N}_{\Lambda\Sigma}\,, \qquad \delta_{\Omega}\mathcal{N}_{\Lambda\Sigma} = X_{\Omega\Lambda}{}^{\Gamma}\mathcal{N}_{\Sigma\Gamma} + X_{\Omega\Sigma}{}^{\Gamma}\mathcal{N}_{\Lambda\Gamma}\,, \tag{61}$$

the kinetic term (59) is obviously gauge invariant. This is what was assumed in the action of general matter-coupled supergravity.

If, one takes into account also other terms in the quantum effective action, a more general transformation rule for $\mathcal{N}_{\Lambda\Sigma}(\bar{z})$ may be allowed

$$\delta_{\Omega}\mathcal{N}_{\Lambda\Sigma} = -X_{\Omega\Lambda\Sigma} + X_{\Omega\Lambda}{}^{\Gamma}\mathcal{N}_{\Sigma\Gamma} + X_{\Omega\Sigma}{}^{\Gamma}\mathcal{N}_{\Lambda\Gamma}\,. \tag{62}$$

Here, $X_{\Omega\Lambda\Sigma}$ is a constant real tensor symmetric in the last two indices, which can be recognized as a natural generalization in the context of symplectic duality transformations. Closure of the gauge algebra requires the constraint

$$X_{\Omega\Lambda\Sigma}X_{\Gamma\Xi}{}^{\Omega} + 2X_{\Sigma[\Xi}{}^{\Omega}X_{\Gamma]\Lambda\Omega} + 2X_{\Lambda[\Xi}{}^{\Omega}X_{\Gamma]\Sigma\Omega} = 0\,. \tag{63}$$

If $X_{\Omega\Lambda\Sigma}$ is non-zero, this leads to a non-gauge invariance of the Peccei-Quinn term in $\mathcal{L}_{\rm g.k.}$:

$$\delta(\Lambda)\mathcal{L}_{\rm g.k.} = \frac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\, X_{\Omega\Lambda\Sigma}\Lambda^{\Omega}\mathcal{F}_{\mu\nu}{}^{\Lambda}\mathcal{F}_{\rho\sigma}{}^{\Sigma}\,. \tag{64}$$

In order to understand how this broken invariance can be restored, it is convenient to split the coefficients $X_{\Omega\Lambda\Sigma}$ into a sum,

$$X_{\Omega\Lambda\Sigma} = X^{(\rm s)}_{\Omega\Lambda\Sigma} + X^{(\rm m)}_{\Omega\Lambda\Sigma}\,, \qquad X^{(\rm s)}_{\Omega\Lambda\Sigma} = X_{(\Omega\Lambda\Sigma)}\,, \qquad X^{(\rm m)}_{(\Omega\Lambda\Sigma)} = 0\,, \tag{65}$$

where $X^{(\rm s)}_{\Omega\Lambda\Sigma}$ is completely symmetric, and $X^{(\rm m)}_{\Omega\Lambda\Sigma}$ denotes the part of mixed symmetry. Terms of the form (64) may then in principle be cancelled by the following two mechanisms, or a combination thereof:

1. As was first realized in a similar context in $\mathcal{N}=2$ supergravity, the gauge variation due to a non-vanishing mixed part, $X^{(\rm m)}_{\Omega\Lambda\Sigma} \neq 0$, may be cancelled by adding a generalized Chern-Simons term (GCS term) that contains a cubic and a quartic part in the vector fields,

$$\mathcal{L}_{\rm GCS} \;=\; \frac{1}{3}X^{(\rm CS)}_{\Omega\Lambda\Sigma}\,\varepsilon^{\mu\nu\rho\sigma}\left(A_{\mu}{}^{\Omega}A_{\nu}{}^{\Lambda}\partial_{\rho}A_{\sigma}^{\Sigma} + \frac{3}{8}X_{\Gamma\Xi}{}^{\Sigma}A_{\mu}{}^{\Omega}A_{\nu}{}^{\Lambda}A_{\rho}{}^{\Gamma}A_{\sigma}{}^{\Xi}\right)\,. \tag{66}$$

   This term depends on a constant tensor $X^{(\rm CS)}_{\Omega\Lambda\Sigma}$, which has the same mixed symmetry structure as $X^{(\rm m)}_{\Omega\Lambda\Sigma}$. The cancellation occurs provided the tensors $X^{(\rm m)}_{\Omega\Lambda\Sigma}$ and $X^{(\rm CS)}_{\Omega\Lambda\Sigma}$ are, in fact, the same.

2. If the chiral fermion spectrum is anomalous under the gauge group, the anomalous triangle diagrams lead to a non-gauge invariance of the quantum effective action Γ for the gauge symmetry: $\delta(\Lambda)\Gamma = \int {\rm d}^4x\Lambda^{\Lambda}\mathcal{A}_{\Lambda}$ of the form

$$\mathcal{A}_{\Lambda} = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\left[2d_{\Omega\Sigma\Lambda}\partial_{\mu}A_{\nu}{}^{\Sigma} + \left(d_{\Omega\Sigma\Gamma}X_{\Lambda\Xi}{}^{\Sigma} + \frac{3}{2}d_{\Omega\Sigma\Lambda}X_{\Gamma\Xi}{}^{\Sigma}\right)A_{\mu}{}^{\Gamma}A_{\nu}{}^{\Xi}\right]\partial_{\rho}A_{\sigma}{}^{\Omega}\,, \tag{67}$$

   with a symmetric tensor $d_{\Omega\Lambda\Sigma}$. If

$$X^{(\rm s)}_{\Omega\Lambda\Sigma} = d_{\Omega\Lambda\Sigma}\,, \tag{68}$$

   this quantum anomaly cancels the symmetric part of (64). This is the Green-Schwarz mechanism.

It was studied to what extent a general gauge theory of the above type with gauged axionic shift symmetries, GCS terms and quantum gauge anomalies can be compatible with $\mathcal{N} = 1$ supersymmetry. The results can be summarized as follows: if one takes as starting point the matter-coupled supergravity Lagrangian, an axionic shift symmetry with $X_{\Lambda\Sigma\Omega} \neq 0$ satisfying the closure condition (63) can be gauged in a way consistent with $\mathcal{N} = 1$ supersymmetry if

1. a GCS term (66) with $X^{(\mathrm{CS})}_{\Omega\Lambda\Sigma} = X^{(\mathrm{m})}_{\Omega\Lambda\Sigma}$ is added,
2. an additional term bilinear in the gaugini, $\lambda^{\Sigma}(x)$, and linear in the vector fields is added:
$$\mathcal{L}_{\mathrm{extra}} = -\frac{1}{4}\mathrm{i}A_\mu{}^{\Omega}X_{\Omega\Lambda\Sigma}\bar{\lambda}^{\Lambda}\gamma_5\gamma^{\mu}\lambda^{\Sigma}, \tag{69}$$
3. the fermions in the chiral multiplets give rise to quantum anomalies with $d_{\Omega\Lambda\Sigma} = X^{(\mathrm{s})}_{\Omega\Lambda\Sigma}$. The consistent gauge anomaly, $\mathcal{A}_\Lambda$ is of the form (121). These quantum anomalies precisely cancel the classical gauge and supersymmetry variation of the new Lagrangian $\mathcal{L}_{\mathrm{old}}+\mathcal{L}_{\mathrm{GCS}}+\mathcal{L}_{\mathrm{extra}}$, where $\mathcal{L}_{\mathrm{old}}$ denotes the original Lagrangian.

5.1 Gauge Transformations

The violation of the Jacobi identity is the prize one has to pay for the symplectically covariant treatment in which both electric and magnetic vector potentials appear at the same time. In order to compensate for this violation ande in addition to the usual non-Abelian transformation $\partial_\mu\Lambda^M + X_{[PQ]}{}^M A_\mu{}^P\Lambda^Q$ and extends the gauge transformation of the vector potentials to

$$\delta A_\mu{}^M = \mathcal{D}_\mu\Lambda^M - X_{(NP)}{}^M\Xi_\mu{}^{NP}\,, \qquad \mathcal{D}_\mu\Lambda^M = \partial_\mu\Lambda^M + X_{PQ}{}^M A_\mu{}^P\Lambda^Q\,, \tag{70}$$

where we introduced the covariant derivative $\mathcal{D}_\mu\Lambda^M$, and new vector-like gauge parameters $\Xi_\mu{}^{NP}$, symmetric in the upper indices. The extra terms $X_{(PQ)}{}^M A_\mu{}^P\Lambda^Q$ and the Ξ-transformations contained in (70) allow one to gauge away the vector fields that correspond to the directions in which the Jacobi identity is violated. The covariant ansatz makes use of the fact that the non-covariant terms appear projected with the tensor $Z^M{}_{PQ}$ and defines the full covariant field strengths as

$$\mathcal{H}^M_{\mu\nu} = \mathcal{F}^M_{\mu\nu} + gZ^M{}_{PQ}\,B^{PQ}_{\mu\nu}\,, \tag{71}$$

upon the introduction of two-form tensor fields of the type $B^{MN}_{\mu\nu} = B^{(MN)}_{[\mu\nu]}$. The non-covariant terms can then be absorbed by postulating the corresponding transformation laws for the two-form fields. The full covariant field strength (71) no longer satisfies the standard Bianchi identies, but rather its deformed version

$$D_{[\mu}\mathcal{H}^M_{\nu\rho]} = \frac{1}{3}\,gZ^M{}_{PQ}\,\mathcal{H}^{PQ}_{\mu\nu\rho}\,, \tag{72}$$

where $\mathcal{H}^{PQ}_{\mu\nu\rho}$ denotes the covariant field strength of the two-forms. The combined transformation

$$\Xi^{MN}_\mu = D_\mu\,\xi^{MN}\,, \qquad \Lambda^M = gZ^M{}_{PQ}\,\xi^{PQ}\,, \tag{73}$$

Explicitly, the new field strength $\mathcal{H}^M_{\mu\nu}$ transforms covariantly under the combined set of gauge transformations

$$\begin{aligned}\delta A^M_\mu &= \mathcal{D}_\mu\Lambda^M - g\,Z^M{}_{PQ}\,\Xi^{PQ}_\mu\,,\\ \delta B^{MN}_{\mu\nu} &= 2\,\mathcal{D}_{[\mu}\Xi^{MN}_{\nu]} - 2\Lambda^{(M}\mathcal{H}^{N)}_{\mu\nu} + 2\,A^{(M}_{[\mu}\,\delta A^{N)}_{\nu]}\,,\end{aligned} \tag{74}$$

where Ξ_μ^{MN} labels the tensor gauge transformations associated with the two-forms. The two-forms $B_{\mu\nu}^{MN}$ introduced in (71) cannot simply be added to the fields of the theory, as the number of degrees of freedom is in general carefully balanced by supersymmetry. Rather, these must be the two-forms that are already present in the ungauged supergravity. They thus can be labeled by indices in the adjoint representation of the global symmetry group G. In other words, out of the two-forms $B_{\mu\nu}^{MN}$ in the symmetric tensor product

$$(\mathcal{R}_{\rm v} \otimes \mathcal{R}_{\rm v})_{\rm sym} \;=\; \mathcal{R}_{\rm adj} \oplus \;\ldots\;, \tag{75}$$

only those transforming in the adjoint representation $\mathcal{R}_{\rm adj}$ are involved in the gauging. This is precisely in accordance with the fact that two-forms in four dimensions are dual to the scalar field isometries as a consequence of the on-shell duality and thus transform in the adjoint representation of G. Their precise representation can be inferred from inspection of the tensor $Z^M{}_{PQ}$ under which they appear.

It is important to notice that the modified gauge transformations still close on the gauge fields and thus form a Lie algebra. Indeed, if we split (70) into two parts,

$$\delta A_\mu{}^M = \delta(\Lambda) A_\mu{}^M + \delta(\Xi) A_\mu{}^M\,, \tag{76}$$

the commutation relations are

$$\begin{aligned} \left[\delta(\Lambda_1), \delta(\Lambda_2)\right] A_\mu{}^M &= \delta(\Lambda) A_\mu{}^M + \delta(\Xi) A_\mu{}^M, \\ \left[\delta(\Lambda), \delta(\Xi)\right] A_\mu{}^M &= \left[\delta(\Xi_1), \delta(\Xi_2)\right] A_\mu{}^M = 0\,, \end{aligned} \tag{77}$$

with

$$\Lambda^M = X_{[NP]}{}^M \Lambda_1^N \Lambda_2^P\,, \qquad\qquad \Xi_\mu{}^{PN} = \Lambda_1^{(P} \mathcal{D}_\mu \Lambda_2^{N)} - \Lambda_2^{(P} \mathcal{D}_\mu \Lambda_1^{N)}\,. \tag{78}$$

In this case the usual properties of the field strength

$$\mathcal{F}_{\mu\nu}{}^M = 2\partial_{[\mu} A_{\nu]}{}^M + \; X_{[PQ]}{}^M A_\mu{}^P A_\nu{}^Q \tag{79}$$

are changed. In particular, it will no longer fulfill the Bianchi identity, which now must be replaced by

$$\mathcal{D}_{[\mu} \mathcal{F}_{\nu\rho]}{}^M \;=\; X_{(NP)}{}^M A_{[\mu}{}^N \mathcal{F}_{\nu\rho]}{}^P - \frac{1}{3} X_{(PN)}{}^M X_{[QR]}{}^P \; A_{[\mu}{}^N A_\nu{}^Q A_{\rho]}{}^R\,. \tag{80}$$

Furthermore, $\mathcal{F}_{\mu\nu}{}^M$ does not transform covariantly under a gauge transformation (70). Instead, we have

$$\begin{aligned} \delta \mathcal{F}_{\mu\nu}{}^M &= 2\mathcal{D}_{[\mu} \delta A_{\nu]}{}^M - 2X_{(PQ)}{}^M A_{[\mu}{}^P \delta A_{\nu]}{}^Q \\ &= X_{NQ}{}^M \; \mathcal{F}_{\mu\nu}{}^N \Lambda^Q - 2X_{(NP)}{}^M \mathcal{D}_{[\mu} \Xi_{\nu]}{}^{NP} - 2X_{(PQ)}{}^M A_{[\mu}{}^P \delta A_{\nu]}{}^Q\,, \end{aligned} \tag{81}$$

where the covariant derivative is

$$\begin{aligned} X_{(NP)}{}^M \mathcal{D}_\mu \Xi_\nu{}^{NP} &= \partial_\mu \left(X_{(NP)}{}^M \Xi_\nu{}^{NP} \right) + A_\mu{}^R X_{RQ}{}^M X_{(NP)}{}^Q \Xi_\nu{}^{NP}\,, \\ \mathcal{D}_\mu \Xi_\nu{}^{NP} &= \partial_\mu \Xi_\nu{}^{NP} + X_{QR}{}^P A_\mu{}^Q \Xi_\nu{}^{NR} + X_{QR}{}^N A_\mu{}^Q \Xi_\nu{}^{PR}\,. \end{aligned} \tag{82}$$

Therefore, if we want to deform the original Lagrangian (59) and accommodate electric and magnetic gauge fields, $\mathcal{F}_{\mu\nu}{}^M$ cannot be used to construct gauge-covariant kinetic terms. For this reason, the authors introduced tensor fields $B_{\mu\nu\,\alpha}$ to be described by $B_{\mu\nu}{}^{MN}$, symmetric in (MN), and with them modified field strengths

$$\mathcal{H}_{\mu\nu}{}^M = \mathcal{F}_{\mu\nu}{}^M + X_{(NP)}{}^M B_{\mu\nu}{}^{NP}\,. \tag{83}$$

We will consider gauge transformations of the antisymmetric tensors of the form

$$\delta B_{\mu\nu}{}^{NP} = 2\mathcal{D}_{[\mu}\Xi_{\nu]}{}^{NP} + 2A_{[\mu}{}^{(N}\delta A_{\nu]}{}^{P)} + \Delta B_{\mu\nu}{}^{NP}\,, \tag{84}$$

where $\Delta B_{\mu\nu}{}^{NP}$ depends on the gauge parameter Λ^Q, but we do not fix it further at this point. Together with (81), this then implies

$$\delta\mathcal{H}_{\mu\nu}{}^{M} = X_{NQ}{}^{M}\Lambda^Q\mathcal{H}_{\mu\nu}{}^{N} + X_{(NP)}{}^{M}\Delta B_{\mu\nu}{}^{NP}\,. \tag{85}$$

5.1.1 The Kinetic Lagrangian

The first step towards a gauge invariant action is to replace $\mathcal{F}_{\mu\nu}{}^{\Lambda}$ in $\mathcal{L}_{\text{g.k.}}$ by $\mathcal{H}_{\mu\nu}{}^{\Lambda}$, which then yields the new kinetic Lagrangian

$$\mathcal{L}_{\text{g.k.}} = \tfrac{1}{4}e\mathcal{I}_{\Lambda\Sigma}\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}^{\mu\nu\Sigma} - \tfrac{1}{8}\mathcal{R}_{\Lambda\Sigma}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{H}_{\rho\sigma}{}^{\Sigma}\,, \tag{86}$$

where again $\mathcal{I}_{\Lambda\Sigma}$ and $\mathcal{R}_{\Lambda\Sigma}$ denote, respectively, $\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}$ and $\mathrm{Re}\,\mathcal{N}_{\Lambda\Sigma}$. Using

$$\mathcal{G}_{\mu\nu\,\Lambda} \equiv \varepsilon_{\mu\nu\rho\sigma}\frac{\partial\mathcal{L}}{\partial\mathcal{H}_{\rho\sigma}{}^{\Lambda}} = \mathcal{R}_{\Lambda\Gamma}\mathcal{H}_{\mu\nu}{}^{\Gamma} + \frac{1}{2}e\varepsilon_{\mu\nu\rho\sigma}\,\mathcal{I}_{\Lambda\Gamma}\,\mathcal{H}^{\rho\sigma\,\Gamma}\,, \tag{87}$$

the Lagrangian and its transformations can be written as

$$\begin{aligned}
\mathcal{L}_{\text{g.k.}} &= -\tfrac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}^{\Lambda}_{\mu\nu}\mathcal{G}_{\rho\sigma\,\Lambda}\,,\\
\delta\mathcal{L}_{\text{g.k.}} &= -\tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\mathcal{G}_{\mu\nu\,\Lambda}\delta\mathcal{H}^{\Lambda}_{\rho\sigma}\\
&\quad+\tfrac{1}{8}\varepsilon^{\mu\nu\rho\sigma}\Lambda^Q\left(\mathcal{H}^{\Lambda}_{\mu\nu}X_{Q\Lambda\Sigma}\mathcal{H}^{\Sigma}_{\rho\sigma} - 2\mathcal{H}^{\Lambda}_{\mu\nu}X_{Q\Lambda}{}^{\Sigma}\mathcal{G}_{\rho\sigma\,\Sigma} - \mathcal{G}_{\mu\nu\,\Lambda}X_{Q}{}^{\Lambda\Sigma}\mathcal{G}_{\rho\sigma\,\Sigma}\right)\,,
\end{aligned} \tag{88}$$

where, in the third line, we used the infinitesimal form of (??):

$$\delta(\Lambda)\mathcal{N}_{\Lambda\Sigma} \;=\; \Lambda^M\Big[-X_{M\Lambda\Sigma} + 2\,X_{M(\Lambda}{}^{\Gamma}\mathcal{N}_{\Sigma)\Gamma} + \mathcal{N}_{\Lambda\Gamma}\,X_M{}^{\Gamma\Xi}\,\mathcal{N}_{\Xi\Sigma}\Big]\,. \tag{89}$$

When we introduce

$$\mathcal{G}_{\mu\nu}{}^{M} = \left(\mathcal{G}_{\mu\nu}{}^{\Lambda}\,,\,\mathcal{G}_{\mu\nu\Lambda}\right) \quad\text{with}\quad \mathcal{G}_{\mu\nu}{}^{\Lambda} \equiv \mathcal{H}_{\mu\nu}{}^{\Lambda}\,, \tag{90}$$

we can rewrite the second line of (88) in a covariant expression, and when we also use (85) we get

$$\delta\mathcal{L}_{\text{g.k.}} = \varepsilon^{\mu\nu\rho\sigma}\left[-\tfrac{1}{4}\mathcal{G}_{\mu\nu\,\Lambda}\left(\Lambda^Q X_{PQ}{}^{\Lambda}\mathcal{H}_{\rho\sigma}{}^{P} + X_{(NP)}{}^{\Lambda}\Delta B_{\rho\sigma}{}^{NP}\right) + \tfrac{1}{8}\mathcal{G}_{\mu\nu}{}^{M}\mathcal{G}_{\rho\sigma}{}^{N}\Lambda^Q X_{QM}{}^{R}\Omega_{NR}\right]\,. \tag{91}$$

Clearly, the newly proposed form for $\mathcal{L}_{\text{g.k.}}$ in (86) is still not gauge invariant. This should not come as a surprise because (89) contains a constant shift, which requires the addition of extra terms to the Lagrangian. Also the last term on the right hand side of (89) gives extra contributions that are quadratic in the kinetic function. In the next steps we will see that besides GCS terms, also terms linear and quadratic in the tensor field are required to restore gauge invariance. We start with the discussion of the latter terms.

5.2 Topological Terms for the B-field and a New Constraint

The second step towards gauge invariance is made by adding topological terms linear and quadratic in the tensor field $B_{\mu\nu}{}^{NP}$ to the gauge kinetic term (86), namely

$$\mathcal{L}_{\text{top},B} = \tfrac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\,X_{(NP)}{}^{\Lambda}\,B_{\mu\nu}{}^{NP}\left(\mathcal{F}_{\rho\sigma\,\Lambda} + \tfrac{1}{2}\,X_{(RS)\Lambda}\,B_{\rho\sigma}{}^{RS}\right)\,. \tag{92}$$

Note that for pure electric gaugings $X_{(NP)}{}^{\Lambda} = 0$. Therefore, in this case this term vanishes, implying that the tensor fields decouple.

We recall that, up to now, only the closure constraint has been used. We are now going to impose one new constraint

$$X_{(NP)}{}^{M}\Omega_{MQ}X_{(RS)}{}^{Q} = 0\,. \tag{93}$$

The constraint thus says that

$$X_{(NP)}{}^{\Lambda}X_{(RS)\Lambda} = X_{(NP)\Lambda}X_{(RS)}{}^{\Lambda}\,. \tag{94}$$

A consequence of this constraint that we will use below follows from the first of (??) and (??):

$$X_{(PQ)}{}^{R}D_{MNR} = 0\,. \tag{95}$$

The variation of $\mathcal{L}_{\text{top},B}$ is

$$\begin{aligned}\delta\mathcal{L}_{\text{top},B} &= \tfrac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}X_{(NP)}{}^{\Lambda}\left[\mathcal{H}_{\mu\nu\Lambda}\,\delta B_{\rho\sigma}{}^{NP} + B_{\rho\sigma}{}^{NP}\delta\mathcal{F}_{\mu\nu\Lambda}\right] \\ &= \tfrac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}X_{(NP)}{}^{\Lambda}\left[\mathcal{H}_{\mu\nu\Lambda}\,\delta B_{\rho\sigma}{}^{NP} + 2B_{\rho\sigma}{}^{NP}\left(\mathcal{D}_{\mu}\delta A_{\nu\Lambda} - X_{(RS)\Lambda}A_{\mu}^{R}\delta A_{\nu}^{S}\right)\right]\,.\end{aligned} \tag{96}$$

5.3 Generalized Chern-Simons Terms

We introduce a generalized Chern-Simons term of the form

$$\mathcal{L}_{\text{GCS}} = \varepsilon^{\mu\nu\rho\sigma}A_{\mu}{}^{M}A_{\nu}{}^{N}\left(\frac{1}{3}\,X_{MN\,\Lambda}\,\partial_{\rho}A_{\sigma}{}^{\Lambda} + \frac{1}{6}X_{MN}{}^{\Lambda}\partial_{\rho}A_{\sigma\Lambda} + \frac{1}{8}X_{MN\,\Lambda}X_{PQ}{}^{\Lambda}A_{\rho}{}^{P}A_{\sigma}{}^{Q}\right)\,. \tag{97}$$

Modulo total derivatives one can write its variation as

$$\begin{aligned}\delta\mathcal{L}_{\text{GCS}} = \varepsilon^{\mu\nu\rho\sigma}&\left[\tfrac{1}{2}\mathcal{F}_{\mu\nu}{}^{\Lambda}\mathcal{D}_{\rho}\delta A_{\sigma\Lambda} - \tfrac{1}{2}\mathcal{F}_{\mu\nu\Lambda}X_{(NP)}{}^{\Lambda}A_{\rho}{}^{N}\delta A_{\sigma}{}^{P}\right.\\ &\left.-D_{MNP}A_{\mu}{}^{M}\delta A_{\nu}{}^{N}\left(\partial_{\rho}A_{\sigma}{}^{P} + \tfrac{3}{8}X_{RS}{}^{P}A_{\rho}{}^{R}A_{\sigma}{}^{S}\right)\right]\,.\end{aligned} \tag{98}$$

These variations can be combined with (96) to

$$\begin{aligned}\delta\left(\mathcal{L}_{\text{top},B} + \mathcal{L}_{\text{GCS}}\right) &= \varepsilon^{\mu\nu\rho\sigma}\left[\tfrac{1}{2}\mathcal{H}_{\mu\nu}{}^{\Lambda}\mathcal{D}_{\rho}\delta A_{\sigma\Lambda} + \tfrac{1}{4}\mathcal{H}_{\mu\nu\Lambda}X_{(NP)}{}^{\Lambda}\left(\delta B_{\rho\sigma}{}^{NP} - 2A_{\rho}{}^{N}\delta A_{\sigma}{}^{P}\right)\right.\\ &\quad\left.-D_{MNP}A_{\mu}{}^{M}\delta A_{\nu}{}^{N}\left(\partial_{\rho}A_{\sigma}{}^{P} + \tfrac{3}{8}X_{RS}{}^{P}A_{\rho}{}^{R}A_{\sigma}{}^{S}\right)\right]\,.\end{aligned} \tag{99}$$

5.4 Variation of the total action

We are now ready to discuss the symmetry variation of the total Lagrangian

$$\mathcal{L}_{\mathcal{VT}} = \mathcal{L}_{\text{g.k.}} + \mathcal{L}_{\text{top},B} + \mathcal{L}_{\text{GCS}}\,, \tag{100}$$

built from (86), (92) and (97). We first check the invariance of (100) with respect to the Ξ-transformations. We see directly from (91) that the gauge-kinetic terms are invariant. The second line of (99) also clearly vanishes inserting (70) and using (95). This leaves us with the first line of (99), which, using (84) and (70), can be written in a symplectically covariant form

$$\delta_{\Xi}\mathcal{L}_{\mathcal{VT}} = -\tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\mathcal{H}_{\mu\nu}{}^{M}X_{(NP)}{}^{Q}\Omega_{MQ}\mathcal{D}_{\rho}\Xi_{\sigma}{}^{NP}\,. \tag{101}$$

The B-terms in $\mathcal{H}$, see (83), are proportional to $X_{(RS)}{}^{M}$ and thus give a vanishing contribution due to our new constraint (93). For the $\mathcal{F}$ terms we can perform an integration by parts and then (80) gives again only terms proportional to $X_{(RS)}{}^{M}$ leading to the same conclusion. We therefore find that the Ξ-variation of the total action vanishes.

We can thus further restrict to the Λ^M gauge transformations. According to (81), the $\mathcal{D}_\rho \delta A_{\sigma\Lambda}$-term in (99) can then be replaced by $\frac{1}{2}\Lambda^Q X_{NQ\Lambda}\mathcal{H}_{\rho\sigma}{}^N$, which can then be combined with the first term of (91) to form a symplectically covariant expression (the first term on the right hand side of (103) below). Adding also the remaining terms of (99) and (91), one obtains, using (84),

$$\begin{aligned}\delta\mathcal{L}_{\mathcal{VT}} = \varepsilon^{\mu\nu\rho\sigma}\big[&\tfrac{1}{4}\mathcal{G}_{\mu\nu}{}^M\Lambda^Q X_{NQ}{}^R\Omega_{MR}\mathcal{H}_{\rho\sigma}{}^N + \tfrac{1}{8}\mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\Lambda^Q X_{QM}{}^R\Omega_{NR} \\ &+\tfrac{1}{4}(\mathcal{H}-\mathcal{G})_{\mu\nu\,\Lambda}X_{(NP)}{}^\Lambda\Delta B_{\rho\sigma}{}^{NP} - D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\big]\ .\end{aligned} \tag{102}$$

We observe that if the $\mathcal{H}$ in the first line was a $\mathcal{G}$, would allow one to write the first line as an expression proportional to D_{MNP}. This leads to the first line in (104) below. The second observation is that the identity $(\mathcal{H}-\mathcal{G})^\Lambda = 0$ allows one to rewrite the second line of (103) in a symplectically covariant way, so that, altogether, we have

$$\begin{aligned}\delta\mathcal{L}_{\mathcal{VT}} = \varepsilon^{\mu\nu\rho\sigma}\big[&\tfrac{1}{4}\mathcal{G}_{\mu\nu}{}^M\Lambda^Q X_{NQ}{}^R\Omega_{MR}(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^N + \tfrac{3}{8}\mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\Lambda^Q D_{QMN} \\ &-\tfrac{1}{4}(\mathcal{H}-\mathcal{G})_{\mu\nu}{}^M\Omega_{MR}X_{(NP)}{}^R\Delta B_{\rho\sigma}{}^{NP} - D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\big]\ .\end{aligned} \tag{103}$$

By choosing

$$\Delta B_{\rho\sigma}{}^{NP} = -\Lambda^N\mathcal{G}_{\rho\sigma}{}^P - \Lambda^P\mathcal{G}_{\rho\sigma}{}^N\ , \tag{104}$$

the result (104) becomes

$$\begin{aligned}\delta\mathcal{L}_{\mathcal{VT}} = \varepsilon^{\mu\nu\rho\sigma}\big[&\tfrac{3}{8}\Lambda^Q D_{MNQ}\left(2\mathcal{G}_{\mu\nu}{}^M(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^N + \mathcal{G}_{\mu\nu}{}^M\mathcal{G}_{\rho\sigma}{}^N\right) \\ &-D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\big]\ ,\end{aligned} \tag{105}$$

which is then proportional to D_{MNP}, and hence zero when the original representation constraint is imposed.

6 The Covariant Anomaly

The superfield version of the covariant anomaly is straightforward,

$$\mathcal{A}_\mathcal{G} \propto \int d^4x\, d^2\theta\, \mathrm{tr}\ i\Lambda W^\alpha W_\alpha + h.c. \tag{106}$$

where W^α denotes the non-abelian superfield vector field strength and Λ is a chiral superfield. Supersymmetric expressions for the difference between the consistent and covariant anomaly for a simple gauge group are complicated.

The covariant (left-)chiral anomaly is

$$\mathcal{A}_\mathcal{G} = (D_\mu j^\mu)^a = \frac{i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[T^a V_{\mu\nu}V_{\rho\sigma}\right] = \frac{i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[T^a\partial_\mu\Big(V_\nu\partial_\rho V_\sigma + \frac{2}{3}V_\nu V_\rho V_\sigma\Big)\right]\ , \tag{107}$$

with $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]$.

Our goal is to generalize this for theories with quantum anomalies. These anomalies depend only on the gauge vectors. The field strengths $\mathcal{G}$, (87), also depend on the matrix $\mathcal{N}$ which itself generically depends on scalar fields. Therefore, we want to consider modified transformations of the antisymmetric tensors such that $\mathcal{G}$ does not appear in the final result.

To achieve this, we would like to replace (104) by a transformation such that

$$X_{(NP)}{}^R\Delta B_{\rho\sigma}{}^{NP} = -2X_{(NP)}{}^R\Lambda^N\mathcal{G}_{\rho\sigma}{}^P + \tfrac{3}{2}\Omega^{RM}D_{MNQ}\Lambda^Q(\mathcal{H}-\mathcal{G})_{\rho\sigma}{}^N\ . \tag{108}$$

Indeed, inserting this in (104) would lead to

$$\delta\mathcal{L}_{\mathcal{VT}} = \varepsilon^{\mu\nu\rho\sigma}\left[\tfrac{3}{8}\Lambda^Q D_{MNQ}\mathcal{F}_{\mu\nu}{}^M\mathcal{F}_{\rho\sigma}{}^N - D_{MNP}A_\mu{}^M\mathcal{D}_\nu\Lambda^N\left(\partial_\rho A_\sigma{}^P + \tfrac{3}{8}X_{RS}{}^P A_\rho{}^R A_\sigma{}^S\right)\right]\ , \tag{109}$$

where we have used (95) to delete contributions coming from the $B_{\mu\nu}{}^{NP}$ term in $\mathcal{H}_{\mu\nu}{}^{M}$.

The first term on the right hand side of (108) would follow from (104), but the second term cannot in general be obtained from assigning transformations to $B_{\rho\sigma}{}^{NP}$. Indeed, self-consistency of (108) requires that the second term on the right hand side be proportional to $X_{(NP)}{}^{R}$, which imposes a further constraint on D_{MNP}. We will see how we can nevertheless justify the transformation law (108) by introducing other antisymmetric tensors. For the moment, we just accept (108) and explore its consequences.

Expanding (109) using (**??**) and (70) and using a partial integration, (109) can be rewritten as

$$\delta\mathcal{L}_{\mathcal{VT}} = -\mathcal{A}_M\,, \tag{110}$$

which describes the quantum gauge anomalies due to anomalous chiral fermions

$$\delta(\Lambda)\Gamma[A] = \int \mathrm{d}^4x\,\Lambda^M\mathcal{A}_M\,, \tag{111}$$

where the covariant anomaly is

$$\begin{aligned}\mathcal{A}_M \;=\;& -\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\Lambda^P D_{MNP}\partial_\mu A_\nu{}^M\partial_\rho A_\sigma{}^N\\ & -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}\Lambda^P\big(D_{MNR}X_{[PS]}{}^N + \frac{3}{2}D_{MNP}X_{[RS]}{}^N\big)\partial_\mu A_\nu{}^M A_\rho{}^R A_\sigma{}^S\,.\end{aligned} \tag{112}$$

This expression formally looks like a symplectically covariant generalization of the electric consistent anomaly (121). To prove (110), one uses (95) and the preservation of D_{MNP} under gauge transformations, which follows from preservation of X, see (**??**), and of Ω, see (**??**), and reads

$$X_{M(N}{}^P\,D_{QR)P} = 0\,. \tag{113}$$

For the terms quartic in the gauge fields, one needs the following consequence of (113):

$$\begin{aligned}(X_{RS}{}^M\,X_{PQ}{}^N\,D_{LMN})_{[RSPL]} &= -(X_{RS}{}^M\,X_{PM}{}^N\,D_{LQN} + X_{RS}{}^M\,X_{PL}{}^N\,D_{QMN})_{[RSPL]}\\ &= -(X_{RS}{}^M\,X_{PL}{}^N\,D_{QMN})_{[RSPL]}\,,\end{aligned} \tag{114}$$

where the final line uses (95).

Let us summarize the result of our calculation up to the present point. We have used the action (100) and considered its transformations under (70) and (84), where $\Delta B_{\mu\nu}{}^{NP}$ was undetermined. We used the closure constraint and one new constraint (93). We showed that the choice (104) leads to invariance if D_{MNP} vanishes, which is the representation constraint used in the anomaly-free case. However, when we use instead the more general transformation (108) in the case $D_{MNP} \neq 0$, we obtain the non-vanishing classical variation (110). The corresponding expression (112) formally looks very similar to a symplectically covariant generalization of the electric consistent quantum anomaly.

7 The Consistent Anomaly

The physical information of a quantum field theory is contained in the Green's functions, which in turn are encoded in an appropriate generating functional. Treating the Yang-Mills fields W_μ as external fields, the generating functional for proper vertices can be written as a path integral over the other matter fields,

$$\mathrm{e}^{-\Gamma[W_\mu]} = \int \mathcal{D}\bar{\phi}\mathcal{D}\phi\,\mathrm{e}^{-\mathcal{S}(W_\mu,\bar{\phi},\phi)}\,. \tag{115}$$

The gauge invariance,

$$\delta_A \Gamma[W_\mu] = 0\,, \tag{116}$$

of the effective action encodes the Ward identities and is crucial for the renormalizability of the theory. Even if the classical action, $\mathcal{S}$, is gauge invariant, a non-invariance of the path integral measure may occur and violate (116), leading to a quantum anomaly. Even though the functional $\Gamma[W_\mu]$ is in general neither a local nor a polynomial functional of the W_μ, the quantum anomaly,

$$\delta(\Lambda)\Gamma[W] = -\int \mathrm{d}^4x\, \Lambda^A \left(\mathcal{D}_\mu \frac{\delta\Gamma[W]}{\delta W_\mu}\right)_A \equiv \int \mathrm{d}^4x\, \Lambda^A \mathcal{A}_A\,, \tag{117}$$

does have this property. More explicitly, for an arbitrary non-Abelian gauge group, the consistent form of the anomaly $\mathcal{A}_A$ is given by

$$\mathcal{A}_A \sim \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Big(T_A \partial_\mu \left(W_\nu \partial_\rho W_\sigma + \tfrac{1}{2} W_\nu W_\rho W_\sigma\right)\Big)\,, \tag{118}$$

where $W_\mu = W_\mu^A T_A$, and T_A denotes the generators in the representation space of the chiral fermions. Similarly there are supersymmetry anomalies, such that the final non-invariance of the one-loop effective action is

$$\mathcal{A} = \delta\Gamma(W) = \delta(\Lambda)\Gamma[W] + \delta(\epsilon)\Gamma[W] = \int \mathrm{d}^4x\, \left(\Lambda^A \mathcal{A}_A + \bar{\epsilon}\mathcal{A}_\epsilon\right)\,. \tag{119}$$

This anomaly should satisfy the Wess-Zumino consistency conditions, which are the statement that these variations should satisfy the symmetry algebra. For the gauge anomalies these are

$$\delta(\Lambda_1)\left(\Lambda_2^A \mathcal{A}_A\right) - \delta(\Lambda_2)\left(\Lambda_1^A \mathcal{A}_A\right) = \Lambda_1^B \Lambda_2^C f_{BC}{}^A \mathcal{A}_A\,. \tag{120}$$

If the effective action is non-invariant under gauge transformations, then also its supersymmetry transformation is non-vanishing. An anomalous spectrum of chiral fermions induces a gauge non-invariance of the quantum effective action, where $\mathcal{A}_C$ denotes the consistent anomaly

$$\begin{aligned}
\mathcal{A}_C &= -\frac{i}{4}\left[d_{ABC}F_{\mu\nu}^B + \left(d_{ABD}f_{CE}{}^B + \tfrac{3}{2}d_{ABC}f_{DE}{}^B\right)W_\mu^D W_\nu^E\right]\tilde{F}^{\mu\nu A}\,, &(121)\\
\bar{\epsilon}\mathcal{A}_\epsilon &= \mathrm{Re}\Big[\tfrac{3}{2}\mathrm{i}d_{ABC}\bar{\epsilon}_R\lambda_R^C\bar{\lambda}_L^A\lambda_L^B + \mathrm{i}d_{ABC}W_\nu^C\tilde{F}^{\mu\nu A}\bar{\epsilon}_L\gamma_\mu\lambda_R^B\\
&\qquad + \tfrac{3}{8}d_{ABC}f_{DE}{}^A\varepsilon^{\mu\nu\rho\sigma}W_\mu^D W_\nu^E W_\sigma^C\bar{\epsilon}_L\gamma_\rho\lambda_R^B\Big]\,. &(122)
\end{aligned}$$

The form of the anomaly depends on the renormalization scheme, which we have chosen such that the anomaly is proportional to d_{ABC}. Choosing a different scheme would change the coefficients in the GCS term. Quantum anomalies can cancel the terms in the variation (64) that are proportional to the symmetric part $C^{(s)}_{AB,C}$, provided we have $d_{ABC} = C^{(s)}_{AB,C}$. This is the Green-Schwarz mechanism. Thus, putting everything together, if we have

$$C_{AB,C} = d_{ABC} + C^{(\mathrm{CS})}_{AB,C}, \tag{123}$$

the variation of the GCS term and the quantum anomaly together cancel the variation (64). The coefficients d_{ABC} form a totally symmetric tensor that is not fixed by the consistency conditions. Comparison with (118) implies that they are of the form

$$d_{ABC} \sim \mathrm{Tr}\left(\{T_A, T_B\}T_C\right)\,. \tag{124}$$

Consequently, the low-energy theory is determined by the Wess-Zumino consistency conditions, rather than by the requirement of supersymmetry, and this procedure does not fix a quartic coupling for the gauginos.

We have shown how gauge invariance can be restored in the presence gauged axionic shift symmetries in general $\mathcal{N}=1$ supersymmetric gauge theories. What we have not yet checked is whether the new action $\hat{S}_f + S_{CS}$ is also invariant under supersymmetry. A careful calculation reveals

$$\delta(\epsilon)\left(\hat{S}_f + S_{\rm CS}\right) = \int \mathrm{d}^4x\ \mathrm{Re}\Big[-\tfrac{3}{2}\mathrm{i}C^{(\mathrm{s})}_{AB,C}\bar{\epsilon}_R\lambda^C_R\bar{\lambda}^A_L\lambda^B_L \\ -\mathrm{i}C^{(\mathrm{s})}_{AB,C}W^C_\nu\tilde{F}^{\mu\nu A}\bar{\epsilon}_L\gamma_\mu\lambda^B_R - \tfrac{3}{8}C^{(\mathrm{s})}_{AB,C}f_{DE}{}^A\varepsilon^{\mu\nu\rho\sigma}W^D_\mu W^E_\nu W^C_\sigma\bar{\epsilon}_L\gamma_\rho\lambda^B_R\Big]\,. \tag{125}$$

This is not zero, and in fact, it should not be zero. The reason is that the above classical action is not gauge invariant. As we are working in the Wess-Zumino gauge, this will also imply a non-invariance under supersymmetry. Thus the classical gauge non-invariance triggers a classical supersymmetry non-invariance. However, this is also true for the quantum gauge anomaly, it also triggers a supersymmetry anomaly of the quantum effective action,

$$\delta(\epsilon)\Gamma[W^A_\mu] = \int d^4x\bar{\epsilon}\mathcal{A}_\epsilon\,. \tag{126}$$

The supersymmetry anomaly has been calculated by Brandt, and it is precisely the negative of equation (125),

$$\delta(\epsilon)\left(\hat{S}_f + S_{\rm CS}\right) + \int d^4x\bar{\epsilon}\mathcal{A}_\epsilon = 0\,. \tag{127}$$

Thus, the entire classical plus quantum theory is indeed supersymmetric.

8 Mixed Anomalies in Supergravity

Having established that the two forms of anomalies are not equivalent for a simple non-abelian gauge group, we point out how one can interpolate between them for a mixed anomaly. Consider a gauge group which is the product of a single $U(1)$ factor and a simple non-abelian group G, $= G\times U(1)$. Write the gauge field $V_\mu = A^a_\mu T^a + iQC_\mu$, where the T^a are anti-hermitian generators of G and Q is the charge under $U(1)$. We use $F^a_{\mu\nu}$ for the non-abelian field strength and $C_{\mu\nu}$ for the abelian field strength. Inserting $V_\mu = A^a_\mu T^a + iQC_\mu$ into the expression for the consistent anomaly and covariant anomaly, we pick up terms which are purely abelian or purely non-abelian anomalies as well as mixed anomalies. We write this

$$(D_\mu j^\mu)^a = \mathcal{A}^a \;=\; \mathcal{A}^a_{\rm non-ab} + \mathcal{A}^a_{\rm mixed}\;, \qquad (D_\mu j^\mu)^Q = \mathcal{A}^Q \;=\; \mathcal{A}^Q_{\rm abel} + \mathcal{A}^Q_{\rm mixed}\;, \tag{128}$$

where $j^a_\mu = -\delta\mathcal{L}/\delta A^a_\mu = -i\bar{\psi}\gamma_\mu T^a L\psi$ and $j^Q_\mu = -\delta\mathcal{L}/\delta C_\mu = -i\bar{\psi}\gamma_\mu iQ\,L\psi$ are the non-abelian and abelian currents. Below subscripts "cov" or "con" indicate whether a given term in the anomalies is written in the covariant form (107) or the consistent form (**??**).

To be explicit, we list the mixed anomalies $\mathcal{A}^a_{\rm mixed}$ and $\mathcal{A}^Q_{\mathcal{M}}$ in covariant and consistent form respectively,

$$\begin{aligned}
\mathcal{A}^Q_{\mathcal{M}\,\mathcal{G}} &= \frac{i}{32\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\ iQ\,F_{\mu\nu}F_{\rho\sigma} = \frac{i}{8\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[iQ\partial_\mu\Big(A_\nu\,\partial_\rho A_\sigma + \frac{2}{3}A_\nu A_\rho A_\sigma\Big)\right],\\
\mathcal{A}^Q_{\mathcal{M}\,\mathcal{C}} &= \frac{i}{24\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[iQ\partial_\mu\Big(A_\nu\,\partial_\rho A_\sigma + \frac{1}{2}A_\nu A_\rho A_\sigma\Big)\right],\\
\mathcal{A}^a_{\mathcal{M}\,\mathcal{G}} &= \frac{i}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[T^a iQ\,C_{\mu\nu}F_{\rho\sigma}\right],\\
\mathcal{A}^a_{\mathcal{M}\,\mathcal{C}} &= \frac{i}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[T^a iQ\partial_\mu\Big(C_\nu\,\partial_\rho A_\sigma + \frac{1}{4}C_\nu A_\rho A_\sigma\Big)\right].
\end{aligned} \tag{129}$$

There are two candidate polynomials in C_μ and A_μ from which finite local counter terms in the Lagrangian can be constructed,

$$\mathcal{L}_1 = -\frac{i}{12\pi^2}\,\epsilon^{\mu\nu\rho\sigma} C_\mu \,\mathrm{tr}\left[iQA_\nu\partial_\rho A_\sigma\right], \qquad \mathcal{L}_2 = -\frac{i}{12\pi^2}\,\epsilon^{\mu\nu\rho\sigma} C_\mu \,\mathrm{tr}\left[iQA_\nu A_\rho A_\sigma\right]. \tag{130}$$

Their gauge variations under non-abelian gauge transformations **??** are

$$\begin{aligned}\delta_\theta \mathcal{L}_1 &= -\frac{i}{12\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[iQ\theta\Big(\partial_\mu C_\nu\,\partial_\rho A_\sigma - 2\partial_\mu(C_\nu A_\rho A_\sigma)\Big)\right],\\ \delta_\theta \mathcal{L}_2 &= -\frac{i}{4\pi^2}\,\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\left[iQ\theta\partial_\mu(C_\nu A_\rho A_\sigma)\right].\end{aligned} \tag{131}$$

One may add these counter terms with arbitrary coefficients to the Lagrangian. This would modify the non-abelian current conservation law by terms proportional to (131). The unique combination

$$\mathcal{L}_{\mathrm{ct}} \;=\; \mathcal{L}_1 + \frac{3}{4}\mathcal{L}_2 \tag{132}$$

precisely cancels the non-abelian mixed anomaly in the consistent form,

$$\delta_\theta \mathcal{L}_{\mathrm{ct}} \;=\; -\,\theta^a \mathcal{A}^a_{\mathcal{M}\,\mathcal{C}}\;. \tag{133}$$

Under abelian gauge variations

$$\delta_\Lambda C_\mu = \partial_\mu \Lambda \tag{134}$$

the counter term gives

$$\delta_\Lambda \mathcal{L}_{\mathrm{ct}} = -\Lambda\big(\mathcal{A}^Q_{\mathcal{M}\,\mathcal{C}} - \mathcal{A}^Q_{\mathcal{M}\,\mathcal{G}}\big)\,, \tag{135}$$

it rotates the consistent form of the abelian mixed anomaly into covariant form. This is essential for the Green-Schwarz mechanism. The gauge variations (131) of the counter terms yield total derivatives, but the covariant mixed anomaly $\mathcal{A}^a_{\mathcal{M}\,\mathcal{G}}$ given in (129) involves $\epsilon^{\mu\nu\rho\sigma}\,\mathrm{tr}\,T^a C_{\mu\nu}F_{\rho\sigma}$ which is not a total derivative. Hence the non-abelian variations of the counter terms (131) could never fully cancel $\mathcal{A}^a_{mixed\ cov}$, it is therefore crucial to use $\mathcal{A}^a_{mixed\ con}$.

9 The Anomalies in $D = 6$ Supergravity

We review some properties of the field equations of six-dimensional (1,0) supergravity coupled to tensor and vector multiplets, and in particular their relation to covariant, consistent and gravitational anomalies. We also describe a lagrangian formulation for this system, obtained applying the Pasti-Sorokin-Tonin prescription. For completeness, we are also including some new interesting results in identification of the supergravity anomalies.

Perturbative six-dimensional string vacua with minimal supersymmetry can arise for instance as compactifications of the heterotic string on $K3$, or as parameter-space orbifolds of $K3$ reductions of the type-IIB string. While in the former case only a single tensor multiplet is present, in the latter one obtains vacua with variable numbers of tensor multiplets, related by string dualities to non-perturbative heterotic and M-theory vacua. In these models, the anomalous contribution due to fermion loops is derived from the residual anomaly polynomial

$$c^r_x\; c^s_y\; \eta_{rs}\; \mathrm{tr}_x F^2\; \mathrm{tr}_y F^2 \quad ,$$

where the c's are a collection of constants (x and y run over the various semi-simple Lie factors in the gauge group and over the Lorentz group) and η is the Minkowski metric for $SO(1, n_T)$, with n_T the number of tensor multiplets. As a consequence, several antisymmetric tensors take part in a generalized Green-Schwarz mechanism. The corresponding Green-Schwarz term has the form

$$B^r\ c_r^x\ \mathrm{tr}_x F^2$$

and, if one considers only gauge anomalies, contains only two derivatives, and thus belongs to the low-energy effective action. Consequently, the resulting low-energy lagrangian has a gauge anomaly, that the Wess-Zumino conditions relate to a supersymmetry anomaly.

The complete coupling of (1,0) six-dimensional supergravity to non-abelian vector and tensor multiplets requiring the closure of the Wess-Zumino conditions, has revealed another related aspect of these six-dimensional models: a quartic coupling for the gauginos is undetermined, and the construction is consistent for any choice of this coupling. Correspondingly, the commutator of two supersymmetry transformations on the gauginos contains an extension, that plays a crucial role in ensuring that the Wess-Zumino consistency conditions close on-shell. The coupling of (1,0) six-dimensional supergravity to non-abelian vectors and self-dual tensors reveals neatly the realization of a peculiar aspect of the physics of branes: singularities in the gauge couplings appear for particular values of the scalars in the tensor multiplets, and can be ascribed to a phase transition in which a string becomes tensionless. In this model the divergence of the energy-momentum tensor is non-vanishing, as is properly the case for a theory that has gauge anomalies but no gravitational anomalies. The whole construction can also be repeated with the inclusion of abelian vectors, that actually allow more general couplings, since in this case the residual anomaly polynomial can have the more general form

$$\mathcal{P}_{\mathcal{A}} = \ c^r_{ab}\ c^s_{cd}\ \eta_{rs} F^a \wedge F^b \wedge F^c \wedge F^d\ , \tag{136}$$

where the indices a, b, c, d run over the different $U(1)$ gauge groups, and where the c^r's are symmetric matrices that may not be simultaneously diagonalized. Notice that these low-energy couplings are obtained by consistency once one includes the Green-Schwarz term in the low-energy theory. The complete theory, supersymmetric and gauge-invariant, would also include additional non-local couplings arising from fermion loops. This is exactly as in the ten-dimensional case, what is peculiar of these six-dimensional models is that here the anomalous terms belong to the low-energy effective action. In order to have an explicit realization of the low-energy dynamics of six-dimensional string vacua, it is of interest to consider how the whole construction is modified by the inclusion of hypermultiplets. The complete coupling to a single tensor multiplet and to vector and charged hypermultiplets was obtained for the case in which no anomalies and no singular couplings are present. More recently, an analysis of the case in which various tensor multiplets are present without taking into account the anomalous terms. Still, this analysis shows that, in correspondence to the phase transition, additional singular terms appear because of the presence of charged hypermultiplets.

9.1 Supergravity in Six Dimensions Coupled to Tensor Multiplets

In formulating the low-energy couplings between tensor and vector multiplets, one has two natural options. The first is related to covariant field equations and to the corresponding covariant anomalies. It has the virtue of respecting gauge covariance and supersymmetry, but the resulting field equations are not integrable. The second is related to consistent, and thus integrable field equations. These may be derived from an action principle that satisfies Wess-Zumino consistency conditions, and as a result embody a supersymmetry anomaly.

We describe minimal $(1, 0)$ six-dimensional supergravity coupled to n tensor multiplets. Simple supersymmetry in six dimensions is generated by an $Sp(2)$ doublet of chiral spinorial charges Q^a $(a = 1, 2)$, obeying the symplectic Majorana condition

$$Q^a = \epsilon^{ab} C \bar{Q}_b^T\ , \tag{137}$$

where ϵ^{ab} is the $Sp(2)$ antisymmetric invariant tensor. Since all fermi fields appear as $Sp(2)$ doublets, from now on we will mostly use Ψ to denote a doublet Ψ^a.

The theory includes the vielbein $e_\mu{}^a$, a left-handed gravitino Ψ_μ, $(n+1)$ antisymmetric tensors $B^r_{\mu\nu}$ $(r=0,...,n)$ obeying (anti)self-duality conditions, n right-handed tensorini χ^m $(m=1,...,n)$, and n scalars. The scalars parameterize the coset space $SO(1,n)/SO(n)$, and are thus associated to the $SO(1,n)$ matrix $(r=0,...n)$

$$V = (v_r x^m_r \quad , (138)$$

whose matrix elements satisfy the constraints

$$v^r v_r = 1\,, \qquad v_r v_s - x^m_r x^m_s = \eta_{rs}\,, \qquad v^r x^m_r = 0\,. \tag{139}$$

Defining

$$G_{rs} = v_r v_s + x^m_r x^m_s\,, \tag{140}$$

the tensor (anti)self-duality conditions can be succinctly written

$$G_{rs} H^{s\mu\nu\rho} = \frac{1}{6e} \epsilon^{\mu\nu\rho\alpha\beta\gamma} H_{r\alpha\beta\gamma}\,, \tag{141}$$

where $H^r_{\mu\nu\rho} = 3\partial_{[\mu} B^r_{\nu\rho]}$. These relations only hold to lowest order in the fermi fields, and imply that $v_r H^r_{\mu\nu\rho}$ is self dual, while the n tensors $x^m_r H^r_{\mu\nu\rho}$ are antiself dual, as one can see using (187). The divergence of (141) yields the second-order tensor equation

$$D_\mu (G_{rs} H^{s\mu\nu\rho}) = 0 \tag{142}$$

while, to lowest order, the fermionic equations are

$$\gamma^{\mu\nu\rho} D_\nu \Psi_\rho + v_r H^{r\mu\nu\rho} \gamma_\nu \Psi_\rho - \frac{i}{2} x^m_r H^{r\mu\nu\rho} \gamma_{\nu\rho} \chi^m + \frac{i}{2} x^m_r \partial_\nu v^r \gamma^\nu \gamma^\mu \chi^m = 0 \tag{143}$$

and

$$\gamma^\mu D_\mu \chi^m - \frac{1}{12} v_r H^{r\mu\nu\rho} \gamma_{\mu\nu\rho} \chi^m - \frac{i}{2} x^m_r H^{r\mu\nu\rho} \gamma_{\mu\nu} \Psi_\rho - \frac{i}{2} x^m_r \partial_\nu v^r \gamma^\mu \gamma^\nu \Psi_\mu = 0\,. \tag{144}$$

Varying the fermi fields in them with the supersymmetry transformations

$$\begin{aligned}
&\delta e_\mu{}^a = -i(\bar\epsilon \gamma^a \Psi_\mu)\,, \qquad \delta B^r_{\mu\nu} = i v^r (\bar\Psi_{[\mu} \gamma_{\nu]} \epsilon) + \frac{1}{2} x^{mr} (\bar\chi^m \gamma_{\mu\nu} \epsilon)\,, \\
&\delta v_r = x^m_r (\bar\epsilon \chi^m)\,, \qquad \delta \Psi_\mu = D_\mu \epsilon + \frac{1}{4} v_r H^r_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon\,, \\
&\delta \chi^m = \frac{i}{2} x^m_r \partial_\mu v^r \gamma^\mu \epsilon + \frac{i}{12} x^m_r H^r_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon\,,
\end{aligned} \tag{145}$$

generates the bosonic equations, using also (141) and (142). Thus, the scalar field equation is

$$x^m_r D_\mu (\partial^\mu v^r) + \frac{2}{3} x^m_r v_s H^r_{\alpha\beta\gamma} H^{s\alpha\beta\gamma} = 0\,, \tag{146}$$

while the Einstein equation is

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \partial_\mu v^r \partial_\nu v_r - \frac{1}{2} g_{\mu\nu} \partial_\alpha v^r \partial^\alpha v_r - G_{rs} H^r_{\mu\alpha\beta} H^s{}_\nu{}^{\alpha\beta} = 0\,. \tag{147}$$

To this order, this amounts to a proof of supersymmetry, and it is also possible to show that the commutator of two supersymmetry transformations on the bosonic fields closes on the local symmetries

$$[\delta_1, \delta_2] = \delta_{gct}(\xi^\mu = -i(\bar{\epsilon}_1 \gamma^\mu \epsilon_2)) + \delta_{tens}(\Lambda^r_\mu = -\frac{1}{2} v^r \xi_\mu - \xi^\nu B^r_{\mu\nu})$$
$$+ \delta_{SO(n)}(A^{mn} = \xi^\mu x^{mr} (\partial_\mu x^n_r)) + \delta_{Lorentz}(\Omega^{ab} = -\xi_\mu(\omega^{\mu ab} - v_r H^{r\mu ab})) \,. \tag{148}$$

To this order, one can not see the local supersymmetry transformation in the gauge algebra, since the expected parameter, $\xi^\mu \Psi_\mu$, is generated by bosonic variations. As usual, the spin connection satisfies its equation of motion, that to lowest order in the fermi fields is

$$D_\mu e_\nu{}^a - D_\nu e_\mu{}^a = 0 \,, \tag{149}$$

and implies the absence of torsion.

Completing these equations will require terms cubic in the fermi fields in the fermionic equations, and terms quadratic in the fermi fields in their supersymmetry transformations. Supersymmetry will then determine corresponding modifications of the bosonic equations, and the (anti)self-duality conditions (141) will also be modified by terms quadratic in the fermi fields. Supercovariance actually fixes all terms containing the gravitino in the first-order equations and in the supersymmetry variations of fermi fields.

The supercovariant forms

$$\hat{\omega}_{\mu\nu\rho} = \omega^0_{\mu\nu\rho} - \frac{i}{2}(\bar{\Psi}_\mu \gamma_\nu \Psi_\rho + \bar{\Psi}_\nu \gamma_\rho \Psi_\mu + \bar{\Psi}_\nu \gamma_\mu \Psi_\rho) \,, \tag{150}$$

$$\hat{H}^r_{\mu\nu\rho} = H^r_{\mu\nu\rho} - \frac{1}{2} x^{mr} (\bar{\chi}^m \gamma_{\mu\nu} \Psi_\rho + \bar{\chi}^m \gamma_{\nu\rho} \Psi_\mu + \bar{\chi}^m \gamma_{\rho\mu} \Psi_\nu) - \frac{i}{2} v^r (\bar{\Psi}_\mu \gamma_\nu \Psi_\rho + \bar{\Psi}_\nu \gamma_\rho \Psi_\mu + \bar{\Psi}_\rho \gamma_\mu \Psi_\nu) \,,$$
$$\widehat{\partial_\mu v^r} = \partial_\mu v^r - x^{mr} (\bar{\chi}^m \Psi_\mu) \,, \tag{151}$$

where

$$\omega^0_{\mu\nu\rho} = \frac{1}{2} e_{\rho a} (\partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a) - \frac{1}{2} e_{\mu a} (\partial_\nu e_\rho{}^a - \partial_\rho e_\nu{}^a) + \frac{1}{2} e_{\nu a} (\partial_\rho e_\mu{}^a - \partial_\mu e_\rho{}^a) \tag{152}$$

is the standard spin connection in the absence of torsion, do not generate derivatives of the parameter under supersymmetry. In the same spirit, one can consider the supercovariant transformations

$$\delta \Psi_\mu = \hat{D}_\mu \epsilon + \frac{1}{4} v_r \hat{H}^r_{\mu\nu\rho} \gamma^{\nu\rho} \epsilon \,, \qquad \delta \chi^m = \frac{i}{2} x^m_r (\widehat{\partial_\mu v^r}) \gamma^\mu \epsilon + \frac{i}{12} x^m_r \hat{H}^r_{\mu\nu\rho} \gamma^{\mu\nu\rho} \epsilon \,. \tag{153}$$

The tensorino transformation is complete, while the gravitino transformation could include additional terms quadratic in the tensorini. On the other hand, one does not expect modifications of the bosonic transformations in the complete theory.

The algebra (148) has been obtained varying only the fermi fields in the bosonic supersymmetry transformations. The next step is to compute the commutator varying the bosonic fields as well. There is no important novelty in the complete commutator on v^r and on the vielbein $e_\mu{}^a$. However, the local Lorentz parameter is modified and takes the form

$$\Omega^{ab} = -\xi^\mu (\hat{\omega}_\mu{}^{ab} - v_r \hat{H}^{r\,ab}_\mu) \tag{154}$$

while, as anticipated, the supersymmetry parameter is

$$\zeta = \xi^\mu \Psi_\mu \quad . \tag{155}$$

These results are obtained using the torsion equation for $\hat{\omega}$,

$$\hat{D}_\mu e_\nu{}^a - \hat{D}_\nu e_\mu{}^a = 2S^a{}_{\mu\nu} = -i(\bar{\Psi}_\mu \gamma^a \Psi_\nu) \quad . \tag{156}$$

One can also compute the commutator on x^m_r and (187) determine its supersymmetry variation

$$\delta x^m_r = v_r(\bar{\epsilon}\chi^m) \quad , \tag{157}$$

and the resulting commutator includes a local $SO(n)$ transformation of parameter

$$A^{mn} = \xi^\mu x^{mr}(\partial_\mu x^n_r) + (\bar{\chi}^m \epsilon_2)(\bar{\chi}^n \epsilon_1) - (\bar{\chi}^m \epsilon_1)(\bar{\chi}^n \epsilon_2) \quad . \tag{158}$$

New results come from the complete commutator on $B^r_{\mu\nu}$, where one needs to use the (anti)self-duality conditions. Supercovariantization is at work here, since these conditions are first-order equations, that become

$$G_{rs}\hat{H}^s_{\mu\nu\rho} = \frac{1}{6e}\epsilon_{\mu\nu\rho\alpha\beta\gamma}\hat{H}^{\alpha\beta\gamma}_r \quad . \tag{159}$$

It is actually possible to alter these conditions demanding that the modified tensor

$$\hat{\mathcal{H}}^r_{\mu\nu\rho} = \hat{H}^r_{\mu\nu\rho} + i\alpha v^r(\bar{\chi}^m \gamma_{\mu\nu\rho}\chi^m) \tag{160}$$

satisfy (anti)self-duality conditions as in (159). Using (187), one can see that the new χ^2 terms contribute only to the self-duality condition, while the tensors $x^m_r \hat{H}^r_{\mu\nu\rho}$ remain antiself dual without extra χ^2 terms. Consequently, since the commutator on $B^r_{\mu\nu}$ uses only the antiself-duality conditions, the result does not contain terms proportional to α. The commutator on the tensor fields generates all local symmetries in the proper form, aside from the extra terms

$$[\delta_1, \delta_2]_{extra} B^r_{\mu\nu} = \frac{1}{2}v^r(\bar{\epsilon}_1\chi^m)(\bar{\chi}^m\gamma_{\mu\nu}\epsilon_2) - \frac{1}{2}v^r(\bar{\epsilon}_2\chi^m)(\bar{\chi}^m\gamma_{\mu\nu}\epsilon_1) \quad , \tag{161}$$

that may be canceled adding χ^2 terms to the transformation of the gravitino. The most general expression one can add is

$$\delta'\Psi_\mu = ia\ \gamma_\mu\chi^m(\bar{\epsilon}\chi^m) + ib\ \gamma_\nu\chi^m(\bar{\epsilon}\gamma_\mu{}^\nu\chi^m) + ic\ \gamma_{\mu\nu\rho}\chi^m(\bar{\epsilon}\gamma^{\nu\rho}\chi^m) \quad , \tag{162}$$

with a, b and c real coefficients. The commutator on $e_\mu{}^a$ now closes with a local Lorentz parameter modified by the addition of

$$\Delta\Omega^{ab} = -\frac{1}{2}[(\bar{\chi}^m\epsilon_1)(\bar{\epsilon}_2\gamma^{ab}\chi^m) - (\bar{\chi}^m\epsilon_2)(\bar{\epsilon}_1\gamma^{ab}\chi^m)] \quad , \tag{163}$$

while the commutators on the scalar fields are not modified.

One can now start to compute the commutators on fermi fields, that as usual close only on shell. We will actually use this result to derive the complete fermionic equations. Supercovariance determines the field equation of the tensorini up to a term proportional to χ^3. Closure of the algebra fixes this additional term, and the end result is

$$\gamma^\mu\hat{D}_\mu\chi^m - \frac{1}{12}v_r\hat{H}^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\chi^m - \frac{i}{2}x^m_r\hat{H}^{r\mu\nu\rho}\gamma_{\mu\nu}\Psi_\rho - \frac{i}{2}x^m_r(\hat{\partial_\nu v^r})\gamma^\mu\gamma^\nu\Psi_\mu - \frac{i}{2}\gamma^\alpha\chi^n(\bar{\chi}^n\gamma_\alpha\chi^m) = 0\,.$$

The complete commutator of two supersymmetry transformations on the tensorini is then

$$[\delta_1, \delta_2]\chi^m = \delta_{gct}\ \chi^m + \delta_{Lorentz}\ \chi^m + \delta_{SO(n)}\ \chi^m + \delta_{susy}\ \chi^m + \frac{1}{4}\gamma^\alpha\xi_\alpha\ [\text{eq. }\chi^m] \quad . \tag{164}$$

A similar result can be obtained for the gravitino. In this case the complete equation,

$$\gamma^{\mu\nu\rho}\hat{D}_\nu\Psi_\rho + \frac{1}{4}v_r\hat{H}^r_{\nu\alpha\beta}\gamma^{\mu\nu\rho}\gamma^{\alpha\beta}\Psi_\rho - \frac{i}{2}x^m_r\hat{H}^{r\mu\nu\rho}\gamma_{\nu\rho}\chi^m + \frac{i}{2}x^m_r(\hat{\partial_\nu v^r})\gamma^\nu\gamma^\mu\chi^m \\ + \frac{3i}{2}\gamma^{\mu\alpha}\chi^m(\bar{\chi}^m\Psi_\alpha) - \frac{i}{4}\gamma^{\mu\alpha}\chi^m(\bar{\chi}^m\gamma_{\alpha\beta}\Psi^\beta) + \frac{i}{4}\gamma_{\alpha\beta}\chi^m(\bar{\chi}^m\gamma^{\mu\alpha}\Psi^\beta) - \frac{i}{2}\chi^m(\bar{\chi}^m\gamma^{\mu\alpha}\Psi_\alpha) = 0\,, \tag{165}$$

is fixed by supercovariance, and the commutator closes up to terms proportional to a particular combination of (166) and its γ-trace. Moreover, a non-trivial symplectic structure makes its first appearance in a commutator, so that the final result is

$$[\delta_1,\delta_2]\Psi^a_\mu = \delta_{gct}\Psi^a_\mu + \delta_{Lorentz}\Psi^a_\mu + \delta_{susy}\Psi^a_\mu + \frac{3}{8}\xi^\alpha\gamma_\alpha([eq.\ \Psi_\mu] - \frac{1}{4}\gamma_\mu[\gamma - trace])^a \\ + \frac{1}{96}\sigma^{ia}_b\gamma^{\alpha\beta\gamma}\xi^i_{\alpha\beta\gamma}([eq.\ \Psi_\mu] - \frac{1}{4}\gamma_\mu[\gamma - trace])^b \quad , \tag{166}$$

where

$$\xi^i_{\alpha\beta\gamma} = -i[\bar{\epsilon}_1\gamma_{\alpha\beta\gamma}\epsilon_2]^i \quad . \tag{167}$$

Summarizing, from the algebra we have obtained the complete fermionic equations of $(1,0)$ six-dimensional supergravity coupled to n tensor multiplets. In addition, the modified 3-form

$$\hat{\mathcal{H}}^r_{\mu\nu\rho} = \hat{H}^r_{\mu\nu\rho} - \frac{i}{8}v^r(\bar{\chi}^m\gamma_{\mu\nu\rho}\chi^m) \tag{168}$$

satisfies the (anti)self-duality conditions

$$G_{rs}\hat{\mathcal{H}}^s_{\mu\nu\rho} = \frac{1}{6e}\epsilon_{\mu\nu\rho\alpha\beta\gamma}\hat{\mathcal{H}}^{\alpha\beta\gamma}_r \quad . \tag{169}$$

We have also identified the complete supersymmetry transformations, that we collect here for convenience:

$$\delta e_\mu{}^a = -i(\bar{\epsilon}\gamma^a\Psi_\mu) \quad , \\ \delta B^r_{\mu\nu} = iv^r(\bar{\Psi}_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2}x^{mr}(\bar{\chi}^m\gamma_{\mu\nu}\epsilon) \quad , \\ \delta v_r = x^m_r(\bar{\chi}^m\epsilon) \quad , \\ \delta\Psi_\mu = \hat{D}_\mu\epsilon + \frac{1}{4}v_r\hat{H}^r_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon - \frac{3i}{8}\gamma_\mu\chi^n(\bar{\epsilon}\chi^n) - \frac{i}{8}\gamma^\nu\chi^n(\bar{\epsilon}\gamma_{\mu\nu}\chi^n) + \frac{i}{16}\gamma_{\mu\nu\rho}\chi^n(\bar{\epsilon}\gamma^{\nu\rho}\chi^n) \quad , \\ \delta\chi^m = \frac{i}{2}x^m_r(\hat{\partial_\alpha v^r})\gamma^\alpha\epsilon + \frac{i}{12}x^m_r\hat{H}^r_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\epsilon \quad . \tag{170}$$

In order to obtain the bosonic equations, it is convenient to associate the fermionic equations to the Lagrangian

$$e^{-1}\mathcal{L}_{\mathcal{F}} = -\frac{i}{2}\bar{\Psi}_\mu\gamma^{\mu\nu\rho}D_\nu[\frac{1}{2}(\omega+\hat{\omega})]\Psi_\rho - \frac{i}{8}v_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho) \\ + \frac{i}{48}v_r[H+\hat{H}]^r_{\alpha\beta\gamma}(\bar{\Psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma}\Psi_\nu) + \frac{i}{2}\bar{\chi}^m\gamma^\mu D_\mu(\hat{\omega})\chi^m - \frac{i}{24}v_r\hat{H}^r_{\mu\nu\rho}(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m) \\ + \frac{1}{4}x^m_r[\partial_\nu v^r + \hat{\partial_\nu v^r}](\bar{\Psi}_\mu\gamma^\nu\gamma^\mu\chi^m) - \frac{1}{8}x^m_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\Psi}_\mu\gamma_{\nu\rho}\chi^m) \\ + \frac{1}{24}x^m_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\Psi}^\alpha\gamma_{\alpha\mu\nu\rho}\chi^m) + \frac{1}{8}(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m)(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho) - \frac{1}{8}(\bar{\chi}^m\gamma^\alpha\chi^n)(\bar{\chi}^m\gamma_\alpha\chi^n)\,, \tag{171}$$

where the spin connection is

$$\omega_{\mu\nu\rho} = \omega^0_{\mu\nu\rho} - \frac{i}{2}(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho + \bar{\Psi}_\nu\gamma_\rho\Psi_\mu + \bar{\Psi}_\nu\gamma_\mu\Psi_\rho) - \frac{i}{4}(\bar{\Psi}^\alpha\gamma_{\mu\nu\rho\alpha\beta}\Psi^\beta) - \frac{i}{4}(\bar{\chi}^m\gamma_{\mu\nu\rho}\chi^m) \tag{172}$$

satisfies its equation of motion, and is thus kept fixed in all variations.

In order to derive the bosonic equations, one can add to (172)

$$e^{-1}\mathcal{L}_{\mathcal{B}} = -\frac{1}{4}R + \frac{1}{12}G_{rs}H^{r\mu\nu\rho}H^{s}_{\mu\nu\rho} - \frac{1}{4}\partial_\mu v^r \partial^\mu v_r \,. \tag{173}$$

One can then obtain from $\mathcal{L}_{fer} + \mathcal{L}_{bose}$ the equations for the vielbein and the scalars, with the prescription that the (anti)self-duality conditions be used only after varying. Actually, ignoring momentarily (169) and varying $\mathcal{L}_{fer} + \mathcal{L}_{bose}$ with respect to the antisymmetric tensor $B^r_{\mu\nu}$ yields the second-order tensor equation, the divergence of (169),

$$D_\mu(G_{rs}\hat{H}^{s\mu\nu\rho}) = \frac{1}{2}D_\mu[x^m_r(\bar{\chi}^m\gamma^{\mu\nu\rho\alpha}\Psi_\alpha)] - \frac{i}{4}D_\mu[v_r(\bar{\Psi}_\alpha\gamma^{\alpha\beta\mu\nu\rho}\Psi_\beta)] + \frac{i}{4}D_\mu[v_r(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m)] \,. \tag{174}$$

In a similar fashion, the scalar equation is

$$\begin{aligned}
&x^m_r[\frac{1}{2}D_\mu(\partial^\mu v^r) + \frac{1}{3}v_s H^{r\mu\nu\rho}H^s_{\mu\nu\rho} - \frac{i}{4}H^{r\mu\nu\rho}(\bar{\Psi}_\mu\gamma_\nu\Psi_\rho) + \frac{i}{24}H^r_{\alpha\beta\gamma}(\bar{\Psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma}\Psi_\nu) \\
&-\frac{i}{24}H^r_{\mu\nu\rho}(\bar{\chi}^n\gamma^{\mu\nu\rho}\chi^n) - \frac{1}{2}D_\nu(x^n_r(\bar{\Psi}_\mu\gamma^\nu\gamma^\mu\chi^n))] + v_r[-\frac{1}{4}H^{r\mu\nu\rho}(\bar{\Psi}_\mu\gamma_{\nu\rho}\chi^m) \\
&+\frac{1}{12}H^{r\mu\nu\rho}(\bar{\Psi}^\alpha\gamma_{\alpha\mu\nu\rho}\chi^m)] = 0 \,,
\end{aligned} \tag{175}$$

while the Einstein equation is

$$\begin{aligned}
&\frac{1}{2}e_{\beta a}[R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R - G_{rs}H^{r\alpha\nu\rho}H^{s\beta}{}_{\nu\rho} + \frac{1}{6}g^{\alpha\beta}G_{rs}H^r_{\mu\nu\rho}H^{s\mu\nu\rho} \\
&+\partial^\alpha v^r\partial^\beta v_r - \frac{1}{2}g^{\alpha\beta}\partial_\mu v^r\partial^\mu v_r] - \frac{i}{2}e^\alpha{}_a(\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\hat{D}_\nu\Psi_\rho) + \frac{i}{2}(\bar{\Psi}_a\gamma^{\alpha\nu\rho}\hat{D}_\nu\Psi_\rho) \\
&+\frac{i}{2}(\bar{\Psi}_\mu\gamma^{\mu\alpha\rho}\hat{D}_a\Psi_\rho) + \frac{i}{2}(\bar{\Psi}_\mu\gamma^{\mu\nu\alpha}\hat{D}_\nu\Psi_a) - \frac{i}{4}e^\alpha{}_a v_r\hat{H}^r_{\mu\nu\rho}(\bar{\Psi}^\mu\gamma^\nu\Psi^\rho) + \frac{i}{4}v_r\hat{H}^r_{\mu a\rho}(\bar{\Psi}^\mu\gamma^\alpha\Psi^\rho) \\
&+\frac{i}{2}v_r\hat{H}^{r\alpha\nu\rho}(\bar{\Psi}_a\gamma_\nu\Psi_\rho) + \frac{i}{2}v_r\hat{H}^r_{a\nu\rho}(\bar{\Psi}^\alpha\gamma^\nu\Psi^\rho) + \frac{i}{24}e^\alpha{}_a v_r\hat{H}^r_{\beta\gamma\delta}(\bar{\Psi}_\mu\gamma^{\mu\nu\beta\gamma\delta}\Psi_\nu) \\
&-\frac{i}{12}v_r\hat{H}^r_{\beta\gamma\delta}(\bar{\Psi}_a\gamma^{\alpha\nu\beta\gamma\delta}\Psi_\nu) - \frac{i}{8}v_r\hat{H}^r_{a\beta\gamma}(\bar{\Psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma}\Psi_\nu) + \frac{i}{2}e^\alpha{}_a(\bar{\chi}^m\gamma^\mu\hat{D}_\mu\chi^m) \\
&-\frac{i}{2}(\bar{\chi}^m\gamma^\alpha\hat{D}_a\chi^m) - \frac{i}{24}e^\alpha{}_a v_r\hat{H}^r_{\mu\nu\rho}(\bar{\chi}^m\gamma^{\mu\nu\rho}\chi^m) + \frac{i}{8}v_r\hat{H}^r_{a\nu\rho}(\bar{\chi}^m\gamma^{\alpha\nu\rho}\chi^m) \\
&+\frac{1}{2}e^\alpha{}_a x^m_r(\hat{\partial_\nu v^r})(\bar{\Psi}_\mu\gamma^\nu\gamma^\mu\chi^m) - \frac{1}{2}x^m_r(\hat{\partial_a v^r})(\bar{\Psi}_\mu\gamma^\alpha\gamma^\mu\chi^m) - \frac{1}{2}x^m_r(\hat{\partial_\nu v^r})(\bar{\Psi}_a\gamma^\nu\gamma^\alpha\chi^m) \\
&-\frac{1}{4}e^\alpha{}_a x^m_r\hat{H}^r_{\mu\nu\rho}(\bar{\Psi}^\mu\gamma^{\nu\rho}\chi^m) + \frac{1}{2}x^m_r\hat{H}^r_{\mu a\rho}(\bar{\Psi}^\mu\gamma^{\alpha\rho}\chi^m) + \frac{1}{4}x^m_r\hat{H}^{r\alpha}{}_{\nu\rho}(\bar{\Psi}_a\gamma^{\nu\rho}\chi^m) \\
&+\frac{1}{4}x^m_r\hat{H}^r_{a\nu\rho}(\bar{\Psi}^\alpha\gamma^{\nu\rho}\chi^m) + \frac{1}{12}e^\alpha{}_a x^m_r\hat{H}^r_{\mu\nu\rho}(\bar{\Psi}_\sigma\gamma^{\sigma\mu\nu\rho}\chi^m) - \frac{1}{12}x^m_r\hat{H}^r_{\mu\nu\rho}(\bar{\Psi}_a\gamma^{\alpha\mu\nu\rho}\chi^m) \\
&-\frac{1}{4}x^m_r\hat{H}^r_{a\nu\rho}(\bar{\Psi}_\sigma\gamma^{\sigma\alpha\nu\rho}\chi^m) + (fermi)^4 = 0 \,.
\end{aligned} \tag{176}$$

For the sake of brevity, a number of quartic fermionic couplings, fully determined by the lagrangian of (172) and (173), are not written explicitly. It then takes a direct, if somewhat tedious, calculation to prove local supersymmetry, showing that

$$\delta S = \int d^6x \left(\delta\mathcal{F}\frac{\delta\mathcal{L}}{\delta\mathcal{F}} + \delta\mathcal{B}\frac{\delta\mathcal{L}}{\delta\mathcal{B}}\right) = 0 \,, \tag{177}$$

where F and B denote collectively the fermi and bose fields aside from the antisymmetric tensors. We would like to stress that the equations for the fermi fields defined from the gauge algebra differ from the lagrangian equations by overall factors that may be simply identified. To see why this works, consider a generic true supergravity theory with a collection of boson and fermion fields $B(x)$ and $F(x)$ and transformation rules which involve arbitrary spinor parameters $\epsilon(x)$. The action $S[B,F]$ is locally supersymmetric, which means that the supersymmetry variation

$$\delta S = \int d^6x \left(\frac{\delta}{\delta B}\delta B + \frac{\delta}{\delta F}\delta F \right) \equiv 0 \tag{178}$$

vanishes identically, for all configurations of $B(x), F(x), \epsilon(x)$. In particular, the terms of each order in F vanish independently. To lowest order, with fermions more specifically described as gravitinos $\psi_\mu(x)$ and Dirac spinors $\lambda(x)$, the fermion transformations have the generic structure

$$(\delta B)_0 \;=\; \bar\epsilon\, \mathcal{G} F \;=\; \bar\epsilon\, (\mathcal{G}^\mu\, \psi_\mu + \mathcal{G}'\, B\, \lambda) \tag{179}$$

$$(\delta F)_0 \;=\; \left\{ \begin{array}{rcl} (\delta\psi_\mu)_0 &=& (D_\mu + \mathcal{G}''{}_\mu\, B)\epsilon \\ (\delta\lambda)_0 &=& (\mathcal{G}^\mu\, \partial_\mu B + \mathcal{G}'''\, B)\epsilon. \end{array} \right. \tag{180}$$

The $\mathcal{G}$ and $\mathcal{G}'$ are matrices of the Clifford algebra with the appropriate tensor structure.

The lowest order term in δS is linear in the fermions

$$(\delta S)_{lin} = \int d^Dx \left[\frac{\delta}{\delta B}\,(\bar\epsilon\,\mathcal{G} F) + \frac{\delta}{\delta F}\,(\delta F)_0 \right] \equiv 0. \tag{181}$$

The variation $\delta/\delta B$ is purely bosonic to this order, and $\delta/\delta F$ is linear in fermions. Note that $(\delta S)_{lin}$ still vanishes for all configurations of $B(x), F(x), \epsilon(x)$. If ϵ is a Killing spinor, then, by definition $(\delta F)_0 = 0$, and linsu then reads

$$(\delta S)_{lin} = \int d^Dx\, \frac{\delta}{\delta B}\,(\bar\epsilon\,\mathcal{G} F) \;=\; 0. \tag{182}$$

It vanishes for all configurations of $B(x)$ which support Killing spinors and all fermion configurations $F(x)$. Thus the sum over all independent boson fields $B_I(x)$ vanishes locally

$$\sum_I \frac{\delta}{\delta B_I}\,(\bar\epsilon\,\mathcal{G} F)_I = 0. \tag{183}$$

If the fermion variations $(\bar\epsilon\,\mathcal{G} F)_I$ are independent, then each boson equation of motion $\delta/\delta B_I = 0$ is satisfied separately. In many cases the fermion variations are independent, in other cases one must supplement the equations 183 with gauge field equations of motion. It is in this way that a bosonic field configuration $B_I(x)$ which supports Killing spinors can give a solution of the bosonic equations of motion of the theory. The first order equations which determine these BPS configurations of $B_I(x)$ are the integrability conditions for the Killing spinor equations $(\delta\psi_\mu)_0 = 0$ and $(\delta\lambda)_0 = 0$.

9.2 Supersymmetry Algebra and Equations of Motion

We describe the full coupling of six-dimensional supergravity to vector, tensor and hypermultiplets. We first summarize the field content of the theory. The gravitational multiplet contains the vielbein $e_\mu{}^m$, a 2-form and a left-handed gravitino ψ_μ^A, the tensor multiplet contains a 2-form, a scalar and a right-handed tensorino, the vector multiplet contains a vector A_μ and a left-handed gaugino λ^A, and finally the hypermultiplet contains four scalars and a right-handed hyperino. In the presence of n_T tensor multiplets, the tensorinos are denoted by χ^{MA} where

$M = 1, ..., n_T$ is an $SO(n_T)$ index. The index $A = 1, 2$ is in the fundamental representation of $USp(2)$, and the gravitino, the tensorinos and the gauginos are $USp(2)$ doublets satisfying the symplectic-Majorana condition

$$\psi^A = \epsilon^{AB} C \bar{\psi}_B^T \quad . \tag{184}$$

The n_T scalars in the tensor multiplets parametrize the coset $SO(1, n_T)/SO(n_T)$, while the $(n_T + 1)$ 2-forms from the gravitational and tensor multiplets are collectively denoted by $B^r_{\mu\nu}$, with $r = 0, ..., n_T$ in the fundamental representation of $SO(1, n_T)$, and their field-strengths satisfy (anti)self-duality conditions. The vector and the gaugino are in the adjoint representation of the gauge group. Finally, taking into account n_H hypermultiplets, the hyperinos are denoted by Ψ^a, where $a = 1, ..., 2n_H$ is a $USp(2n_H)$ index, and the symplectic-Majorana condition for these spinors is

$$\Psi^a = \Omega^{ab} C \bar{\Psi}_b^T \quad , \tag{185}$$

where Ω^{ab} is the antisymmetric invariant tensor of $USp(2n_H)$. The hyper-scalars ϕ^α, $\alpha = 1, ..., 4n_H$, are coordinates of a quaternionic manifold, that is a manifold whose holonomy group is contained in $USp(2) \times USp(2n_H)$.

If the quaternionic manifold parametrized by the hyper-scalars has isometries, these correspond to global symmetries of the supergravity theory. Then the global symmetry group, or a subgroup can be gauged. We will consider without loss of generality the case in which the scalars parametrize the symmetric manifold $USp(2, 2n_H)/USp(2) \times USp(2n_H)$, whose isometry group is $USp(2, 2n_H)$. We will then describe the gauging of the maximal compact subgroup $USp(2) \times USp(2n_H)$ of the isometry group. All the results can be naturally generalized to other symmetric quaternionic spaces.

The scalars in the tensor multiplets can be described in terms of the $SO(1, n_T)$ matrix

$$V = \left(v_r x_r^M \right. \quad , \tag{186}$$

whose elements satisfy the constraints

$$v^r v_r = 1 \quad , \qquad v_r v_s - x_r^M x_s^M = \eta_{rs} \quad , \qquad v^r x_r^M = 0 \quad . \tag{187}$$

We will take v_r and x_r^M, with the constraints of (187), as fundamental fields, so that the composite $SO(n_T)$ connection that appears in the covariant derivative of the tensorinos will be $x_r^N \partial_\mu x^{Mr}$. On the other hand, the notation in which the fundamental fields are the scalars $\Phi^{\bar{\alpha}}$ ($\bar{\alpha} = 1, ..., n_T$) parametrizing the coset manifold, adds to the supersymmetry variation of the tensorinos χ^{MA} the term

$$-\delta\Phi^{\bar{\alpha}} \mathcal{A}_{\bar{\alpha}}^{MN} \chi^{NA} \quad , \tag{188}$$

where $\mathcal{A}_{\bar{\alpha}}^{MN}$ is the composite connection of $SO(n_T)$. In this notation, the commutator of two supersymmetry transformations on the tensorinos does not generate a local $SO(n_T)$ transformation.

We now recall the notations used to describe the scalars in the hypermultiplets. We denote by $V_\alpha^{aA}(\phi)$ the vielbein of the quaternionic manifold, where the index structure corresponds to the requirement that the holomony be contained in $USp(2) \times USp(2n_H)$. The internal $USp(2)$ and $USp(2n_H)$ connections are then denoted, respectively, by $\mathcal{A}_{\alpha B}^A$ and $\mathcal{A}_{\alpha b}^a$, that in our conventions are anti-hermitian matrices. The index $\alpha = 1, ..., 4n_H$ is a curved index on the quaternionic manifold. The field-strengths of the connections are

$$\mathcal{F}_{\alpha\beta}{}^A{}_B = \partial_\alpha \mathcal{A}_{\beta B}^A - \partial_\beta \mathcal{A}_{\alpha B}^A + [\mathcal{A}_\alpha, \mathcal{A}_\beta]^A{}_B \quad , \mathcal{F}_{\alpha\beta}{}^a{}_b = \partial_\alpha \mathcal{A}_{\beta b}^a - \partial_\beta \mathcal{A}_{\alpha b}^a + [\mathcal{A}_\alpha, \mathcal{A}_\beta]^a{}_b \quad ,$$

where $\partial_\alpha = \partial/\partial\phi^\alpha$. The request that the vielbein $V_\alpha^{aA}(\phi)$ be covariantly constant gives the following relations

$$V_{aA}^\alpha V_{bB}^\beta g_{\alpha\beta} = \Omega_{ab} \epsilon_{AB} \, , V_{aA}^\alpha V^{\beta bA} + V_{aA}^\beta V^{\alpha bA} = \frac{1}{n_H} g^{\alpha\beta} \delta_b^a \quad , V_{aA}^\alpha V^{\beta aB} + V_{aA}^\beta V^{\alpha aB} = g^{\alpha\beta} \delta_B^A \, ,$$

where Ω_{ab} is the antisymmetric invariant tensor of $USp(2n_H)$. The field-strength of the $USp(2)$ connection $\mathcal{A}^A_{\alpha\,B}$ is naturally constructed in terms of V^{aA}_α by the relation:

$$\mathcal{F}_{\alpha\beta AB} = V_{\alpha aA} V^a_{\beta\ B} + V_{\alpha aB} V^a_{\beta\ A} \, , \tag{189}$$

and then the cyclic identity for the internal curvature tensor implies that the field-strength of the $USp(2n_H)$ connection $\mathcal{A}^a_{\alpha b}$ has the form

$$\mathcal{F}_{\alpha\beta ab} = V_{\alpha aA} V_{\beta b}{}^A + V_{\alpha bA} V_{\beta a}{}^A + \Omega_{abcd} V^{dA}_\alpha V^c_{\beta\, A} \quad , \tag{190}$$

where Ω_{abcd} is totally symmetric in its indices.

Now, assuming that the scalars parametrize the coset manifold $USp(2,2n_H)/USp(2)\times USp(2n_H)$, we describe the gauging of the hypermultiplets under the group $USp(2)\times USp(2n_H)$, that is the maximal compact subgroup of the isometry group. We denote the gauge fields of this group by A^i_μ and A^I_μ, where i and I take values in the adjoint representation of $USp(2)$ and $USp(2n_H)$, and the corresponding field-strengths are

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + \epsilon^{ijk} A^j_\mu A^k_\nu \, , \qquad F^I_{\mu\nu} = \partial_\mu A^I_\nu - \partial_\nu A^I_\mu + f^{IJK} A^J_\mu A^K_\nu \, , \tag{191}$$

where ϵ^{ijk} and f^{IJK} are the structure constants of $USp(2)$ and $USp(2n_H)$. Under the gauge transformations

$$\delta A^i_\mu = D_\mu \Lambda^i \, , \qquad \delta A^I_\mu = D_\mu \Lambda^I \tag{192}$$

the scalars transform as

$$\delta\phi^\alpha = \Lambda^i \xi^{\alpha i} + \Lambda^I \xi^{\alpha I} \, , \tag{193}$$

where $\xi^{\alpha i}$ and $\xi^{\alpha I}$ are the Killing vectors corresponding to the $USp(2)$ and $USp(2n_H)$ isometries. The covariant derivative for the scalars is then

$$D_\mu \phi^\alpha = \partial_\mu \phi^\alpha - A^i_\mu \xi^{\alpha i} - A^I_\mu \xi^{\alpha I} \, . \tag{194}$$

One can correspondingly define the covariant derivatives for the spinors in a natural way, adding the composite connections $D_\mu\phi^\alpha \mathcal{A}_\alpha$. For instance, the covariant derivative for the hyperinos Ψ^a will contain the connections $D_\mu \phi^\alpha \mathcal{A}^a_{\alpha b}$, while the covariant derivative for the gravitino and the tensorinos will contain the connections $D_\mu\phi^\alpha \mathcal{A}^A_{\alpha\,B}$. The covariant derivatives for the gauginos λ^{iA}, λ^{IA} are

$$\begin{aligned} D_\mu \lambda^{iA} &= \partial_\mu \lambda^{iA} + \frac{1}{4}\omega_{\mu mn}\gamma^{mn}\lambda^{iA} + D_\mu\phi^\alpha \mathcal{A}^A_{\alpha\,B}\lambda^{iB} + \epsilon^{ijk} A^j_\mu \lambda^{kA} \, , \\ D_\mu \lambda^{IA} &= \partial_\mu \lambda^{IA} + \frac{1}{4}\omega_{\mu mn}\gamma^{mn}\lambda^{IA} + D_\mu\phi^\alpha \mathcal{A}^A_{\alpha\,B}\lambda^{IB} + f^{IJK} A^J_\mu \lambda^{KA} \, . \end{aligned} \tag{195}$$

Notice that the gravitino, the tensorinos and the hyperinos are not coupled to the gauge vectors through terms that do not contain the hyperscalars.

We now proceed to the construction of the model. We assume that the gauge group has the form $\mathcal{G} = \prod_z \mathcal{G}_z$, with $\mathcal{G}_z$ semi-simple. The scalars in the hypermultiplets are charged with respect to $\mathcal{G}_1 = USp(2)$ and $\mathcal{G}_2 = USp(2n_H)$. The field-strengths of the 2-forms $B^r_{\mu\nu}$ are

$$H^r_{\mu\nu\rho} = 3\partial_{[\mu} B^r_{\nu\rho]} + c^{rz}\omega^z_{\mu\nu\rho} \, , \tag{196}$$

where c^{rz} are constants and ω^z are the Chern-Simons 3-forms

$$\omega^z = tr_z(AdA + \frac{2}{3}A^3) \, . \tag{197}$$

These 3-form field-strengths satisfy (anti)self-duality conditions, that to lowest order in the fermi fields are

$$G_{rs} H^{s\mu\nu\rho} = \frac{1}{6e} \epsilon^{\mu\nu\rho\sigma\delta\tau} H_{r\sigma\delta\tau} \,, \tag{198}$$

where $G_{rs} = v_r v_s + x_r^M x_s^M$. Gauge invariance of H^r requires that B^r transform under vector gauge transformations according to

$$\delta B^r = -c^{rz} tr_z(\Lambda dA) \,. \tag{199}$$

To lowest order in the fermi fields, we produce the construction adding the hypermultiplet couplings. The equations for all fields, with the exception of the 2-forms, can be obtained from the lagrangian

$$\begin{aligned}
e^{-1}\mathcal{L} = &-\frac{1}{4}R + \frac{1}{12} G_{rs} H^{r\mu\nu\rho} H^s_{\mu\nu\rho} - \frac{1}{4}\partial_\mu v^r \partial^\mu v_r + \frac{1}{2} v_r c^{rz} tr_z(F_{\mu\nu}F^{\mu\nu}) \\
&+\frac{1}{8e}\epsilon^{\mu\nu\rho\sigma\delta\tau} B^r_{\mu\nu} c^z_r tr_z(F_{\rho\sigma}F_{\delta\tau}) + \frac{1}{2} g_{\alpha\beta}(\phi) D_\mu \phi^\alpha D^\mu \phi^\beta + \frac{1}{4 v_r c^{r1}} \mathcal{A}^A_{\alpha\, B} \mathcal{A}^B_{\beta\, A} \xi^{\alpha i} \xi^{\beta i} \\
&+\frac{1}{4 v_r c^{r2}} \mathcal{A}^A_{\alpha\, B} \mathcal{A}^B_{\beta\, A} \xi^{\alpha I} \xi^{\beta I} - \frac{i}{2}(\bar\psi_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho) - \frac{i}{2} v_r H^{r\mu\nu\rho}(\bar\psi_\mu \gamma_\nu \psi_\rho) + \frac{i}{2}(\bar\chi^M \gamma^\mu D_\mu \chi^M) \\
&-\frac{i}{24} v_r H^r_{\mu\nu\rho}(\bar\chi^M \gamma^{\mu\nu\rho}\chi^M) + \frac{1}{2} x^M_r \partial_\nu v^r (\bar\psi_\mu \gamma^\nu \gamma^\mu \chi^M) - \frac{1}{2} x^M_r H^{r\mu\nu\rho}(\bar\psi_\mu \gamma_{\nu\rho}\chi^M) \\
&+\frac{i}{2}(\bar\Psi_a \gamma^\mu D_\mu \Psi^a) + \frac{i}{24} v_r H^r_{\mu\nu\rho}(\bar\Psi_a \gamma^{\mu\nu\rho}\Psi^a) - V^{aA}_\alpha D_\nu \phi^\alpha (\bar\psi_{\mu A} \gamma^\nu \gamma^\mu \Psi_a) - i v_r c^{rz} tr_z(\bar\lambda \gamma^\mu D_\mu \lambda) \\
&-\frac{i}{\sqrt{2}} v_r c^{rz} tr_z[F_{\nu\rho}(\bar\psi_\mu \gamma^{\nu\rho}\gamma^\mu \lambda)] - \frac{1}{\sqrt{2}} x^M_r c^{rz} tr_z[F_{\mu\nu}(\bar\chi^M \gamma^{\mu\nu}\lambda)] + \frac{i}{12} c^z_r H^r_{\mu\nu\rho} tr_z(\bar\lambda \gamma^{\mu\nu\rho}\lambda) \\
&-\sqrt{2} V^{aA}_\alpha [\xi^{\alpha i}(\bar\lambda^i_A \Psi_a) + \xi^{\alpha I}(\bar\lambda^I_A \Psi_a)] + \frac{i}{\sqrt{2}} \mathcal{A}^A_{\alpha\, B}[\xi^{\alpha i}(\bar\lambda^i_A \gamma^\mu \psi^B_\mu) + \xi^{\alpha I}(\bar\lambda^I_A \gamma^\mu \psi^B_\mu)] \\
&+\frac{1}{\sqrt{2}} \mathcal{A}^A_{\alpha\, B}[\frac{x^M_r c^{r1}}{v_s c^{s1}} \xi^{\alpha i}(\bar\lambda^i_A \chi^{MB}) + \frac{x^M_r c^{r2}}{v_s c^{s2}} \xi^{\alpha I}(\bar\lambda^I_A \chi^{MB})] \,,
\end{aligned} \tag{200}$$

after imposing the (anti)self-duality conditions. With this prescription, its variation under the supersymmetry transformations

$$\begin{aligned}
&\delta e_\mu{}^m = -i(\bar\epsilon \gamma^m \psi_\mu) \quad , \\
&\delta B^r_{\mu\nu} = i v^r (\bar\psi_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2} x^{Mr}(\bar\chi^M \gamma_{\mu\nu}\epsilon) + 2c^{rz} tr_z(A_{[\mu}\delta A_{\nu]}) \quad , \\
&\delta v_r = x^M_r (\bar\epsilon \chi^M) \quad , \qquad \delta x^M_r = v_r(\bar\epsilon \chi^M) \quad , \\
&\delta \phi^\alpha = V^\alpha_{aA}(\bar\epsilon^A \Psi^a) \quad , \\
&\delta A_\mu = -\frac{i}{\sqrt{2}}(\bar\epsilon \gamma_\mu \lambda) \quad , \\
&\delta \psi^A_\mu = D_\mu \epsilon^A + \frac{1}{4} v_r H^r_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon^A \quad , \\
&\delta \chi^{MA} = \frac{i}{2} x^M_r \partial_\mu v^r \gamma^\mu \epsilon^A + \frac{i}{12} x^M_r H^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon^A \quad , \\
&\delta \Psi^a = i\gamma^\mu \epsilon_A V^{aA}_\alpha D_\mu \phi^\alpha \quad , \\
&\delta \lambda^A = -\frac{1}{2\sqrt{2}} F_{\mu\nu}\gamma^{\mu\nu}\epsilon^A \qquad (z \neq 1,2) \quad , \\
&\delta \lambda^{iA} = -\frac{1}{2\sqrt{2}} F^i_{\mu\nu}\gamma^{\mu\nu}\epsilon^A - \frac{1}{\sqrt{2} v_r c^{r1}} \mathcal{A}^A_{\alpha\, B}\xi^{\alpha i}\epsilon^B \\
&\delta \lambda^{IA} = -\frac{1}{2\sqrt{2}} F^I_{\mu\nu}\gamma^{\mu\nu}\epsilon^A - \frac{1}{\sqrt{2} v_r c^{r2}} \mathcal{A}^A_{\alpha\, B}\xi^{\alpha I}\epsilon^B
\end{aligned} \tag{201}$$

gives the supersymmetry anomaly

$$\mathcal{A}_\epsilon = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma\delta\tau}c^z_r c^{rz'} tr_z(\delta_\epsilon A_\mu A_\nu) tr_{z'}(F_{\rho\sigma}F_{\delta\tau}) - \frac{1}{6}\epsilon^{\mu\nu\rho\sigma\delta\tau}c^z_r c^{rz'} tr_z(\delta_\epsilon A_\mu F_{\nu\rho})\omega^{z'}_{\sigma\delta\tau}\ , \tag{202}$$

related by the Wess-Zumino conditions to the consistent gauge anomaly

$$\mathcal{A}_\Lambda = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma\delta\tau}c^z_r c^{rz'} tr_z(\Lambda\partial_\mu A_\nu) tr_{z'}(F_{\rho\sigma}F_{\delta\tau})\,. \tag{203}$$

Notice the presence in the lagrangian of the scalar potential

$$\mathcal{V}(\phi) = -\frac{1}{4v_r c^{r1}}\mathcal{A}^A_{\alpha\,B}\mathcal{A}^B_{\beta\,A}\xi^{\alpha i}\xi^{\beta i} - \frac{1}{4v_r c^{r2}}\mathcal{A}^A_{\alpha\,B}\mathcal{A}^B_{\beta\,A}\xi^{\alpha I}\xi^{\beta I} \quad . \tag{204}$$

As in rather more conventional gauged models, the potential contains interesting informations, and it may be very instructive to study its extrema in special cases.

The complete supersymmetry transformations of the fermi fields are

$$\begin{aligned}
\delta\psi^A_\mu &= D_\mu(\hat\omega)\epsilon^A + \frac{1}{4}v_r\hat{H}^r_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon^A - \frac{3i}{8}\gamma_\mu\chi^{MA}(\bar\epsilon\chi^M) - \frac{i}{8}\gamma^\nu\chi^{MA}(\bar\epsilon\gamma_{\mu\nu}\chi^M)\\
&+\frac{i}{16}\gamma_{\mu\nu\rho}\chi^{MA}(\bar\epsilon\gamma^{\nu\rho}\chi^M) + \frac{9i}{8}v_r c^{rz} tr_z[\lambda^A(\bar\epsilon\gamma_\mu\lambda)] - \frac{i}{8}v_r c^{rz} tr_z[\gamma_{\mu\nu}\lambda^A(\bar\epsilon\gamma^\nu\lambda)]\\
&+\frac{i}{16}v_r c^{rz} tr_z[\gamma^{\nu\rho}\lambda^A(\bar\epsilon\gamma_{\mu\nu\rho}\lambda)] - \delta\phi^\alpha\mathcal{A}^A_{\alpha\,B}\psi^B_\mu \quad ,\\
\delta\chi^{MA} &= \frac{i}{2}x^M_r(\hat{\partial_\mu v^r})\gamma^\mu\epsilon^A + \frac{i}{12}x^M_r\hat{H}^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon^A - \frac{1}{2}x^M_r c^{rz} tr_z[\gamma_\mu\lambda^A(\bar\epsilon\gamma^\mu\lambda)] - \delta\phi^\alpha\mathcal{A}^A_{\alpha\,B}\chi^{MB}\,,\\
\delta\Psi^a &= i\gamma^\mu\epsilon_A V^{aA}_\alpha \hat{D_\mu\phi^\alpha} - \delta\phi^\alpha\mathcal{A}^a_{\alpha b}\Psi^b\,,\\
\delta\lambda^A &= -\frac{1}{2\sqrt{2}}\hat{F}_{\mu\nu}\gamma^{\mu\nu}\epsilon^A - \frac{x^M_r c^{rz}}{2v_s c^{sz}}(\bar\chi^M\lambda)\epsilon^A - \frac{x^M_r c^{rz}}{4v_s c^{sz}}(\bar\chi^M\epsilon)\lambda^A\\
&+\frac{x^M_r c^{rz}}{8v_s c^{sz}}(\bar\chi^M\gamma_{\mu\nu}\epsilon)\gamma^{\mu\nu}\lambda^A - \delta\phi^\alpha\mathcal{A}^A_{\alpha\,B}\lambda^B \qquad\qquad (z\neq 1,2) \quad ,\\
\delta\lambda^{iA} &= -\frac{1}{2\sqrt{2}}\hat{F}^i_{\mu\nu}\gamma^{\mu\nu}\epsilon^A - \frac{x^M_r c^{r1}}{2v_s c^{s1}}(\bar\chi^M\lambda^i)\epsilon^A - \frac{x^M_r c^{r1}}{4v_s c^{s1}}(\bar\chi^M\epsilon)\lambda^{iA}\\
&+\frac{x^M_r c^{r1}}{8v_s c^{s1}}(\bar\chi^M\gamma_{\mu\nu}\epsilon)\gamma^{\mu\nu}\lambda^{iA} - \delta\phi^\alpha\mathcal{A}^A_{\alpha\,B}\lambda^{iB} - \frac{1}{\sqrt{2}v_r c^{r1}}\mathcal{A}^A_{\alpha\,B}\xi^{\alpha i}\epsilon^B \quad ,\\
\delta\lambda^{IA} &= -\frac{1}{2\sqrt{2}}\hat{F}^I_{\mu\nu}\gamma^{\mu\nu}\epsilon^A - \frac{x^M_r c^{r2}}{2v_s c^{s2}}(\bar\chi^M\lambda^I)\epsilon^A - \frac{x^M_r c^{r2}}{4v_s c^{s2}}(\bar\chi^M\epsilon)\lambda^{IA}\\
&+\frac{x^M_r c^{r2}}{8v_s c^{s2}}(\bar\chi^M\gamma_{\mu\nu}\epsilon)\gamma^{\mu\nu}\lambda^{IA} - \delta\phi^\alpha\mathcal{A}^A_{\alpha\,B}\lambda^{IB} - \frac{1}{\sqrt{2}v_r c^{r2}}\mathcal{A}^A_{\alpha\,B}\xi^{\alpha I}\epsilon^B \quad .
\end{aligned} \tag{205}$$

We now want to extend the results to all orders in the fermi fields. First of all, we define the supercovariant quantities

$$\begin{aligned}
\hat\omega_{\mu\nu\rho} &= \omega^0_{\mu\nu\rho} - \frac{i}{2}(\bar\psi_\mu\gamma_\nu\psi_\rho + \bar\psi_\nu\gamma_\rho\psi_\mu + \bar\psi_\nu\gamma_\mu\psi_\rho)\,,\\
\hat{H}^r_{\mu\nu\rho} &= H^r_{\mu\nu\rho} - \frac{1}{2}x^{Mr}(\bar\chi^M\gamma_{\mu\nu}\psi_\rho + \bar\chi^M\gamma_{\nu\rho}\psi_\mu + \bar\chi^M\gamma_{\rho\mu}\psi_\nu)\\
&\quad -\frac{i}{2}v^r(\bar\psi_\mu\gamma_\nu\psi_\rho + \bar\psi_\nu\gamma_\rho\psi_\mu + \bar\psi_\rho\gamma_\mu\psi_\nu)\,,\\
\hat{\partial_\mu v^r} &= \partial_\mu v^r - x^{Mr}(\bar\chi^M\psi_\mu)\,, \qquad\qquad \hat{D_\mu\phi^\alpha} = D_\mu\phi^a - V^\alpha_{aA}(\bar\psi^A_\mu\Psi^a)\,,\\
\hat{F}_{\mu\nu} &= F_{\mu\nu} + \frac{i}{\sqrt{2}}(\bar\lambda\gamma_\mu\psi_\nu) - \frac{i}{\sqrt{2}}(\bar\lambda\gamma_\nu\psi_\mu)\,,
\end{aligned} \tag{206}$$

and require that the transformation rules for the fermi fields be supercovariant. All fermionic terms in the supersymmetry transformations of the fermi fields that are not determined by supercovariance are then obtained requiring the closure of the supersymmetry algebra on bose and fermi fields. Moreover, since the supersymmetry algebra on the fermi fields closes only on-shell, in this way one can determine the complete fermionic field equations, and from these the complete lagrangian, up to some subtleties related to the (anti)self-dual forms.

One can compute the commutators of two supersymmetry transformations on the bose fields using these relations, and show that they generate the local symmetries:

$$[\delta_1, \delta_2] = \delta_{gct} + \delta_{Lorentz} + \delta_{susy} + \delta_{tens} + \delta_{gauge} + \delta_{SO(n)} \quad , \tag{207}$$

where the parameters of generic coordinate, local Lorentz, supersymmetry, tensor gauge, vector gauge and composite $SO(n)$ transformations are respectively

$$\begin{aligned}
&\xi_\mu = -i(\bar{\epsilon}_1 \gamma_\mu \epsilon_2) \quad , \\
&\Omega^{mn} = -i\xi^\mu(\hat{\omega}_\mu{}^{mn} - v_r \hat{H}_\mu^{rmn}) - \frac{1}{2}[(\bar{\chi}^M \epsilon_1)(\bar{\epsilon}_2 \gamma^{mn} \chi^M) - (\bar{\chi}^M \epsilon_2)(\bar{\epsilon}_1 \gamma^{mn} \chi^M)] \\
&\qquad -v_r c^{rz} tr_z[(\bar{\epsilon}_1 \gamma^m \lambda)(\bar{\epsilon}_2 \gamma^n \lambda) - (\bar{\epsilon}_2 \gamma^m \lambda)(\bar{\epsilon}_1 \gamma^n \lambda)] \quad , \\
&\zeta^A = \xi^\mu \psi_\mu^A + V_{aC}^\alpha \mathcal{A}_\alpha^A{}_B \epsilon_2^B (\bar{\epsilon}_1^C \Psi^a) - V_{aC}^\alpha \mathcal{A}_\alpha^A{}_B \epsilon_1^B (\bar{\epsilon}_2^C \Psi^a) \quad , \\
&\Lambda_\mu^r = -\frac{1}{2} v^r \xi_\mu - \xi^\nu B_{\mu\nu}^r \quad , \\
&\Lambda = \xi^\mu A_\mu \quad , \\
&A^{MN} = \xi^\mu x^{Mr}(\partial_\mu x_r^N) + (\bar{\chi}^M \epsilon_2)(\bar{\chi}^N \epsilon_1) - (\bar{\chi}^M \epsilon_1)(\bar{\chi}^N \epsilon_2) \quad .
\end{aligned} \tag{208}$$

In order to prove this result, one has to use the (anti)self-duality condition for the tensor fields, that to all orders in the fermi fields is

$$G_{rs} \hat{\mathcal{H}}_{\mu\nu\rho}^s = \frac{1}{6e} \epsilon_{\mu\nu\rho\sigma\delta\tau} \hat{\mathcal{H}}_r^{\sigma\delta\tau} \tag{209}$$

in terms of the 3-forms

$$\hat{\mathcal{H}}_{\mu\nu\rho}^r = \hat{H}_{\mu\nu\rho}^r - \frac{i}{8} v^r (\bar{\chi}^M \gamma_{\mu\nu\rho} \chi^M) + \frac{i}{8} v^r (\bar{\Psi}_a \gamma_{\mu\nu\rho} \Psi^a) - \frac{i}{4} c^{rz} tr_z (\bar{\lambda} \gamma_{\mu\nu\rho} \lambda) \quad . \tag{210}$$

Requiring that the commutator of two supersymmetry transformations on the fermi fields close on-shell then determines the complete fermi field equations. The equations obtained in this way are

$$\begin{aligned}
&-i\gamma^{\mu\nu\rho} D_\nu(\hat{\omega}) \psi_\rho^A - \frac{i}{4} v_r \hat{H}_{\nu\sigma\delta}^r \gamma^{\mu\nu\rho} \gamma^{\sigma\delta} \psi_\rho^A - \frac{1}{12} x_r^M \hat{H}^{r\nu\rho\sigma} \gamma_{\nu\rho\sigma} \gamma^\mu \chi^{MA} \\
&+\frac{1}{2} x_r^M (\hat{\partial_\nu v^r}) \gamma^\nu \gamma^\mu \chi^{MA} + \frac{3}{2} \gamma^{\mu\nu} \chi^{MA} (\bar{\chi}^M \psi_\nu) - \frac{1}{4} \gamma^{\mu\nu} \chi^{MA} (\bar{\chi}^M \gamma_{\nu\rho} \psi^\rho) \\
&+\frac{1}{4} \gamma_{\nu\rho} \chi^{MA} (\bar{\chi}^M \gamma^{\mu\nu} \psi^\rho) - \frac{1}{2} \chi^{MA} (\bar{\chi}^M \gamma^{\mu\nu} \psi_\nu) - i v_r c^{rz} \,\mathrm{tr}_z [-\frac{1}{\sqrt{2}} \gamma^{\nu\rho} \gamma^\mu \lambda^A \hat{F}_{\nu\rho} \\
&+\frac{3i}{4} \gamma^{\mu\nu\rho} \lambda^A (\bar{\psi}_\nu \gamma_\rho \lambda) - \frac{i}{2} \gamma^\mu \lambda^A (\bar{\psi}_\nu \gamma^\nu \lambda) + \frac{i}{2} \gamma^\nu \lambda^A (\bar{\psi}_\nu \gamma^\mu \lambda) + \frac{i}{4} \gamma_\rho \lambda^A (\bar{\psi}_\nu \gamma^{\mu\nu\rho} \lambda)] \\
&-\frac{i}{2} x_r^M c^{rz} \,\mathrm{tr}_z [\gamma_\nu \lambda^A (\bar{\chi}^M \gamma^\nu \gamma^\mu \lambda)] - V_\alpha^{aA} \hat{D_\nu \phi^\alpha} \gamma^\nu \gamma^\mu \Psi_a \\
&+\frac{i}{\sqrt{2}} \mathcal{A}_{\alpha B}^a \xi^{\alpha i} \gamma^\mu \lambda^{iB} = 0
\end{aligned} \tag{211}$$

for the gravitino,

$$
\begin{aligned}
& i\gamma^\mu D_\mu(\hat\omega)\chi^{MA} - \frac{i}{12} v_r \hat H^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\chi^{MA} + \frac{1}{12} x^M_r \hat H^r_{\mu\nu\rho}\gamma^\sigma\gamma^{\mu\nu\rho}\psi^A_\sigma + \frac{1}{2} x^M_r (\partial_\nu \hat{v}^r)\gamma^\mu\gamma^\nu\psi^A_\mu \\
& + \frac{1}{\sqrt{2}} x^M_r c^{rz}\,\mathrm{tr}_z(\hat F_{\mu\nu}\gamma^{\mu\nu}\lambda^A) - \frac{i}{2} x^M_r c^{rz}\,\mathrm{tr}_z[\gamma^\mu\gamma^\nu\lambda^A(\bar\psi_\mu\gamma_\nu\lambda)] + \frac{1}{2}\gamma^\mu\chi^{NA}(\bar\chi^N\gamma_\mu\chi^M) \\
& - \frac{3}{8} v_r c^{rz}\,\mathrm{tr}_z[(\bar\chi^M\gamma_{\mu\nu}\lambda)\gamma^{\mu\nu}\lambda^A] - \frac{1}{4} v_r c^{rz}\,\mathrm{tr}_z[(\bar\chi^M\lambda)\lambda^A] \\
& - \frac{3}{2}\frac{x^M_r c^{rz} x^N_s c^{sz}}{v_t c^{tz}}\,\mathrm{tr}_z[(\bar\chi^N\lambda)\lambda^A] + \frac{1}{4}\frac{x^M_r c^{rz} x^N_s c^{sz}}{v_t c^{tz}}\,\mathrm{tr}_z[(\bar\chi^N\gamma_{\mu\nu}\lambda)\gamma^{\mu\nu}\lambda^A] \\
& - \frac{x^M_r c^{r1}}{\sqrt{2} v_s c^{s1}} \mathcal{A}^A_{\alpha\, B}\xi^{\alpha i}\lambda^{iB} = 0
\end{aligned}
\tag{212}
$$

for the tensorinos, and

$$
\begin{aligned}
& i\gamma^\mu D_\mu(\hat\omega)\Psi^a + \frac{i}{12} v_r \hat H^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\Psi^a + \gamma^\mu\gamma^\nu\psi_{\mu A} V^{aA}_\alpha \hat{D_\nu\phi^\alpha} - \frac{1}{48} v_r c^{rz}\,\mathrm{tr}_z(\bar\lambda\gamma_{\mu\nu\rho}\lambda)\gamma^{\mu\nu\rho}\Psi^a \\
& + \frac{1}{12}\Omega^{abcd}\gamma^\mu\Psi_b(\bar\Psi_c\gamma_\mu\Psi_d) + \sqrt{2} V^{aA}_\alpha \xi^{\alpha i}\lambda^i_A = 0
\end{aligned}
\tag{213}
$$

for the hyperinos.

9.3 Wess-Zumino Consistency Conditions

In general, the Wess-Zumino consistency conditions follow from the requirement that the symmetry algebra be realized on the effective action. For locally supersymmetric theories this implies

$$
\delta_{\Lambda_1}\mathcal{A}_{\Lambda_2} - \delta_{\Lambda_2}\mathcal{A}_{\Lambda_1} = \mathcal{A}_{[\Lambda_1,\Lambda_2]}\,, \quad \delta_\epsilon\mathcal{A}_\Lambda = \delta_\Lambda\mathcal{A}_\epsilon\,, \quad \delta_{\epsilon_1}\mathcal{A}_{\epsilon_2} - \delta_{\epsilon_2}\mathcal{A}_{\epsilon_1} = \mathcal{A}_{\tilde\epsilon} + \mathcal{A}_{\tilde\Lambda}\,, \tag{214}
$$

where only gauge and supersymmetry anomalies are considered, and where $\tilde\epsilon$ and $\tilde\Lambda$ are the parameters of supersymmetry and gauge transformations determined by the supersymmetry algebra.

The operator $\hat d$ makes it possible to understand in a simple way why the Stora–Zumino descent represents the most general non-trivial solution of the Wess–Zumino consistency condition (**??**). For the case of gauge anomalies we are considering here, $\mathcal{I}(v) = \int \mathrm{tr}(v\, a(A))$, the latter reads

$$
\delta_{v_1}\int \mathrm{tr}(v_2\, a(A)) - \delta_{v_2}\int \mathrm{tr}(v_1\, a(A)) - \int \mathrm{tr}([v_1, v_2]\, a(A)) = 0\,. \tag{215}
$$

The two transformations with parameters v_1 and v_2 can be incorporated into a family of transformations parametrized by θ^1 and θ^2, with parameter $\hat v = v_\alpha d\theta^\alpha$. In this way, $v_\alpha = g^{-1}\partial_\alpha g$. At $\theta^\alpha = 0$, $g(x,0) = 1$ and therefore $\bar A(x,0) = A(x)$ and $\bar F(x,0) = F(x)$. At that point, $\hat d$ generates ordinary gauge transformations on A and F, with $\hat d = d\theta^\alpha\delta_{v_\alpha}$. The condition (215) can then be multiplied by $d\theta^1 d\theta^2$ and rewritten as

$$
\int \mathrm{tr}(\hat v\, \hat d\, a(A)) + \int \mathrm{tr}(\hat v^2\, a(A)) = 0\,. \tag{216}
$$

In global supersymmetry the analysis is somewhat simpler, since the r.h.s. of the last of (214) does not contain the (global) supersymmetry anomaly. Let us therefore begin by reviewing the case of supersymmetric Yang-Mills theory in four dimensions. From the 6-form anomaly polynomial

$$
\mathcal{P}_\mathcal{A} = I_6 = trF^3\,, \tag{217}
$$

in the language of forms, one obtains the four-dimensional gauge anomaly

$$\mathcal{A}^{(4)}_{\Lambda} = tr[\Lambda(dA)^2 + \frac{ig}{2}\, d\Lambda A^3] \quad , \tag{218}$$

and from (214) one can determine the form of the global supersymmetry anomaly. With the classical lagrangian

$$\mathcal{L}_{SYM} = \mathrm{tr}\left[-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + 2i\bar{\lambda}\gamma^{\mu}D_{\mu}\lambda\right] \quad , \tag{219}$$

and λ a right-handed Weyl spinor, the supersymmetry transformations are

$$\delta A_{\mu} = \frac{i}{\sqrt{2}}(\bar{\epsilon}\gamma_{\mu}\lambda - \bar{\lambda}\gamma_{\mu}\epsilon)\,, \qquad \delta\lambda = \frac{1}{2\sqrt{2}}F_{\mu\nu}\gamma^{\mu\nu}\epsilon\,. \tag{220}$$

The second of (214) (with $\mathcal{A}_{\tilde{\epsilon}}$ absent in this global case), then determines the supersymmetry anomaly up to terms cubic in λ,

$$\mathcal{A}^{(4)}_{\epsilon} = tr[\delta_{\epsilon}AA(dA) + \delta_{\epsilon}A(dA)A - \frac{3ig}{2}\delta_{\epsilon}AA^3] \quad , \tag{221}$$

and indeed

$$\delta_{\epsilon_2}\mathcal{A}^{(4)}_{\epsilon_1} - \delta_{\epsilon_1}\mathcal{A}^{(4)}_{\epsilon_2} = \mathcal{A}^{(4)}_{\tilde{\Lambda}} + 3tr[\delta_{\epsilon_1}A\delta_{\epsilon_2}AF - \delta_{\epsilon_2}A\delta_{\epsilon_1}AF] \quad . \tag{222}$$

In order to compensate the second term in (222), one is to add to $\mathcal{A}^{(4)}_{\epsilon}$ the gauge-invariant term

$$\Delta\mathcal{A}^{(4)}_{\epsilon} = -\frac{i}{2}tr[\delta_{\epsilon}A\bar{\lambda}\gamma^{(3)}\lambda + \bar{\lambda}\delta_{\epsilon}A\gamma^{(3)}\lambda] \quad , \tag{223}$$

so that $\mathcal{A}^{(4)}_{\epsilon} + \Delta\mathcal{A}^{(4)}_{\epsilon}$ is the proper global supersymmetry anomaly. Although the supersymmetry algebra closes only on the field equation of λ, in four dimensions a simple dimensional counting shows that (214) can not generate a term proportional to $\gamma^{\mu}D_{\mu}\lambda$. Therefore, in this case the Wess-Zumino consistency conditions close accidentally even off-shell.

9.4 Inclusion of Vector Multiplets

Up to now, we have always considered the case in which the gauge group is non-abelian. In the abelian case, the couplings can actually have a more general form, since gauge invariance allows non-diagonal kinetic and Chern-Simons terms, in which the constants c^r_z are substituted by generic symmetric matrices c^r_{IJ}, with I, J running over the various $U(1)$ factors. We now want to generalize it to the case in which also charged hypermultiplets are present, and therefore we will consider the gauging with respect to abelian subgroups of $USp(2) \times USp(2n_H)$. There are no subtleties when the symmetric matrices c^r_{IJ} are diagonal, since in this situation the previous results can be straightforwardly applied. We are thus interested in the case in which the c^r_{IJ} can not be simultaneously diagonalized. To this end, we will consider a model in which only these abelian gauge groups are present.

We denote with A^I_{μ}, $I = 1, ..., m$, the set of abelian vectors, and the gauginos are correspondingly denoted by λ^{IA}. We collect here only the final results, since the construction follows the same lines as in the non-abelian case. All the field equations may then be derived from the lagrangian

$$
\begin{aligned}
e^{-1}\mathcal{L} = &-\frac{1}{4}R+\frac{1}{12}G_{rs}H^{r\mu\nu\rho}H^{s}_{\mu\nu\rho}-\frac{1}{4}\partial_\mu v^r\partial^\mu v_r-\frac{1}{4}v_r c^{rIJ}F^I_{\mu\nu}F^{J\mu\nu}\\
&-\frac{1}{16e}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_r^{IJ}B^r_{\mu\nu}F^I_{\rho\sigma}F^J_{\delta\tau}+\frac{1}{2}g_{\alpha\beta}(\phi)D_\mu\phi^\alpha D^\mu\phi^\beta+\frac{1}{4}[(v\cdot c)^{-1}]^{IJ}\mathcal{A}^A_{\alpha\,B}\mathcal{A}^B_{\beta\,A}\xi^{\alpha I}\xi^{\beta J}\\
&-\frac{i}{2}(\bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu[\frac{1}{2}(\omega+\hat\omega)]\psi_\rho)-\frac{i}{8}v_r[H+\hat H]^{r\mu\nu\rho}(\bar\psi_\mu\gamma_\nu\psi_\rho)+\frac{i}{48}v_r[H+\hat H]^r_{\rho\sigma\delta}(\bar\psi_\mu\gamma^{\mu\nu\rho\sigma\delta}\psi_\nu)\\
&+\frac{i}{2}(\bar\chi^M\gamma^\mu D_\mu(\hat\omega)\chi^M)-\frac{i}{24}v_r\hat H^r_{\mu\nu\rho}(\bar\chi^M\gamma^{\mu\nu\rho}\chi^M)+\frac{1}{4}x_r^M[\partial_\nu v^r+\hat{\partial_\nu v^r}](\bar\psi_\mu\gamma^\nu\gamma^\mu\chi^M)\\
&-\frac{1}{8}x_r^M[H+\hat H]^{r\mu\nu\rho}(\bar\psi_\mu\gamma_{\nu\rho}\chi^M)+\frac{1}{24}x_r^M[H+\hat H]^{r\mu\nu\rho}(\bar\psi^\sigma\gamma_{\sigma\mu\nu\rho}\chi^M)+\frac{i}{2}(\bar\Psi_a\gamma^\mu D_\mu(\hat\omega)\Psi^a)\\
&+\frac{i}{24}v_r\hat H^r_{\mu\nu\rho}(\bar\Psi_a\gamma^{\mu\nu\rho}\Psi^a)-\frac{1}{2}V^{aA}_\alpha[D_\nu\phi^\alpha+\hat{D_\nu\phi^\alpha}](\bar\psi_{\mu A}\gamma^\nu\gamma^\mu\Psi_a)+\frac{i}{2}v_r c^{rIJ}(\bar\lambda^I\gamma^\mu D_\mu(\hat\omega)\lambda^J)\\
&+\frac{i}{24}x_r^M x_s^M\hat H^r_{\mu\nu\rho}c^{sIJ}(\bar\lambda^I\gamma^{\mu\nu\rho}\lambda^J)+\frac{i}{4\sqrt2}v_r c^{rIJ}(F+\hat F)^I_{\nu\rho}(\bar\psi_\mu\gamma^{\nu\rho}\gamma^\mu\lambda^J)\\
&+\frac{1}{2\sqrt2}x_r^M c^{rIJ}(\bar\chi^M\gamma^{\mu\nu}\lambda^I)\hat F^J_{\mu\nu}-\sqrt2 V^{aA}_\alpha\xi^{\alpha I}(\bar\lambda^I_A\Psi_a)+\frac{i}{\sqrt2}\mathcal{A}^A_{\alpha\,B}\xi^{\alpha I}(\bar\lambda^I_A\gamma^\mu\psi^B_\mu)\\
&+\frac{1}{\sqrt2}[(v\cdot c)^{-1}(x^M\cdot c)]^{IJ}\mathcal{A}^A_{\alpha\,B}\xi^{\alpha I}(\bar\lambda^J_A\chi^{MB})+\frac{1}{8}(\bar\chi^M\gamma^{\mu\nu\rho}\chi^M)(\bar\psi_\mu\gamma_\nu\psi_\rho)\\
&-\frac{1}{8}(\bar\chi^M\gamma^\mu\chi^N)(\bar\chi^M\gamma_\mu\chi^N)+\frac{1}{8}(\bar\Psi_a\gamma^{\mu\nu\rho}\Psi^a)(\psi_\mu\gamma_\nu\psi_\rho)+\frac{1}{48}\Omega^{abcd}(\bar\Psi_a\gamma_\mu\Psi_b)(\bar\Psi_c\gamma^\mu\Psi_d)\\
&+\frac{1}{32}v_r c^{rIJ}(\bar\lambda^I\gamma_{\mu\nu\rho}\lambda^J)(\bar\chi^M\gamma^{\mu\nu\rho}\chi^M)-\frac{i}{16}(\bar\chi^M\gamma_{\mu\nu}\psi_\rho)x_r^M c^{rIJ}(\bar\lambda^I\gamma^{\mu\nu\rho}\lambda^J)\\
&-\frac{i}{4}x_r^M c^{rIJ}(\bar\chi^M\gamma^\mu\gamma^\nu\lambda^I)(\bar\psi_\mu\gamma_\nu\lambda^J)+\frac{1}{8}(\bar\psi_\mu\gamma_\nu\psi_\rho)v_r c^{rIJ}(\bar\lambda^I\gamma^{\mu\nu\rho}\lambda^J)-\frac{1}{16}v_r c^{rIJ}(\bar\chi^M\lambda^I)(\bar\chi^M\lambda^J)\\
&-\frac{3}{32}v_r c^{rIJ}(\bar\chi^M\gamma_{\mu\nu}\lambda^I)(\bar\chi^M\gamma^{\mu\nu}\lambda^J)+[(x^M\cdot c)(v\cdot c)^{-1}(x^N\cdot c)]^{IJ}[-\frac{1}{4}(\bar\chi^M\lambda^I)(\bar\chi^N\lambda^J)\\
&+\frac{1}{16}(\bar\chi^N\gamma^{\mu\nu}\lambda^I)(\bar\chi^M\gamma_{\mu\nu}\lambda^J)-\frac{1}{8}(\bar\chi^N\lambda^I)(\bar\chi^M\lambda^J)]+\frac{5}{192}v_r c^{rIJ}(\bar\lambda^I\gamma_{\mu\nu\rho}\lambda^J)(\bar\Psi_a\gamma^{\mu\nu\rho}\Psi^a)\\
&-\frac{1}{8}v_r v_s c^{rIJ}c^{sKL}(\bar\lambda^I\gamma_\mu\lambda^K)(\bar\lambda^J\gamma^\mu\lambda^L)+\frac{\alpha}{8}c^{rIJ}c_r^{KL}(\bar\lambda^I\gamma_\mu\lambda^K)(\bar\lambda^J\gamma^\mu\lambda^L)]\,. \qquad (224)
\end{aligned}
$$

The variation of this lagrangian with respect to gauge transformations gives the abelian gauge anomaly

$$
\mathcal{A}_\Lambda=-\frac{1}{32}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_r^{IJ}c^{rKL}\Lambda^I F^J_{\mu\nu}F^K_{\rho\sigma}F^L_{\delta\tau}\,, \qquad (225)
$$

Once again, in the case of the gauginos, aside from local symmetry transformations and field equations, the commutator of two supersymmetry transformations generates the additional two-cocycle

$$
\begin{aligned}
\delta_{(\alpha)}\lambda^I = &[(v\cdot c)^{-1}c_r]^{IJ}c^{rKL}[-\frac{1}{8}(\bar\epsilon_1\gamma_\mu\lambda^K)(\bar\epsilon_2\gamma_\nu\lambda^L)\gamma^{\mu\nu}\lambda^J-\frac{\alpha}{4}(\bar\lambda^J\gamma_\mu\lambda^K)(\bar\epsilon_1\gamma_\nu\lambda^L)\gamma^{\mu\nu}\epsilon_2\\
&+\frac{\alpha}{32}(\bar\lambda^J\gamma_{\mu\nu\rho}\lambda^K)(\bar\epsilon_1\gamma^\rho\lambda^L)\gamma^{\mu\nu}\epsilon_2+\frac{\alpha}{32}(\bar\lambda^J\gamma_\rho\lambda^K)(\bar\epsilon_1\gamma^{\mu\nu\rho}\lambda^L)\gamma_{\mu\nu}\epsilon_2\\
&+\frac{1-\alpha}{8}(\bar\lambda^J\gamma_\mu\lambda^K)(\bar\epsilon_1\gamma^\mu\lambda^L)\epsilon_2-(1\leftrightarrow2)\\
&+\frac{1-\alpha}{32}(\bar\epsilon_1\gamma^\mu\epsilon_2)(\bar\lambda^K\gamma_{\mu\nu\rho}\lambda^L)\gamma^{\nu\rho}\lambda^J]\,. \qquad (226)
\end{aligned}
$$

The tensionless string phase transition point in the moduli space of the scalars in the tensor multiplets now would correspond to the vanishing of some of the eigenvalues of the matrix $(v\cdot c)^{IJ}$.

In the general case variation with respect to the supersymmetry transformations

$$
\begin{aligned}
&\delta e_\mu{}^m = -i(\bar\epsilon\gamma^m\psi_\mu) \quad , \\
&\delta B^r_{\mu\nu} = iv^r(\bar\psi_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2}x^{Mr}(\bar\chi^M\gamma_{\mu\nu}\epsilon) + 2c^{rIJ}A^I_{[\mu}\delta A^J_{\nu]}\,, \\
&\delta v_r = x^M_r(\bar\epsilon\chi^M)\,, \qquad \delta x^M_r = v_r(\bar\epsilon\chi^M) \quad , \\
&\delta\phi^\alpha = V^\alpha_{aA}(\bar\epsilon^A\Psi^a)\,, \qquad \delta A^I_\mu = -\frac{i}{\sqrt{2}}(\bar\epsilon\gamma_\mu\lambda^I) \quad , \\
&\delta\psi^A_\mu = D_\mu(\hat\omega)\epsilon^A + \frac{1}{4}v_r\hat H^r_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon^A - \frac{3i}{8}\gamma_\mu\chi^{MA}(\bar\epsilon\chi^M) - \frac{i}{8}\gamma^\nu\chi^{MA}(\bar\epsilon\gamma_{\mu\nu}\chi^M) \\
&\quad +\frac{i}{16}\gamma_{\mu\nu\rho}\chi^{MA}(\bar\epsilon\gamma^{\nu\rho}\chi^M) - \frac{9i}{16}v_rc^{rIJ}\lambda^{IA}(\bar\epsilon\gamma_\mu\lambda^J) + \frac{i}{16}v_rc^{rIJ}\gamma_{\mu\nu}\lambda^{IA}(\bar\epsilon\gamma^\nu\lambda^J) \\
&\quad -\frac{i}{32}v_rc^{rIJ}\gamma^{\nu\rho}\lambda^{IA}(\bar\epsilon\gamma_{\mu\nu\rho}\lambda^J) - \delta\phi^\alpha\mathcal{A}^A_{\alpha\,B}\psi^B_\mu \quad , \\
&\delta\chi^{MA} = \frac{i}{2}x^M_r(\hat{\partial_\mu v^r})\gamma^\mu\epsilon^A + \frac{i}{12}x^M_r\hat H^r_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon^A + \frac{1}{4}x^M_rc^{rIJ}\gamma_\mu\lambda^{IA}(\bar\epsilon\gamma^\mu\lambda^J) - \delta\phi^\alpha\mathcal{A}^A_{\alpha\,B}\chi^{MB} \quad , \\
&\delta\Psi^a = i\gamma^\mu\epsilon_AV^{aA}_\alpha\hat{D_\mu\phi^\alpha} - \delta\phi^\alpha\mathcal{A}^a_{\alpha b}\Psi^b \quad , \\
&\delta\lambda^{IA} = -\frac{1}{2\sqrt{2}}\hat F^I_{\mu\nu}\gamma^{\mu\nu}\epsilon^A + [(v\cdot c)^{-1}(x^M\cdot c)]^{IJ}[-\frac{1}{2}(\bar\chi^M\lambda^J)\epsilon^A - \frac{1}{4}(\bar\chi^M\epsilon)\lambda^{JA} \\
&\quad +\frac{1}{8}(\bar\chi^M\gamma_{\mu\nu}\epsilon)\gamma^{\mu\nu}\lambda^{JA}] - \delta\phi^\alpha\mathcal{A}^A_{\alpha\,B}\lambda^{IB} - \frac{1}{\sqrt{2}}[(v\cdot c)^{-1}]^{IJ}\mathcal{A}^A_{\alpha\,B}\xi^{\alpha J}\epsilon^B
\end{aligned} \tag{227}
$$

gives the supersymmetry anomaly

$$
\begin{aligned}
\mathcal{A}_\epsilon &= c^{IJ}_rc^{rKL}\{-\frac{1}{16}\epsilon^{\mu\nu\rho\sigma\delta\tau}\delta_\epsilon A^I_\mu A^J_\nu F^K_{\rho\sigma}F^L_{\delta\tau} - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\delta\tau}\delta_\epsilon A^I_\mu F^J_{\nu\rho}A^K_\sigma F^L_{\delta\tau} + \frac{ie}{8}\delta_\epsilon A^I_\mu F^J_{\nu\rho}(\bar\lambda^K\gamma^{\mu\nu\rho}\lambda^L) \\
&+\frac{ie}{8}\delta_\epsilon A^I_\mu(\bar\lambda^J\gamma^{\mu\nu\rho}\lambda^K)F^L_{\nu\rho} + \frac{ie}{4}\delta_\epsilon A^I_\mu(\bar\lambda^J\gamma_\nu\lambda^K)F^{L\mu\nu} + \frac{e}{128}\delta_\epsilon e_\mu{}^m(\bar\lambda^I\gamma^{\mu\nu\rho}\lambda^J)(\bar\lambda^K\gamma_{m\nu\rho}\lambda^L) \\
&-\frac{e}{8\sqrt{2}}\delta_\epsilon A^I_\mu(\bar\lambda^J\gamma^\mu\gamma^\nu\gamma^\rho\lambda^K)(\bar\lambda^L\gamma_\nu\psi_\rho)\} + ec^{IJ}_r[c^r(v\cdot c)^{-1}(x^M\cdot c)]^{KL}\{-\frac{i}{4\sqrt{2}}\delta_\epsilon A^I_\mu(\bar\lambda^J\gamma^\mu\lambda^K)(\bar\chi^M\lambda^L) \\
&+\frac{i}{16\sqrt{2}}\delta_\epsilon A^I_\mu(\bar\lambda^J\gamma^\mu\gamma^{\nu\rho}\lambda^L)(\bar\chi^M\gamma_{\nu\rho}\lambda^K) - \frac{i}{8\sqrt{2}}\delta_\epsilon A^I_\mu(\bar\lambda^J\gamma^\mu\lambda^L)(\bar\chi^M\lambda^K)\} \\
&-\frac{ie}{4}c^{IJ}_r[(v\cdot c)^{-1}c^r]^{KL}\delta_\epsilon A^I_\mu\mathcal{A}^A_{\alpha\,B}\xi^{\alpha K}(\bar\lambda^L_A\gamma^\mu\lambda^{JB}) \\
&+\frac{\alpha}{8}c^{IJ}_rc^{rKL}\delta_\epsilon[e(\bar\lambda^I\gamma_\mu\lambda^K)(\bar\lambda^J\gamma^\mu\lambda^L)]\,.
\end{aligned} \tag{228}
$$

All the observations made for the non-abelian case are naturally valid also here: the theory is obtained by the requirement that the Wess-Zumino conditions close on-shell, and, as we have already shown, it is determined up to an arbitrary quartic coupling for the gauginos. In the case of a single vector multiplet, in which this quartic coupling vanishes, the two-cocycle of (226) is still present, although it is properly independent of α.

9.5 Covariant Anomaly

It is well known that consistent and covariant gauge anomalies are related by the divergence of a local functional. In six dimensions the residual covariant gauge anomaly is

$$
\mathcal{A}^{cov}_\Lambda = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta\gamma\delta}c^{rz}c^{z'}_r\,\mathrm{tr}_z(\Lambda F_{\mu\nu})\,\mathrm{tr}_{z'}(F'_{\alpha\beta}F'_{\gamma\delta})\,, \tag{229}
$$

and is related to the consistent anomaly by a local counterterm,

$$\mathcal{A}^{cons}_{\Lambda} + \mathrm{tr}[\Lambda D_\mu f^\mu] = \mathcal{A}^{cov}_{\Lambda}\,, \tag{230}$$

where

$$f^\mu = c^z_r c^{rz'}\{-\frac{1}{4}\epsilon^{\mu\nu\alpha\beta\gamma\delta}A_\nu\,\mathrm{tr}_{z'}(F'_{\alpha\beta}F'_{\gamma\delta}) - \frac{1}{6}\epsilon^{\mu\nu\alpha\beta\gamma\delta}F_{\nu\alpha}\;\omega'_{\beta\gamma\delta}\}\,. \tag{231}$$

Comparing (231) with (**??**) one can see that, to lowest order in the Fermi fields,

$$\mathcal{A}_\epsilon = \mathrm{tr}(\delta_\epsilon A_\mu f^\mu)\,, \tag{232}$$

and this implies that the transition from consistent to covariant anomalies turns a model with a supersymmetry anomaly into another. Indeed, six-dimensional supergravity coupled to vector and tensor multiplets was originally formulated in this fashion to lowest order in the Fermi fields. The resulting vector equation is not integrable. Moreover, the corresponding gauge anomaly is not the gauge variation of a local functional and does not satisfy Wess-Zumino consistency conditions.

This result can be generalized naturally, if somewhat tediously, to include terms of all orders in the Fermi fields. The complete supersymmetry anomaly of (**??**) has the form

$$\mathcal{A}_\epsilon = \mathrm{tr}(\delta_\epsilon A_\mu f^\mu) + \delta_\epsilon e_\mu{}^a g^\mu{}_a \quad , \tag{233}$$

where to lowest order f^μ is defined in (231). Modifying the vector equation so that

$$A^\mu{}_{(cov)} \equiv \mathcal{J}^\mu_{(cov)} = \frac{\delta\mathcal{L}}{\delta A_\mu} - f^\mu \quad , \tag{234}$$

and similarly for the Einstein equation, the resulting theory is supersymmetric but no longer integrable. The covariant vector field equation is

$$\begin{aligned}
\mathcal{J}^\mu_{(cov)} &= 2D_\nu(v_r F^{\mu\nu}) - 2G_{rs}\hat{H}^{s\mu\nu\rho}F_{\nu\rho} - \frac{i}{2}v_r(\bar\psi_\alpha\gamma^{\alpha\beta\mu\nu\rho}\psi_\beta)F_{\nu\rho} + \frac{i}{2}v_r(\bar\chi^m\gamma^{\mu\nu\rho}\chi^m)F_{\nu\rho}\\
&-x^M_r(\bar\psi_\alpha\gamma^{\alpha\mu\nu\rho}\chi^M)F_{\nu\rho} - ix^M_r x^M_s c^{sz'}tr_{z'}(\bar\lambda'\gamma^{\mu\nu\rho}\lambda')F_{\nu\rho} + i\sqrt{2}D_\nu[v_r(\bar\psi_\rho\gamma^{\mu\nu}\gamma^\rho\lambda)]\\
&+\sqrt{2}D_\nu[x^M_r(\bar\chi^M\gamma^{\mu\nu}\lambda)] - \frac{i}{2}F_{\nu\rho}c^{z'}_r\,\mathrm{tr}_{z'}(\bar\lambda'\gamma^{\mu\nu\rho}\lambda') - \frac{i}{2}c^{z'}_r\,\mathrm{tr}_{z'}[(\bar\lambda\gamma^{\mu\nu\rho}\lambda')F'_{\nu\rho}]\\
&-ic^{z'}_r[(\bar\lambda\gamma_\nu\lambda')F'^{\mu\nu}] + \frac{1}{2\sqrt{2}}c^{z'}_r\,\mathrm{tr}_{z'}[(\bar\lambda\gamma^\mu\gamma^\nu\gamma^\rho\lambda')(\bar\lambda'\gamma_\nu\psi_\rho)]\\
&+\frac{x^M_s c^{sz'}}{v_t c^{tz'}}c^{z'}_r\,\mathrm{tr}_{z'}[\frac{3i}{2\sqrt{2}}(\bar\lambda\gamma^\mu\lambda')(\bar\lambda'\chi^M) + \frac{i}{4\sqrt{2}}(\bar\lambda\gamma^{\mu\nu\rho}\lambda')(\bar\lambda'\gamma_{\nu\rho}\chi^M)\\
&+\frac{i}{2\sqrt{2}}(\bar\lambda\gamma_\nu\lambda')(\bar\lambda'\gamma^{\mu\nu}\chi^M)] + c^{z'}_r\,\mathrm{tr}_{z'}[i\alpha\hat{F}_{\nu\rho}(\bar\lambda'\gamma^{\mu\nu\rho}\lambda') - i\alpha(\bar\lambda\gamma^{\mu\nu\rho}\lambda')\hat{F}'_{\nu\rho}\\
&+6i\alpha(\bar\lambda\gamma^\nu\lambda')\hat{F}'_{\mu\nu}] + c^{z'}_r\frac{x^M_s c^{sz'}}{v_t c^{tz'}}\,\mathrm{tr}_{z'}[i\alpha\sqrt{2}(\bar\lambda\gamma^\mu\lambda')(\bar\lambda'\chi^M) - \frac{i\alpha}{2\sqrt{2}}(\bar\lambda\gamma_{\nu\rho}\chi^M)(\bar\lambda'\gamma^{\mu\nu\rho}\lambda')]\\
&+c^{z'}_r\frac{x^M_s c^{sz}}{v_t c^{tz}}\,\mathrm{tr}_{z'}[-\frac{i\alpha}{\sqrt{2}}(\bar\lambda\gamma^\mu\lambda')(\bar\lambda'\chi^M) + \frac{i\alpha}{2\sqrt{2}}(\bar\lambda\gamma^{\mu\nu\rho}\lambda')(\bar\lambda'\gamma_{\nu\rho}\chi^M)\\
&-\frac{i\alpha}{\sqrt{2}}(\bar\lambda\gamma_\nu\lambda')(\bar\lambda'\gamma^{\mu\nu}\chi^M)] = 0\,,
\end{aligned} \tag{235}$$

and completes the results to all orders in the Fermi fields. Analogous to the consistent anomaly the covariant anomaly is defined by the covariant divergence of the covariant current

$$tr(\Lambda D_\mu \mathcal{J}^\mu_{(cov)}) = -\mathcal{A}^{cov}_\Lambda \quad , \tag{236}$$

where $\mathcal{A}_\Lambda^{cov}$ contains higher-order Fermi terms

$$
\begin{aligned}
\mathcal{A}_\Lambda^{cov} = c^{rz} c_r^{z'} \operatorname{tr}_{z,z'}\{ & \frac{1}{2}\epsilon^{\mu\nu\alpha\beta\gamma\delta}(\Lambda F_{\mu\nu})(F'_{\alpha\beta}F'_{\gamma\delta}) + ie\Lambda F_{\nu\rho}(\bar{\lambda}'\gamma^{\mu\nu\rho}D_\mu\lambda') + \frac{ie}{2}\Lambda D_\mu(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda')F'_{\nu\rho} \\
& + ie\Lambda D_\mu[(\bar{\lambda}\gamma_\nu\lambda')F'^{\mu\nu}] - \frac{e}{2\sqrt{2}}\Lambda D_\mu[(\bar{\lambda}\gamma^\mu\gamma^\nu\gamma^\rho\lambda')(\bar{\lambda}'\gamma_\nu\psi_\rho)] \\
& + e\Lambda D_\mu\{\frac{x_s^M c^{sz'}}{v_t c^{tz'}}[-\frac{3i}{2\sqrt{2}}(\bar{\lambda}\gamma^\mu\lambda')(\bar{\lambda}'\chi^M) - \frac{i}{4\sqrt{2}}(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda')(\bar{\lambda}'\gamma_{\nu\rho}\chi^M) \\
& - \frac{i}{2\sqrt{2}}(\bar{\lambda}\gamma_\nu\lambda')(\bar{\lambda}'\gamma^{\mu\nu}\chi^M)]\} + e\Lambda D_\mu[-i\alpha\hat{F}_{\nu\rho}(\bar{\lambda}'\gamma^{\mu\nu\rho}\lambda') + i\alpha(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda')\hat{F}'_{\nu\rho} \\
& - 6i\alpha(\bar{\lambda}\gamma^\nu\lambda')\hat{F}'_{\mu\nu}] + e\Lambda D_\mu\{\frac{x_s^M c^{sz'}}{v_t c^{tz'}}[-i\alpha\sqrt{2}(\bar{\lambda}\gamma^\mu\lambda')(\bar{\lambda}'\chi^M) + \frac{i\alpha}{2\sqrt{2}}(\bar{\lambda}\gamma_{\nu\rho}\chi^M)(\bar{\lambda}'\gamma^{\mu\nu\rho}\lambda')]\} \\
& + e\Lambda D_\mu\{\frac{x_s^M c^{sz}}{v_t c^{tz}}[\frac{i\alpha}{\sqrt{2}}(\bar{\lambda}\gamma^\mu\lambda')(\bar{\lambda}'\chi^M) - \frac{i\alpha}{2\sqrt{2}}(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda')(\bar{\lambda}'\gamma_{\nu\rho}\chi^M) \\
& + \frac{i\alpha}{\sqrt{2}}(\bar{\lambda}\gamma_\nu\lambda')(\bar{\lambda}'\gamma^{\mu\nu}\chi^M)]\}\} \,. \qquad (237)
\end{aligned}
$$

9.6 Gravitational Anomaly

The gauge anomaly $\mathcal{A}_\Lambda = \delta_\Lambda \mathcal{L}$ naturally satisfies the condition

$$\mathcal{A}_\Lambda = -tr(\Lambda \mathcal{D}_\mu J^\mu)\,, \qquad (238)$$

where $J^\mu = 0$ is the complete field equation of the vector field. One can similarly show that the supersymmetry anomaly is related to the field equation of the gravitino, that we write succinctly $\mathcal{J}^\mu = 0$, according to

$$\mathcal{A}_\epsilon = -(\bar{\epsilon}\mathcal{D}_\mu \mathcal{J}^\mu)\,. \qquad (239)$$

We would like to stress that the Noether identities (238) and (239) relate the anomalies to the equations of the fields whose transformations contain derivatives. This observation has a natural application to gravitational anomalies, that we would now like to elucidate. In fact, in analogy with the previous cases one would expect that

$$\mathcal{A}_\xi = \delta_\xi \mathcal{L} = 2\xi_\mu \mathcal{D}_\nu T^{\mu\nu}\,, \qquad (240)$$

where the variation of the metric under general coordinate transformations is

$$\delta g_{\mu\nu} = -\xi^\alpha \partial_\alpha g_{\mu\nu} - g_{\alpha\nu}\partial_\mu \xi^\alpha - g_{\mu\alpha}\partial_\nu \xi^\alpha\,. \qquad (241)$$

Thus, for models without gravitational anomalies one would expect that the divergence of the energy-momentum tensor vanish. Actually, this is no longer true if other anomalies are present, since all fields, not only the metric, have derivative variations under coordinate transformations. For instance, in a theory with gauge and supersymmetry anomalies, the gravitational anomaly is actually

$$\mathcal{A}_\xi = \delta_\xi \mathcal{L} = 2\xi_\nu D_\mu T^{\mu\nu} + \xi_\nu \operatorname{tr}(A^\nu D_\mu J^\mu) + \xi_\nu(\bar{\psi}^\nu D_\mu \mathcal{J}^\mu)\,. \qquad (242)$$

In particular, in our case we are not accounting for gravitational anomalies, that would result in higher-derivative couplings, and indeed one can verify that the divergence of the energy-momentum tensor does not vanish, but satisfies the relation

$$D_\mu T^{\mu\nu} = -\frac{1}{2}\operatorname{tr}(A^\nu D_\mu J^\mu) - \frac{1}{2}(\bar{\psi}^\nu D_\mu \mathcal{J}^\mu)\,. \qquad (243)$$

Finally, one can study the divergence of the Rarita-Schwinger and Einstein equations in the covariant model. To this end, let us begin by stating that the derivation of Noether identities for a system of non-integrable equations does not present difficulties of principle, since these involve only first variations. Indeed, the only difference with respect to the standard case of integrable equations is that now $\delta\mathcal{L}$ is not an exact differential in field space. Still, all invariance principles reflect themselves in linear dependencies of the field equations. Thus, for instance, with the covariant equations obtained from the consistent ones by the redefinition of (234) and by

$$(eq.\ e^{\mu}{}_{a})_{(cov)} = \frac{\delta\mathcal{L}}{\delta e_{\mu}{}^{a}} - g^{\mu}{}_{a}\,, \tag{244}$$

the total $\delta_\epsilon\mathcal{L}$ vanishes by construction. The usual procedure then proves that the divergence of the Rarita-Schwinger equation vanishes for any value of the parameter α. On the other hand, the divergence of the energy-momentum tensor presents some subtleties that we would now like to describe. In particular, it vanishes to lowest order in the fermi couplings, while it gives a covariant non-vanishing result if all fermion couplings are taken into account. The subtlety has to do with the transformation of the vector under general coordinate transformations,

$$\delta_\xi A_\mu = -\xi^\alpha \partial_\alpha A_\mu - \partial_\mu \xi^\alpha A_\alpha\,, \tag{245}$$

and with the corresponding full form of the identity of (242). Starting again from the consistent equations, one finds

$$\mathcal{A}_\xi = \delta_\xi \mathcal{L} = 2\xi_\nu D_\mu T^{\mu\nu} + \xi_\nu tr(A^\nu D_\mu J^\mu) + \xi_\nu tr(F^{\mu\nu} J_\mu) + \xi_\nu (\bar{\Psi}^\nu D_\mu \mathcal{J}^\mu)\,. \tag{246}$$

Reverting to the covariant form eliminates the divergence of the Rarita-Schwinger equation and alters the vector equation, so that the third term has to be retained. The final result is then

$$D_\mu T^{\mu\nu}_{(cov)} = -\frac{1}{2} tr(A^\nu D_\mu J^\mu_{(cov)}) - \frac{1}{2} tr(f_\mu F^{\mu\nu}) - \frac{1}{2} tr(A^\nu D_\mu f^\mu) - \frac{1}{2} e^{\nu a} D_\mu g^{\mu}{}_{a}\,, \tag{247}$$

and is nicely verified by our equations. In particular, this implies that, to lowest order in the fermi couplings, the divergence of $T^{\mu\nu}_{(cov)}$ vanishes.

10 PST Construction

We have reviewed a number of properties of six-dimensional $(1,0)$ supergravity coupled to vector and tensor multiplets. We have always confined our attention to the field equations, thus evading the traditional difficulties met with the action principles for (anti)self-dual tensor fields. We would like to complete our discussion, presenting an action principle for the consistent field equations. What follows is an application of a general method introduced by Pasti, Sorokin and Tonin (PST), that have shown how to obtain Lorentz-covariant Lagrangians for (anti)self-dual tensors with a single auxiliary field. Alternative constructions, some of which preceded the work of PST, need an infinite number of auxiliary fields, and bear a closer relationship to the BRST formulation of closed-string spectra. This method has already been applied to a number of systems, including $(1,0)$ six-dimensional supergravity coupled to tensor multiplets and type IIB ten-dimensional supergravity, whose local gravitational anomaly has been shown to reproduce the well-known results of Alvarez-Gaumé and Witten.

Let us begin by considering a single 2-form with a self-dual field strength in six-dimensional Minkowski space. The PST lagrangian

$$\mathcal{L} = \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} - \frac{1}{4(\partial\Xi)^2} \partial^\mu \Xi H^{-}_{\mu\nu\rho} H^{-\sigma\nu\rho} \partial_\sigma \Xi \quad , \tag{248}$$

where $H = dB$ and $H^- = H - *H$, is invariant under the gauge transformation $\delta B = d\Lambda$, as well as under the additional gauge transformations

$$\delta B = (d\Xi)\Lambda' \tag{249}$$

and

$$\delta\Xi = \Lambda'' , \quad \delta B_{\mu\nu} = \frac{\Lambda''}{(\partial\Xi)^2} H^-_{\mu\nu\rho}\partial^\rho \Xi \,. \tag{250}$$

The last two types of gauge transformations can be used to recover the usual field equation of a self-dual 2-form. Indeed, the scalar equation results from the tensor equation contracted with

$$\frac{H^-_{\mu\nu\rho}\partial^\rho\Xi}{(\partial\Xi)^2} \quad , \tag{251}$$

and consequently does not introduce any additional degrees of freedom. The invariance of (250) can then be used to eliminate the scalar field. This field can not be set to zero, since this choice would clearly make the Lagrangian of (249) inconsistent. With this condition, using (249) one can see that the only solution of the tensor equation is precisely the self-duality condition for its field strength.

We now want to apply this construction to six-dimensional supergravity coupled to vector and tensor multiplets. The theory describes a single self-dual 2-form

$$\hat{\mathcal{H}}_{\mu\nu\rho} = v_r \hat{H}^r_{\mu\nu\rho} - \frac{i}{8}(\bar{\chi}^m \gamma_{\mu\nu\rho}\chi^m) \tag{252}$$

and n antiself-dual 2-forms

$$\hat{\mathcal{H}}^M_{\mu\nu\rho} = x^M_r \hat{H}^r_{\mu\nu\rho} + \frac{i}{4} x^M_r c^{rz}\, \mathrm{tr}_z(\bar{\lambda}\gamma_{\mu\nu\rho}\lambda) \quad . \tag{253}$$

The complete Lagrangian is obtained adding the term

$$-\frac{1}{4}\frac{\partial^\mu\Xi\partial^\sigma\Xi}{(\partial\Xi)^2}[\hat{\mathcal{H}}^-_{\mu\nu\rho}\hat{\mathcal{H}}^{-\nu\rho}_{\sigma} + \hat{\mathcal{H}}^{M+}_{\mu\nu\rho}\hat{\mathcal{H}}^{M+}{}_{\sigma}{}^{\nu\rho}] \tag{254}$$

It can be shown that the 3-form

$$\hat{K}_{\mu\nu\rho} = \hat{\mathcal{H}}_{\mu\nu\rho} - 3\frac{\partial_{[\mu}\Xi\partial^\sigma\Xi}{(\partial\Xi)^2}\hat{\mathcal{H}}^-_{\nu\rho]\sigma} \tag{255}$$

is identically self-dual, while the 3-forms

$$\hat{K}^M_{\mu\nu\rho} = \hat{\mathcal{H}}^M_{\mu\nu\rho} - 3\frac{\partial_{[\mu}\Xi\partial^\sigma\Xi}{(\partial\Xi)^2}\hat{\mathcal{H}}^{M+}_{\nu\rho]\sigma} \tag{256}$$

are identically antiself-dual. With these definitions, we can display rather simply the complete supersymmetry transformations of the fields. Actually, only the transformations of the gravitino and of the tensorinos are affected, and become

$$\begin{aligned}
\delta\psi_\mu &= \hat{D}_\mu\epsilon + \frac{1}{4}\hat{K}_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon + \frac{i}{32}(\bar{\chi}^M\gamma_{\mu\nu\rho}\chi^M)\gamma^{\nu\rho}\epsilon - \frac{3i}{8}(\bar{\epsilon}\chi^M)\gamma_\mu\chi^M \\
&\quad - \frac{i}{8}(\bar{\epsilon}\gamma_{\mu\nu}\chi^M)\gamma^\nu\chi^M + \frac{i}{16}(\bar{\epsilon}\gamma^{\nu\rho}\chi^M)\gamma_{\mu\nu\rho}\chi^M - \frac{9i}{8}v_r c^{rz}\,\mathrm{tr}_z[(\bar{\epsilon}\gamma_\mu\lambda)\lambda] \\
&\quad + \frac{i}{8}v_r c^{rz}\,\mathrm{tr}_z[(\bar{\epsilon}\gamma^\nu\lambda)\gamma_{\mu\nu}\lambda] - \frac{i}{16}v_r c^{rz}\,\mathrm{tr}_z[(\bar{\epsilon}\gamma_{\mu\nu\rho}\lambda)\gamma^{\nu\rho}\lambda] \quad , \\
\delta\chi^M &= \frac{i}{2}x^M_r\hat{\partial_\mu v^r}\gamma^\mu\epsilon + \frac{i}{12}\hat{K}^M_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon + \frac{1}{2}x^M_r c^{rz}\,\mathrm{tr}_z[(\bar{\epsilon}\gamma_\mu\lambda)\gamma^\mu\lambda] \quad ,
\end{aligned} \tag{257}$$

while the scalar field Ξ is invariant under supersymmetry. It can be shown that the complete lagrangian transforms under supersymmetry as dictated by the Wess-Zumino consistency conditions.

We now turn to describe the corresponding modifications of the supersymmetry algebra. In addition to general coordinate, gauge and supersymmetry transformations, the commutator of two supersymmetry transformations on $B^r_{\mu\nu}$ now generates two local PST transformations with parameters

$$\Lambda'_{r\mu} = \frac{\partial^\sigma \Xi}{(\partial\Xi)^2}(v_r \hat{\mathcal{H}}^-_{\sigma\mu\rho} - x^M_r \hat{\mathcal{H}}^{M+}_{\sigma\mu\rho})\xi^\rho \,, \qquad \Lambda = \xi^\mu \partial_\mu \Xi \,. \tag{258}$$

The transformation of (320) on the scalar field ϕ is opposite to its coordinate transformation, and this gives an interpretation of the corresponding commutator

$$[\delta_1, \delta_2]\Xi = \delta_{gct}\Xi + \delta_{PST}\Xi = 0 \,, \tag{259}$$

that vanishes consistently with the invariance of Ξ under supersymmetry. Finally, the commutator on the vielbein determines the parameter of the local Lorentz transformation, that is now

$$\begin{aligned}\Omega_{mn} = &-\xi^\nu(\omega_{\nu mn} - \hat{K}_{\nu mn} - \frac{i}{8}(\bar{\chi}^M \gamma_{\nu mn}\chi^M)) + \frac{1}{2}(\bar{\chi}^M \epsilon_1)(\bar{\chi}^M \gamma_{mn}\epsilon_2) \\ &-\frac{1}{2}(\bar{\chi}^M \epsilon_2)(\bar{\chi}^M \gamma_{mn}\epsilon_1) + v_r c^{rz}\,\mathrm{tr}_z[(\bar{\epsilon}_1\gamma_m\lambda)(\bar{\epsilon}_2\gamma_n\lambda) - (\bar{\epsilon}_2\gamma_m\lambda)(\bar{\epsilon}_1\gamma_n\lambda)] \,.\end{aligned} \tag{260}$$

All other parameters remain unchanged while the algebra closes on-shell on the modified field equations of the Fermi fields.

For completeness, we conclude by displaying the lagrangian of six-dimensional supergravity coupled to vector and tensor multiplets with the inclusion of the PST term,

$$\begin{aligned}e^{-1}\mathcal{L} = &-\frac{1}{4}R + \frac{1}{12}G_{rs}H^{r\mu\nu\rho}H^s_{\mu\nu\rho} - \frac{1}{4}\partial_\mu v^r \partial^\mu v_r - \frac{1}{2}v_r c^{rz}\,\mathrm{tr}_z(F_{\mu\nu}F^{\mu\nu}) \\
&-\frac{1}{8e}\epsilon^{\mu\nu\alpha\beta\gamma\delta}c^z_r B^r_{\mu\nu}\,\mathrm{tr}_z(F_{\alpha\beta}F_{\gamma\delta}) - \frac{i}{2}(\bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu[\frac{1}{2}(\omega+\hat{\omega})]\psi_\rho) - \frac{i}{8}v_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\psi}_\mu\gamma_\nu\psi_\rho) \\
&+\frac{i}{48}v_r[H+\hat{H}]^r_{\alpha\beta\gamma}(\bar{\psi}_\mu\gamma^{\mu\nu\alpha\beta\gamma}\psi_\nu) + \frac{i}{2}(\bar{\chi}^M\gamma^\mu D_\mu(\hat{\omega})\chi^M) - \frac{i}{24}v_r\hat{H}^r_{\mu\nu\rho}(\bar{\chi}^M\gamma^{\mu\nu\rho}\chi^M) \\
&+\frac{1}{4}x^M_r[\partial_\nu v^r + \hat{\partial_\nu v^r}](\bar{\psi}_\mu\gamma^\nu\gamma^\mu\chi^M) - \frac{1}{8}x^M_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\psi}_\mu\gamma_{\nu\rho}\chi^M) \\
&+\frac{1}{24}x^M_r[H+\hat{H}]^{r\mu\nu\rho}(\bar{\psi}^\alpha\gamma_{\alpha\mu\nu\rho}\chi^M) + iv_r c^{rz}\,\mathrm{tr}_z(\bar{\lambda}\gamma^\mu\hat{D}_\mu\lambda) + \frac{i}{2\sqrt{2}}v_r c^{rz}\,\mathrm{tr}_z[(F+\hat{F})_{\nu\rho}(\bar{\psi}_\mu\gamma^{\nu\rho}\gamma^\mu\lambda)] \\
&+\frac{i}{12}x^M_r x^M_s \hat{H}^r_{\mu\nu\rho}c^{sz}\,\mathrm{tr}_z(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda) - \frac{i}{2}x^M_r c^{rz}\,\mathrm{tr}_z[(\bar{\chi}^M\gamma^\mu\gamma^\nu\lambda)(\bar{\psi}_\mu\gamma_\nu\lambda)] \\
&+\frac{1}{16}v_r c^{rz}\,\mathrm{tr}_z(\bar{\lambda}\gamma_{\mu\nu\rho}\lambda)(\bar{\chi}^M\gamma^{\mu\nu\rho}\chi^M) + \frac{1}{\sqrt{2}}x^M_r c^{rz}\,\mathrm{tr}_z[(\bar{\chi}^M\gamma^{\mu\nu}\lambda)\hat{F}_{\mu\nu}] \\
&-\frac{i}{8}(\bar{\chi}^M\gamma_{\mu\nu}\psi_\rho)x^M_r c^{rz}\,\mathrm{tr}_z(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda) - \frac{3}{16}v_r c^{rz}\,\mathrm{tr}_z[(\bar{\chi}^M\gamma_{\mu\nu}\lambda)(\bar{\chi}^M\gamma^{\mu\nu}\lambda)] - \frac{1}{8}v_r c^{rz}\,\mathrm{tr}_z[(\bar{\chi}^M\lambda)(\bar{\chi}^M\lambda)] \\
&-\frac{3}{4}\frac{x^M_r c^{rz}x^N_s c^{sz}}{v_t c^{tz}}\,\mathrm{tr}_z[(\bar{\chi}^M\lambda)(\bar{\chi}^N\lambda)] + \frac{1}{8}(\bar{\chi}^M\gamma^{\mu\nu\rho}\chi^M)(\bar{\psi}_\mu\gamma_\nu\psi_\rho) \\
&+\frac{1}{8}\frac{x^M_r c^{rz}x^N_s c^{sz}}{v_t c^{tz}}\,\mathrm{tr}_z[(\bar{\chi}^M\gamma_{\mu\nu}\lambda)(\bar{\chi}^N\gamma^{\mu\nu}\lambda)] - \frac{1}{8}(\bar{\chi}^M\gamma^\mu\chi^N)(\bar{\chi}^M\gamma_\mu\chi^N) + \frac{1}{4}(\bar{\psi}_\mu\gamma_\nu\psi_\rho)v_r c^{rz}\,\mathrm{tr}_z(\bar{\lambda}\gamma^{\mu\nu\rho}\lambda) \\
&-\frac{1}{2}v_r v_s c^{rz}c^{sz'}\,\mathrm{tr}_{z,z'}[(\bar{\lambda}\gamma_\mu\lambda')(\bar{\lambda}\gamma^\mu\lambda')] + \frac{\alpha}{2}c^{rz}c^{z'}_r\,\mathrm{tr}_{z,z'}[(\bar{\lambda}\gamma_\mu\lambda')(\bar{\lambda}\gamma\mu\lambda')] \\
&-\frac{\partial^\mu\Xi\partial^\sigma\Xi}{4(\partial\Xi)^2}[\hat{\mathcal{H}}^-_{\mu\nu\rho}\hat{\mathcal{H}}^{-\nu\rho}_\sigma + \hat{\mathcal{H}}^{M+}_{\mu\nu\rho}\hat{\mathcal{H}}^{M+}{}_\sigma{}^{\nu\rho}] \,,\end{aligned} \tag{261}$$

where α is the coefficient of the quartic coupling for the gauginos, and the corresponding supersymmetry transformations

$$
\begin{aligned}
&\delta e_\mu{}^m = -i(\bar\epsilon\gamma^m\psi_\mu) \quad ,\\
&\delta B^r_{\mu\nu} = iv^r(\bar\psi_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2}x^{Mr}(\bar\chi^M\gamma_{\mu\nu}\epsilon) - 2c^{rz}\,\mathrm{tr}_z(A_{[\mu}\delta A_{\nu]}) \quad ,\\
&\delta v_r = x^M_r(\bar\chi^M\epsilon) \quad ,\\
&\delta A_\mu = -\frac{i}{\sqrt{2}}(\bar\epsilon\gamma_\mu\lambda) \quad ,\\
&\delta\psi_\mu = \hat D_\mu\epsilon + \frac{1}{4}\hat K_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon + \frac{i}{32}(\bar\chi^M\gamma_{\mu\nu\rho}\chi^M)\gamma^{\nu\rho}\epsilon - \frac{3i}{8}(\bar\epsilon\chi^M)\gamma_\mu\chi^M\\
&\qquad -\frac{i}{8}(\bar\epsilon\gamma_{\mu\nu}\chi^M)\gamma^\nu\chi^M + \frac{i}{16}(\bar\epsilon\gamma^{\nu\rho}\chi^M)\gamma_{\mu\nu\rho}\chi^M - \frac{9i}{8}v_rc^{rz}\,\mathrm{tr}_z[(\bar\epsilon\gamma_\mu\lambda)\lambda]\\
&\qquad +\frac{i}{8}v_rc^{rz}\,\mathrm{tr}_z[(\bar\epsilon\gamma^\nu\lambda)\gamma_{\mu\nu}\lambda] - \frac{i}{16}v_rc^{rz}\,\mathrm{tr}_z[(\bar\epsilon\gamma_{\mu\nu\rho}\lambda)\gamma^{\nu\rho}\lambda] \quad ,\\
&\delta\chi^M = \frac{i}{2}x^M_r\hat{\partial_\mu}v^r\gamma^\mu\epsilon + \frac{i}{12}\hat K^M_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon + \frac{1}{2}x^M_r c^{rz}\,\mathrm{tr}_z[(\bar\epsilon\gamma_\mu\lambda)\gamma^\mu\lambda] \quad ,\\
&\delta\lambda = -\frac{1}{2\sqrt{2}}\hat F_{\mu\nu}\gamma^{\mu\nu}\epsilon - \frac{1}{2}\frac{x^M_rc^{rz}}{v_sc^{sz}}(\bar\chi^M\lambda)\epsilon - \frac{1}{4}\frac{x^M_rc^{rz}}{v_sc^{sz}}(\bar\chi^M\epsilon)\lambda\\
&\qquad +\frac{1}{8}\frac{x^M_rc^{rz}}{v_sc^{sz}}(\bar\chi^M\gamma_{\mu\nu}\epsilon)\gamma^{\mu\nu}\lambda \quad .
\end{aligned}
\tag{262}
$$

We want now to apply to our case the general method introduced by Pasti, Sorokin and Tonin for obtaining Lorentz-covariant lagrangians for (anti)self-dual tensors using a single auxiliary field. Our theory describes a single self-dual 3-form

$$
\hat{\mathcal{H}}_{\mu\nu\rho} = v_r\hat H^r_{\mu\nu\rho} - \frac{i}{8}(\bar\chi^M\gamma_{\mu\nu\rho}\chi^M) + \frac{i}{8}(\bar\Psi_a\gamma_{\mu\nu\rho}\Psi^a) \tag{263}
$$

and n_T antiself-dual 3-forms

$$
\hat{\mathcal{H}}^M_{\mu\nu\rho} = x^M_r\hat H^r_{\mu\nu\rho} - \frac{i}{4}x^M_rc^{rz}tr_z(\bar\lambda\gamma_{\mu\nu\rho}\lambda)\,. \tag{264}
$$

The complete Lagrangian is obtained adding to (**??**) the term

$$
-\frac{\partial^\mu\phi\partial^\sigma\phi}{4(\partial\phi)^2}[\hat{\mathcal{H}}^-_{\mu\nu\rho}\hat{\mathcal{H}}_\sigma^{-\nu\rho} + \hat{\mathcal{H}}^{M+}_{\mu\nu\rho}\hat{\mathcal{H}}^{M+}{}_\sigma{}^{\nu\rho}]\,, \tag{265}
$$

where ϕ is an auxiliary field and $H^\pm = H \pm *H$. The resulting lagrangian is invariant under the additional gauge transformations

$$
\delta B^r_{\mu\nu} = (\partial_\mu\phi)\Lambda^r_\nu - (\partial_\nu\phi)\Lambda^r_\mu \tag{266}
$$

and

$$
\delta\phi = \Lambda\,, \qquad \delta B^r_{\mu\nu} = \frac{\Lambda}{(\partial\phi)^2}[v^r\hat{\mathcal{H}}^-_{\mu\nu\rho} - x^{Mr}\hat{\mathcal{H}}^{M+}_{\mu\nu\rho}]\partial^\rho\phi\,, \tag{267}
$$

used to recover the usual field equations for (anti)self-dual forms. The 3-form

$$
\hat K_{\mu\nu\rho} = \hat{\mathcal{H}}_{\mu\nu\rho} - 3\frac{\partial_{[\mu}\phi\partial^\sigma\phi}{(\partial\phi)^2}\hat{\mathcal{H}}^-_{\nu\rho]\sigma} \tag{268}
$$

is identically self-dual, while the 3-forms

$$\hat{K}^M_{\mu\nu\rho} = \hat{\mathcal{H}}^M_{\mu\nu\rho} - 3\frac{\partial_{[\mu}\phi\partial^\sigma\phi}{(\partial\phi)^2}\hat{\mathcal{H}}^{M+}_{\nu\rho]\sigma} \tag{269}$$

are identically antiself-dual. In order to obtain the complete supersymmetry transformations, we have to substitute $\hat{\mathcal{H}}$ with $\hat{K}$ in the transformation of the gravitino and $\hat{\mathcal{H}}^M$ with $\hat{K}^M$ in the transformations of the tensorinos. Moreover, the auxiliary scalar is invariant under supersymmetry. It can be shown that the complete lagrangian transforms under supersymmetry as dictated by the Wess-Zumino consistency conditions. The commutator of two supersymmetry transformations on $B^r_{\mu\nu}$ now generates the local PST transformations with parameters

$$\Lambda_{r\mu} = \frac{\partial^\sigma\phi}{(\partial\phi)^2}(v_r\hat{\mathcal{H}}^-_{\sigma\mu\nu} - x^M_r\hat{\mathcal{H}}^{M+}_{\sigma\mu\nu})\xi^\nu\,, \qquad \Lambda = \xi^\mu\partial_\mu\phi\,, \tag{270}$$

while in the parameter of the local Lorentz transformation the term $\hat{\mathcal{H}}$ is replaced by $\hat{K}$. All other parameters remain unchanged.

It would be interesting to study in some detail the vacua of the lagrangian (**??**), analyzing the extrema of the potential (204). As a simple example, consider the model without hypermultiplets, in which one can gauge the global R-symmetry group $USp(2)$ of the theory. Formally, the gauged theory without hypermultiplets is obtained from the theory described previously putting $n_H = 0$ and making the identification

$$\mathcal{A}^A_{\alpha\,B}\xi^{\alpha i} \to -T^{iA}{}_B\,, \tag{271}$$

where T^i are the anti-hermitian generators of $USp(2)$. This corresponds to the replacement of the previous couplings between gauge fields and spinors, dressed by the scalars in the case $n_H \neq 0$, with ordinary minimal couplings

$$D_\mu\phi^\alpha\mathcal{A}^A_{\alpha\,B}\epsilon^B \to A^A_{\mu\,B}\epsilon^B\,. \tag{272}$$

Implementing this identification gives in this case the positive-definite potential

$$V = \frac{3}{8v_r c^{r1}} \tag{273}$$

for the scalars in the tensor multiplets. One would thus expect that in these models supersymmetry be spontaneously broken. Notice that this potential diverges at the tensionless string phase transition point. Similarly, one could try to study explicitly the behavior of the potential in simple models containing charged hypermultiplets. Their dimensional reduction gives $N = 2$ supergravity coupled to vector and hypermultiplets in five dimensions, and in the context of the AdS/CFT correspondence and its generalizations there is a renewed interest in studying the explicit gauging of these five-dimensional models.

11 Geometric Couplings in Six-Dimensional Models

We construct the low-energy couplings for six-dimensional type-I models with brane supersymmetry breaking. All the features of brane supersymmetry breaking are present in the T^4/Z_2 orientifold, where a change of the orientifold projection leads to $D9$ branes and $\bar{D}5$ branes. The spectrum has $(1,0)$ supersymmetry in the closed and 9-9 sectors, while supersymmetry is broken in the 9-$\bar{5}$ and $\bar{5}$-$\bar{5}$ sectors. The gauge group is $SO(16)\times SO(16)$ on the $D9$ branes and $USp(16)\times USp(16)$ on the $\bar{D}5$ branes, if all the $\bar{D}5$ branes are at a fixed point. One of the peculiar features of low-energy effective actions for six-dimensional type-I models with minimal supersymmetry is the fact that they embody reducible gauge and supersymmetry anomalies, to be canceled by fermion loops. Consequently, the Lagrangian is determined imposing the closure of the Wess-Zumino consistency conditions, rather than by the requirement of supersymmetry. We use the notations of the previous chapter, and we denote the vector multiplet from the 9-9 sector as $A_\mu^{(9)}, \lambda^{(9)A}$ Denoting with $\Phi^{\bar{\alpha}}$ ($\bar{\alpha}=1,...,n_T$) the scalars in the tensor multiplets, parametrizing the coset $SO(1,n_T)/SO(n_T)$, the vielbein $V_{\bar{\alpha}}^M$ of the internal manifold is related to v^r and x^{Mr} of (187) by

$$V_{\bar{\alpha}}^M = v^r \partial_{\bar{\alpha}} x_r^M \quad , \tag{274}$$

where $\partial_{\bar{\alpha}} = \partial/\partial\Phi^{\bar{\alpha}}$. The metric of the internal manifold is $g_{\bar{\alpha}\bar{\beta}} = V_{\bar{\alpha}}^M V_{\bar{\beta}}^M$.

Denoting with $A_\mu^{(9)i}$ the gauge fields under which the hypermultiplets are charged. The index i runs in the adjoint representation of the gauge group under the gauge transformations

$$\delta A_\mu^{(9)i} = D_\mu \Lambda^{(9)i} \tag{275}$$

the scalars transform as

$$\delta\phi^\alpha = \Lambda^{(9)i}\xi^{\alpha i} \quad , \tag{276}$$

where $\xi^{\alpha i}$ are the Killing vectors corresponding to the isometry that we are gauging. The covariant derivative for the scalars is then

$$D_\mu\phi^\alpha = \partial_\mu\phi^\alpha - A_\mu^{(9)i}\xi^{\alpha i} \quad . \tag{277}$$

The covariant derivatives for the gauginos $\lambda^{(9)iA}$ are

$$D_\mu\lambda^{(9)iA} = \partial_\mu\lambda^{(9)iA} + \frac{1}{4}\omega_{\mu mn}\gamma^{mn}\lambda^{(9)iA} + D_\mu\phi^\alpha \mathcal{A}_{\alpha}{}^{A}{}_{B}\lambda^{(9)iB} + f^{ijk}A_\mu^{(9)j}\lambda^{(9)kA} \quad , \tag{278}$$

where f^{ijk} are the structure constants of the group.

We use the method of Pasti, Sorokin and Tonin (PST) in order to write a covariant action for fields that satisfy self-duality conditions. For a self-dual 3-form in six dimensions the PST action

$$\mathcal{L}_{PST} = \frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho} - \frac{1}{4}\frac{\partial^\mu\Xi\partial^\sigma\Xi}{(\partial\Xi)^2}H^-_{\mu\nu\rho}H_\sigma^{-\nu\rho} \quad , \tag{279}$$

where $H^- = H - *H$ and Ξ is a scalar auxiliary field, is invariant under the standard gauge transformations for a 2-form,

$$\delta B = d\Lambda, \tag{280}$$

and under the additional PST gauge transformations

$$\delta B_{\mu\nu} = (\partial_\mu\Xi)\Lambda_\nu - (\partial_\nu\Xi)\Lambda_\mu \tag{281}$$

and

$$\delta\Xi = \Lambda \quad , \qquad \delta B_{\mu\nu} = \frac{\Lambda}{(\partial\Xi)^2}H^-_{\mu\nu\rho}\partial^\rho\Xi \quad . \tag{282}$$

We have a single self-dual 3-form and n_T antiself-dual 3-forms. These forms are obtained dressing with the scalars in the tensor multiplets the 3-forms

$$H^r = dB^r - c^{rz}\omega^{(9)z} \quad , \tag{283}$$

where the index z runs over the various semi-simple factors of the gauge group in the 9-9 sector, ω is the Chern-Simons 3-form and the c's are constants. We denote with $z=1$ the group under which the hypermultiplets are charged. More precisely, the combinations

$$H_{\mu\nu\rho} = v_r H^r_{\mu\nu\rho} \tag{284}$$

and

$$H^M_{\mu\nu\rho} = x^M_r H^r_{\mu\nu\rho} \tag{285}$$

are respectively self-dual and antiself-dual, to lowest order in the Fermi fields, although in the complete lagrangian these conditions are modified by the inclusion of fermionic bilinears. As in ten dimensions, the gauge invariance of H^r in (283) implies that B^r vary as

$$\delta B^r = c^{rz}\,\mathrm{tr}_z(\Lambda^{(9)} dA^{(9)}) \tag{286}$$

under gauge transformations.

To lowest order in the Fermi fields, the Lagrangian describing the coupling of the supergravity multiplet to n_T tensor multiplets, vector multiplets and n_H hypermultiplets is

$$\begin{aligned}
e^{-1}\mathcal{L}_{SUSY} =& -\frac{1}{4}R + \frac{1}{12}G_{rs}H^{r\mu\nu\rho}H^s_{\mu\nu\rho} + \frac{1}{4}g_{\bar\alpha\bar\beta}\partial_\mu\Phi^{\bar\alpha}\partial^\mu\Phi^{\bar\beta} - \frac{1}{2}v_r c^{rz}\,\mathrm{tr}_z(\mathcal{F}^{(9)}_{\mu\nu}\mathcal{F}^{(9)\mu\nu}) \\
& -\frac{\partial^\mu\Xi\partial^\sigma\Xi}{4(\partial\Xi)^2}[\mathcal{H}^-_{\mu\nu\rho}\mathcal{H}^{-\nu\rho}_\sigma + \mathcal{H}^{M+}_{\mu\nu\rho}\mathcal{H}^{M+}{}_\sigma{}^{\nu\rho}] - \frac{1}{8e}\epsilon^{\mu\nu\rho\sigma\delta\tau}B^r_{\mu\nu}c^z_r\,\mathrm{tr}_z(\mathcal{F}^{(9)}_{\rho\sigma}\mathcal{F}^{(9)}_{\delta\tau}) \\
& +\frac{1}{2}g_{\alpha\beta}(\phi)D_\mu\phi^\alpha D^\mu\phi^\beta + \frac{1}{4v_r c^{r1}}\mathcal{A}^A_{\alpha\,B}\mathcal{A}^B_{\beta\,A}\xi^{\alpha i}\xi^{\beta i} - \frac{i}{2}(\bar\psi_\mu\gamma^{\mu\nu\rho}D_\nu\psi_\rho) \\
& -\frac{i}{2}v_r H^{r\mu\nu\rho}(\bar\psi_\mu\gamma_\nu\psi_\rho) + \frac{i}{2}(\bar\chi^M\gamma^\mu D_\mu\chi^M) - \frac{i}{24}v_r H^r_{\mu\nu\rho}(\bar\chi^M\gamma^{\mu\nu\rho}\chi^M) \\
& +\frac{1}{2}x^M_r\partial_\nu v^r(\bar\psi_\mu\gamma^\nu\gamma^\mu\chi^M) - \frac{1}{2}x^M_r H^{r\mu\nu\rho}(\bar\psi_\mu\gamma_{\nu\rho}\chi^M) + \frac{i}{2}(\bar\Psi_a\gamma^\mu D_\mu\Psi^a) \\
& +\frac{i}{24}v_r H^r_{\mu\nu\rho}(\bar\Psi_a\gamma^{\mu\nu\rho}\Psi^a) - V^{aA}_\alpha D_\nu\phi^\alpha(\bar\psi_{\mu A}\gamma^\nu\gamma^\mu\Psi_a) + iv_r c^{rz}\,\mathrm{tr}_z(\bar\lambda^{(9)}\gamma^\mu D_\mu\lambda^{(9)}) \\
& +\frac{i}{\sqrt{2}}v_r c^{rz}\,\mathrm{tr}_z[\mathcal{F}^{(9)}_{\nu\rho}(\bar\psi_\mu\gamma^{\nu\rho}\gamma^\mu\lambda^{(9)})] + \frac{1}{\sqrt{2}}x^M_r c^{rz}\,\mathrm{tr}_z[\mathcal{F}^{(9)}_{\mu\nu}(\bar\chi^M\gamma^{\mu\nu}\lambda^{(9)})] \\
& -\frac{i}{12}c^z_r H^r_{\mu\nu\rho}\,\mathrm{tr}_z(\bar\lambda^{(9)}\gamma^{\mu\nu\rho}\lambda^{(9)}) - \sqrt{2}V^{aA}_\alpha\xi^{\alpha i}(\bar\lambda^{(9)i}_A\Psi_a) \\
& +\frac{i}{\sqrt{2}}\mathcal{A}^A_{\alpha\,B}\xi^{\alpha i}(\bar\lambda^{(9)i}_A\gamma^\mu\psi^B_\mu) + \frac{1}{\sqrt{2}}\mathcal{A}^A_{\alpha\,B}\frac{x^M_r c^{r1}}{v_s c^{s1}}\xi^{\alpha i}(\bar\lambda^{(9)i}_A\chi^{MB})\ ,
\end{aligned} \tag{287}$$

where $G_{rs} = v_r v_s + x^M_r x^M_s$, while

$$\mathcal{H}_{\mu\nu\rho} = v_r H^r_{\mu\nu\rho} - \frac{3i}{2}(\bar\psi_{[\mu}\gamma_\nu\psi_{\rho]}) - \frac{i}{8}(\bar\chi^M\gamma_{\mu\nu\rho}\chi^M) + \frac{i}{8}(\bar\Psi_a\gamma_{\mu\nu\rho}\Psi^a) \tag{288}$$

and

$$\mathcal{H}^M_{\mu\nu\rho} = x^M_r H^r_{\mu\nu\rho} + \frac{3}{2}(\bar\chi^M\gamma_{[\mu\nu}\psi_{\rho]}) + \frac{i}{4}x^M_r c^{rz}\,\mathrm{tr}_z(\bar\lambda^{(9)}\gamma_{\mu\nu\rho}\lambda^{(9)}) \tag{289}$$

satisfy on-shell self-duality and antiself-duality conditions, respectively. Finally, Ξ is the PST auxiliary field.

Due to (286), the Wess-Zumino term $B\wedge F\wedge F$ is not gauge invariant, and thus the variation of (287) under gauge transformations produces the consistent gauge anomaly

$$\mathcal{A}_\Lambda = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma\delta\tau}c^z_r c^{rz'}\,\mathrm{tr}_z(\Lambda^{(9)}\partial_\mu A^{(9)}_\nu)\,\mathrm{tr}_{z'}(F^{(9)}_{\rho\sigma}F^{(9)}_{\delta\tau}) \quad , \tag{290}$$

related by the Wess-Zumino conditions to the supersymmetry anomaly

$$\mathcal{A}_\epsilon = \epsilon^{\mu\nu\rho\sigma\delta\tau}c^z_r c^{rz'}[-\frac{1}{4}\,\mathrm{tr}_z(\delta_\epsilon A^{(9)}_\mu A^{(9)}_\nu)\,\mathrm{tr}_{z'}(F^{(9)}_{\rho\sigma}F^{(9)}_{\delta\tau}) - \frac{1}{6}\,\mathrm{tr}_z(\delta_\epsilon A^{(9)}_\mu F^{(9)}_{\nu\rho})\omega^{(9)z'}_{\sigma\delta\tau}]\ , \tag{291}$$

that one can recover varying the Lagrangian of (287) under the supersymmetry transformations

$$\begin{aligned}
&\delta e_\mu{}^m = -i(\bar\epsilon\gamma^m\psi_\mu) \quad ,\\
&\delta B^r_{\mu\nu} = iv^r(\bar\psi_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2}x^{Mr}(\bar\chi^M\gamma_{\mu\nu}\epsilon) - 2c^{rz}\,\mathrm{tr}_z(A^{(9)}_{[\mu}\delta A^{(9)}_{\nu]}) \quad ,\\
&\delta\Phi^{\bar\alpha} = V^{\bar\alpha M}(\bar\chi^M\epsilon) \quad ,\\
&\delta\phi^\alpha = V^\alpha_{aA}(\bar\epsilon^A\Psi^a) \quad ,\\
&\delta\Xi = 0 \quad ,\\
&\delta A^{(9)}_\mu = -\frac{i}{\sqrt2}(\bar\epsilon\gamma_\mu\lambda^{(9)}) \quad ,\\
&\delta\psi^A_\mu = D_\mu\epsilon^A + \frac{1}{4}K_{\mu\nu\rho}\gamma^{\nu\rho}\epsilon^A \quad ,\\
&\delta\chi^{MA} = -\frac{i}{2}V^M_{\bar\alpha}\partial_\mu\Phi^{\bar\alpha}\gamma^\mu\epsilon^A + \frac{i}{12}K^M_{\mu\nu\rho}\gamma^{\mu\nu\rho}\epsilon^A \quad ,\\
&\delta\Psi^a = i\gamma^\mu\epsilon_A V^{aA}_\alpha D_\mu\phi^\alpha \quad ,\\
&\delta\lambda^{(9)A} = -\frac{1}{2\sqrt2}\mathcal{F}^{(9)}_{\mu\nu}\gamma^{\mu\nu}\epsilon^A \qquad\qquad (z\neq 1) \quad ,\\
&\delta\lambda^{(9)iA} = -\frac{1}{2\sqrt2}\mathcal{F}^{(9)i}_{\mu\nu}\gamma^{\mu\nu}\epsilon^A - \frac{1}{\sqrt2 v_r c^{r1}}\mathcal{A}^A_{\alpha\,B}\xi^{\alpha i}\epsilon^B
\end{aligned} \tag{292}$$

where

$$K_{\mu\nu\rho} = \mathcal{H}_{\mu\nu\rho} - 3\frac{\partial_{[\mu}\Xi\partial^\sigma\Xi}{(\partial\Xi)^2}\mathcal{H}^-_{\sigma\nu\rho]}\ , \qquad K^M_{\mu\nu\rho} = \mathcal{H}^M_{\mu\nu\rho} - 3\frac{\partial_{[\mu}\Xi\partial^\sigma\Xi}{(\partial\Xi)^2}\mathcal{H}^{M+}_{\sigma\nu\rho]} \tag{293}$$

are identically self-dual and antiself-dual, respectively. In the complete theory, the anomalous terms would be exactly canceled by the anomalous contributions of fermion loops.

Following the same reasoning as for the ten dimensional case, we can describe the couplings to non-supersymmetric matter requiring that local supersymmetry be non-linearly realized on the $\bar D5$-branes, and requiring that the supersymmetry variation of the non-supersymmetric fields be as in (**??**). The supersymmetry variation of the dressed scalars in the tensor multiplets is

$$\hat\Phi^{\bar\alpha} = \Phi^{\bar\alpha} - V^{\bar\alpha M}(\bar\theta\chi^M) + \frac{i}{24}V^{\bar\alpha M}x^M_r H^r_{\mu\nu\rho}(\bar\theta\gamma^{\mu\nu\rho}\theta) \tag{294}$$

is a general coordinate transformation of parameter

$$\xi_\mu = -\frac{i}{2}(\bar\theta\gamma_\mu\epsilon) \quad . \tag{295}$$

This definition of $\hat\Phi$ then induces the corresponding dressing

$$\hat v^r = v^r - x^{Mr}(\bar\theta\chi^M) - \frac{i}{24}H^r_{\mu\nu\rho}(\bar\theta\gamma^{\mu\nu\rho}\theta) \quad , \tag{296}$$

and, in a similar fashion, the supersymmetry transformation of

$$\hat\phi^\alpha = \phi^\alpha - V^\alpha_{aA}(\bar\theta^A\Psi^a) - \frac{i}{2}V_{\beta aA}V^{\alpha aB}(\bar\theta^A\gamma^\mu\theta_B)D_\mu\phi^\beta \tag{297}$$

is again a coordinate transformation with the same parameter, together with an additional gauge transformation of parameter

$$\Lambda^{(9)} = \xi^\mu A^{(9)}_\mu \quad . \tag{298}$$

Similarly, the supersymmetry variation of

$$\hat{e}_\mu{}^m = e_\mu{}^m + i(\bar\theta\gamma^m\psi_\mu) - \frac{i}{2}(\bar\theta\gamma^m D_\mu\theta) - \frac{i}{8}v_r H^r_{\mu\nu\rho}(\bar\theta\gamma^{m\nu\rho}\theta) \tag{299}$$

contains also an additional local Lorentz transformation of parameter

$$\Lambda^{mn} = -\xi^\mu[\omega_\mu{}^{mn} - v_r H_\mu^{rmn}] \tag{300}$$

where ω denotes the spin connection. Since the scalars in the non-supersymmetric 9-$\bar{5}$ sector are charged with respect to the vectors in the 9-9 sector, we define also

$$\begin{aligned}
\hat{A}^{(9)}_\mu &= A^{(9)}_\mu + \frac{i}{\sqrt{2}}(\bar\theta\gamma_\mu\lambda^{(9)}) + \frac{i}{8}\mathcal{F}^{(9)\nu\rho}(\bar\theta\gamma_{\mu\nu\rho}\theta) \quad (z \neq 1) \ , \\
\hat{A}^{(9)i}_\mu &= A^{(9)i}_\mu + \frac{i}{\sqrt{2}}(\bar\theta\gamma_\mu\lambda^{(9)i}) + \frac{i}{8}\mathcal{F}^{(9)i\nu\rho}(\bar\theta\gamma_{\mu\nu\rho}\theta) + \frac{i}{4v_r c^{r1}}\mathcal{A}^A_{\alpha\, B}\xi^{\alpha i}(\bar\theta_A\gamma_\mu\theta^B) \ ,
\end{aligned} \tag{301}$$

whose supersymmetry transformation is a general coordinate transformation of parameter as in (295), aside from a gauge transformation of parameter as in (298). If one requires that the supersymmetry variation of the vector $A^{(5)}_\mu$ from the non-supersymmetric $\bar{5}$-$\bar{5}$ sector be

$$\delta A^{(5)}_\mu = \mathcal{F}^{(5)}_{\mu\nu}\xi^\nu \ , \tag{302}$$

namely a general coordinate transformation together with a gauge transformation of parameter

$$\Lambda^{(5)} = \xi^\mu A^{(5)}_\mu \ , \tag{303}$$

one obtains a supersymmetrization of the kinetic term for $A^{(5)}_\mu$ writing

$$-\frac{1}{2}\hat{e}\hat{v}^r c^w_r \,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\mu\nu}\mathcal{F}^{(5)}_{\rho\sigma})\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma} \quad , \tag{304}$$

where

$$\hat{g}_{\mu\nu} = \hat{e}_\mu{}^m\hat{e}_{\nu m} \ , \tag{305}$$

and the index w runs over the various semi-simple factors of the gauge group in the $\bar{5}$-$\bar{5}$ sector. In analogy with the ten-dimensional case, the uncanceled NS-NS tadpole translates, in the low-energy theory, in the presence of a term

$$-\Lambda\hat{e}f(\hat\Phi^{\bar\alpha},\hat\phi^\alpha) \ , \tag{306}$$

that depends on the scalars of the closed sector and contains the dilaton, that belongs to a hypermultiplet in type-I vacua. Thus, supersymmetry breaking naturally corresponds in this case also to a breaking of the isometries of the scalar manifolds.

Denoting with S the scalars in the 9-$\bar{5}$ sector, charged with respect to the gauge fields in both the 9-9 and $\bar{5}$-$\bar{5}$ sectors, we define their covariant derivative as

$$\hat{D}_\mu S = \partial_\mu S - i\hat{A}^{(9)}_\mu S - iA^{(5)}_\mu S \quad , \tag{307}$$

so that the term

$$\frac{1}{2}\hat{e}(\hat{D}_\mu S)^\dagger(\hat{D}_\nu S)\hat{g}^{\mu\nu} \tag{308}$$

is supersymmetric, if again the supersymmetry transformation of S is a general coordinate transformation, together with a gauge transformation of the right parameters. As in the ten-dimensional case, if one considers terms up to quartic couplings in the fermionic fields, one does not have to supersymmetrize terms that are quadratic in the additional fermions from the non-supersymmetric $\bar{5}$-$\bar{5}$ and 9-$\bar{5}$ sectors. Denoting with $\lambda^{(5)}$ these fermions, the coupling of $\lambda^{(5)2}$ to the 3-forms is not determined by supersymmetry, and can only be determined by string considerations.

The inclusion of additional non-supersymmetric vectors modifies H^r, that now includes the Chern-Simons 3-forms corresponding to these fields, so that (283) becomes

$$H^r = dB^r - c^{rz}\omega^{(9)z} - c^{rw}\omega^{(5)w} \quad . \tag{309}$$

The gauge invariance of H^r then requires that

$$\delta B^r = c^{rw}\,\mathrm{tr}_w(\Lambda^{(5)} dA^{(5)}) \tag{310}$$

under gauge transformations in the $\bar{5}$-$\bar{5}$ sector. Consequently, the supersymmetry transformation of B^r is also modified, and becomes

$$\delta B^r_{\mu\nu} = iv^r(\bar{\psi}_{[\mu}\gamma_{\nu]}\epsilon) + \frac{1}{2}x^{Mr}(\bar{\chi}^M\gamma_{\mu\nu}\epsilon) - 2c^{rz}\,\mathrm{tr}_z(A^{(9)}_{[\mu}\delta A^{(9)}_{\nu]}) - 2c^{rw}\,\mathrm{tr}_w(A^{(5)}_{[\mu}\delta A^{(5)}_{\nu]}) \; . \tag{311}$$

The complete reducible gauge anomaly

$$\begin{aligned}\mathcal{A}_\Lambda = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma\delta\tau}\{&c^z_r c^{rz'}\,\mathrm{tr}_z(\Lambda^{(9)}\partial_\mu A^{(9)}_\nu)\,\mathrm{tr}_{z'}(\mathcal{F}^{(9)}_{\rho\sigma}\mathcal{F}^{(9)}_{\delta\tau}) + c^z_r c^{rw}\,\mathrm{tr}_z(\Lambda^{(9)}\partial_\mu A^{(9)}_\nu)\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau})\\ &+c^w_r c^{rz}\,\mathrm{tr}_w(\Lambda^{(5)}\partial_\mu A^{(5)}_\nu)\,\mathrm{tr}_z(\mathcal{F}^{(9)}_{\rho\sigma}\mathcal{F}^{(9)}_{\delta\tau}) + c^w_r c^{rw'}\,\mathrm{tr}_w(\Lambda^{(5)}\partial_\mu A^{(5)}_\nu)\,\mathrm{tr}_{w'}(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau})\}\;,\end{aligned} \tag{312}$$

related by the Wess-Zumino conditions to the supersymmetry anomaly

$$\begin{aligned}\mathcal{A}_\epsilon = \epsilon^{\mu\nu\rho\sigma\delta\tau}\{&c^z_r c^{rz'}[-\frac{1}{4}\,\mathrm{tr}_z(\delta_\epsilon A^{(9)}_\mu A^{(9)}_\nu)\,\mathrm{tr}_{z'}(\mathcal{F}^{(9)}_{\rho\sigma}\mathcal{F}^{(9)}_{\delta\tau}) - \frac{1}{6}\,\mathrm{tr}_z(\delta_\epsilon A^{(9)}_\mu \mathcal{F}^{(9)}_{\nu\rho})\omega^{(9)z'}_{\sigma\delta\tau}]\\ &+c^z_r c^{rw}[-\frac{1}{4}\,\mathrm{tr}_z(\delta_\epsilon A^{(9)}_\mu A^{(9)}_\nu)\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau}) - \frac{1}{6}\,\mathrm{tr}_z(\delta_\epsilon A^{(9)}_\mu \mathcal{F}^{(9)}_{\nu\rho})\omega^{(5)w}_{\sigma\delta\tau}]\\ &+c^w_r c^{rz}[-\frac{1}{4}\,\mathrm{tr}_w(\delta_\epsilon A^{(5)}_\mu A^{(5)}_\nu)\,\mathrm{tr}_z(\mathcal{F}^{(9)}_{\rho\sigma}\mathcal{F}^{(9)}_{\delta\tau}) - \frac{1}{6}\,\mathrm{tr}_w(\delta_\epsilon A^{(5)}_\mu \mathcal{F}^{(5)}_{\nu\rho})\omega^{(9)z}_{\sigma\delta\tau}]\\ &+c^w_r c^{rw'}[-\frac{1}{4}\,\mathrm{tr}_w(\delta_\epsilon A^{(5)}_\mu A^{(5)}_\nu)\,\mathrm{tr}_{w'}(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau}) - \frac{1}{6}\,\mathrm{tr}_w(\delta_\epsilon A^{(5)}_\mu \mathcal{F}^{(5)}_{\nu\rho})\omega^{(5)w'}_{\sigma\delta\tau}]\}\;,\end{aligned} \tag{313}$$

is induced by the Wess-Zumino term

$$-\frac{1}{8}\epsilon^{\mu\nu\rho\sigma\delta\tau}B^r_{\mu\nu}c^w_r\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau}) \quad . \tag{314}$$

It should be noticed that, as in the case of linearly realized supersymmetry, (312) and (313) satisfy the Wess-Zumino condition

$$\delta_\Lambda \mathcal{A}_\epsilon = \delta_\epsilon \mathcal{A}_\Lambda \quad , \tag{315}$$

since the explicit form of the gauge field supersymmetry variation plays no role in its proof. We expect that, to higher order in the fermions, the supersymmetry anomaly will be modified by gauge-invariant terms. From the definition of H^r, one can deduce the Bianchi identities

$$\partial_{[\mu}H^r_{\nu\rho\sigma]} = -\frac{3}{2}c^r_z\,\mathrm{tr}_z(\mathcal{F}^{(9)}_{[\mu\nu}\mathcal{F}^{(9)}_{\rho\sigma]}) - \frac{3}{2}c^r_w\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{[\mu\nu}\mathcal{F}^{(5)}_{\rho\sigma]}) \quad . \tag{316}$$

We now want to determine the terms proportional to $F \wedge F$ containing the goldstino that one has to add for the consistency of the model. Unlike the ten dimensional case, where duality maps the 2-form theory with Chern-Simons couplings to the 6-form theory with Wess-Zumino couplings, in this case the low-energy effective action contains both Chern-Simons and Wess-Zumino couplings. First of all, we observe that for the quantity

$$\begin{aligned}\hat{B}^r_{\mu\nu} &= B^r_{\mu\nu} - iv^r(\bar{\psi}_{[\mu}\gamma_{\nu]}\theta) - \frac{1}{2}x^{Mr}(\bar{\chi}^M\gamma_{\mu\nu}\theta) - \frac{2i}{\sqrt{2}}c^{rz}\,\mathrm{tr}_z[A^{(9)}_{[\mu}(\bar{\theta}\gamma_{\nu]}\lambda^{(9)})] \\ &+ \frac{i}{8}(\partial_\rho v^r)(\bar{\theta}\gamma_{\mu\nu}{}^{\rho}\theta) + \frac{i}{8}x^{Mr}H^{M\rho\sigma}_{[\mu}(\bar{\theta}\gamma_{\nu]\rho\sigma}\theta) + \frac{i}{2}v^r(\bar{\theta}\gamma_{[\mu}D_{\nu]}\theta) \\ &- \frac{i}{4}c^{rz}\,\mathrm{tr}_z[A^{(9)}_{[\mu}\mathcal{F}^{(9)\rho\sigma}](\bar{\theta}\gamma_{\nu]\rho\sigma}\theta) - \frac{ic^{r1}}{4v_s c^{s1}}\mathcal{A}_{\alpha}{}^{A}{}_{B}\xi^{\alpha i}A^{(9)i}_{[\mu}(\bar{\theta}^A\gamma_{\nu]}\theta^B)\end{aligned} \tag{317}$$

the supersymmetry variation is a general coordinate transformation of the correct parameter, together with an additional tensor gauge transformation of parameter

$$\Lambda^r_\mu = -\frac{1}{2}v^r\xi_\mu - \xi^\nu B^r_{\mu\nu} \quad , \tag{318}$$

as well as PST gauge transformations of parameters

$$\Lambda^{(PST)r}_\mu = \frac{\partial^\sigma\Xi}{(\partial\Xi)^2}[v^r v_s H^{s-}_{\sigma\mu\rho} - x^{Mr}x^M_s H^{s+}_{\sigma\mu\rho}]\xi^\rho \tag{319}$$

and

$$\Lambda^{(PST)} = \xi^\mu\partial_\mu\Xi \tag{320}$$

and gauge transformations of the form (286) and (310) whose parameters are as in (298) and (303). We should now consider all the terms proportional to $F\wedge F$ that arise, those directly introduced by the inclusion of the Chern-Simons 3-form for the fields in the $\bar{5}$-$\bar{5}$ sector, those originating from the consequent modification of the Bianchi identities, and finally those introduced by the variation of the Wess-Zumino term.

The end result is that the variation of all these contributions gives

$$\begin{aligned}\delta\mathcal{L} &= \epsilon^{\mu\nu\rho\sigma\delta\tau}\{-\frac{i}{4}v_r(\bar{\epsilon}\gamma_\mu\psi_\nu) + \frac{1}{8}x^M_r(\bar{\epsilon}\gamma_{\mu\nu}\chi^M)\}c^{rw}\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau}) \\ &- 2v_r c^{rw}\,\mathrm{tr}_w(\delta A^{(5)}_\mu\mathcal{F}^{(5)}_{\nu\rho})K^{\mu\nu\rho} - 2x^M_r c^{rw}\,\mathrm{tr}_w(\delta A^{(5)}_\mu\mathcal{F}^{(5)}_{\nu\rho})K^{M\mu\nu\rho} \quad .\end{aligned} \tag{321}$$

The first two terms are canceled by the goldstino variation in the additional couplings

$$\mathcal{L}' = \epsilon^{\mu\nu\rho\sigma\delta\tau}\{\frac{i}{4}v_r(\bar{\theta}\gamma_\mu\psi_\nu) - \frac{1}{8}x^M_r(\bar{\theta}\gamma_{\mu\nu}\chi^M)\}c^{rw}\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau})\,, \tag{322}$$

where, however, the variations of the gravitino and the tensorinos produce additional terms. Some of these cancel the last two terms in (321), while the remaining ones are canceled by the goldstino variation in

$$\mathcal{L}'' = \epsilon^{\mu\nu\rho\sigma\delta\tau}\{-\frac{i}{32}\partial^\rho v_r(\bar{\theta}\gamma_{\mu\nu\rho}\theta) - \frac{i}{8}v_r(\bar{\theta}\gamma_\mu D_\nu\theta) - \frac{i}{32}x^M_r K^{M\alpha\beta}_\mu(\bar{\theta}\gamma_{\nu\alpha\beta}\theta)\}c^{rw}\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau})\,. \tag{323}$$

If one restricts the attention to terms up to quartic fermion couplings, no further contributions are produced. We can thus conclude that the non-linear realization of supersymmetry is granted by the inclusion of $\mathcal{L}'$ and $\mathcal{L}''$ in the low-energy effective action. From (317) we also see that these two contributions can be written in the compact form

$$\mathcal{L}' + \mathcal{L}'' = -\frac{1}{4}\epsilon^{\mu\nu\rho\sigma\delta\tau}\mathcal{B}^r_{\mu\nu}c^w_r\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau})\,, \tag{324}$$

where

$$\mathcal{B}^r_{\mu\nu} = -iv^r(\bar\psi_{[\mu}\gamma_{\nu]}\theta) - \frac{1}{2}x^{Mr}(\bar\chi^M\gamma_{\mu\nu}\theta) - \frac{2i}{\sqrt{2}}c^{rz}\,\mathrm{tr}_z[A^{(9)}_{[\mu}(\bar\theta\gamma_{\nu]}\lambda^{(9)})] + \frac{i}{8}(\partial_\rho v^r)(\bar\theta\gamma_{\mu\nu}{}^\rho\theta) + \frac{i}{8}x^{Mr}K^{M\rho\sigma}_{[\mu}(\bar\theta\gamma_{\nu]\rho\sigma}\theta) + \frac{i}{2}v^r(\bar\theta\gamma_{[\mu}D_{\nu]}\theta) \tag{325}$$

coincides with the counterterm of B^r only if no 9-9 vectors are present.

To this end, observe that, if no 9-9 vectors are present, (324) is exactly twice the term that one should add to (314) in order to geometrize the Wess-Zumino term, substituting B with $\hat{B}$. This means, roughly speaking, that half of the contribution in (324) comes from the Green-Schwarz term, and half from the Chern-Simons couplings. This interpretation is in perfect agreement with self-duality, and thus in six dimensions there is no duality transformation that can give a fully geometric Lagrangian. If also 9-9 vectors are in the spectrum, no additional terms are produced in the lagrangian, in agreement with the fact that the additional terms of $\hat{B}^r$ in (317) are not gauge invariant.

To resume, the Lagrangian for supergravity coupled to tensor multiplets, hypermultiplets and non-supersymmetric vectors is

$$\mathcal{L}_{SUGRA} = \mathcal{L}_{SUSY} - \frac{1}{2}\hat{e}\hat{v}^r c^w_r\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\mu\nu}\mathcal{F}^{(5)}_{\rho\sigma})\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma} - \Lambda\hat{e}f(\hat{\Phi}^{\bar\alpha},\hat{\phi}^\alpha) + \frac{1}{2}\hat{e}(\hat{D}_\mu S)^\dagger(\hat{D}_\nu S)\hat{g}^{\mu\nu} - \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\delta\tau}B^r_{\mu\nu}c^w_r\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau}) - \frac{1}{4}\epsilon^{\mu\nu\rho\sigma\delta\tau}\mathcal{B}^r_{\mu\nu}c^w_r\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau}) \ . \tag{326}$$

Since the supersymmetry transformation of other non-supersymmetric fermions is of higher order in the Fermi fields, at this level we can always add them in the construction, while the couplings that can not be determined by supersymmetry could in principle be determined by string inputs.

Finally, it is important to observe that without 9-9 vectors, although the Lagrangian (326) is not completely geometric. Indeed, if one fixes the PST gauge in such a way that the 3-forms satisfy the standard (anti)self-duality conditions, the equation for the vector fields, up to terms quartic in the fermions, is

$$\hat{e}D_\nu[\hat{v}^r c_{rw}\mathcal{F}^{(5)}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}] + \frac{1}{6}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_{rw}\mathcal{F}^{(5)}_{\nu\rho}\hat{H}^r_{\sigma\delta\tau} + \frac{1}{12}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_{rw}\mathcal{F}^{(5)}_{\nu\rho}c^{rw'}\omega^{(5)w'}_{\sigma\delta\tau} + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_{rw}A^{(5)}_\nu c^{rw'}\,\mathrm{tr}_{w'}(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau}) = 0 \ , \tag{327}$$

where

$$\hat{H}^r_{\mu\nu\rho} = 3\partial_{[\mu}\hat{B}^r_{\nu\rho]} - c^r_w\omega^{(5)w}_{\mu\nu\rho} \ , \tag{328}$$

and this is nicely of geometric form. It should be noticed that no additional counterterms containing the goldstino have to be added if also 9-9 vectors are present. In fact, all the terms in $\hat{B}^r$ induced by $A^{(9)}$ are not gauge invariant, and their inclusion in the lagrangian is forbidden because it would modify the gauge anomaly. The resulting equation for $A^{(5)}$ is then

$$\hat{e}D_\nu[\hat{v}^r c_{rw}\mathcal{F}^{(5)}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}] + \frac{1}{6}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_{rw}\mathcal{F}^{(5)}_{\nu\rho}\tilde{H}^r_{\sigma\delta\tau} + \frac{1}{12}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_{rw}\mathcal{F}^{(5)}_{\nu\rho}c^{rz}\omega^{(9)z}_{\sigma\delta\tau} + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_{rw}A^{(5)}_\nu c^{rz}\,\mathrm{tr}_z(\mathcal{F}^{(9)}_{\rho\sigma}\mathcal{F}^{(9)}_{\delta\tau}) + \frac{1}{12}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_{rw}\mathcal{F}^{(5)}_{\nu\rho}c^{rw'}\omega^{(5)w'}_{\sigma\delta\tau} + \frac{1}{8}\epsilon^{\mu\nu\rho\sigma\delta\tau}c_{rw}A^{(5)}_\nu c^{rw'}\,\mathrm{tr}_{w'}(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau}) = 0 \ , \tag{329}$$

where

$$\tilde{H}^r_{\mu\nu\rho} = 3\partial_{[\mu}B^r_{\nu\rho]} + 3\partial_{[\mu}B^r_{\nu\rho]}{}^{extra} - c^{rz}\omega^{(9)z}_{\mu\nu\rho} - c^{rw}\omega^{(5)w}_{\mu\nu\rho} \tag{330}$$

is geometric up to gauge-invariant terms proportional to c^{rz}. The result is thus in agreement with what expected by anomaly considerations. If gauge and supersymmetry anomalies are absent, the $A^{(5)}$ equation is mapped into itself by supersymmetry: this is the very reason why this equation is geometric. In the presence of gauge and supersymmetry anomalies, as long as 9-9 vectors are absent, the equation for $A^{(5)}$ is still geometric, albeit not gauge invariant. The supersymmetry anomaly, in this case, results from the gauge transformation contained in (302). When also 9-9 vectors are present, these arguments do not apply, and thus in (329) the geometric structure is violated by terms proportional to $c^{rz}c^w_r$.

The consistent formulation described above can be reverted to a supersymmetric formulation in terms of covariant non-integrable field equations, that embody the corresponding covariant gauge anomaly

$$\begin{aligned}\mathcal{A}^{cov}_\Lambda = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma\delta\tau}[&c^{rz}c^{z'}_r\,\mathrm{tr}_z(\Lambda^{(9)}\mathcal{F}^{(9)}_{\mu\nu})\,\mathrm{tr}_{z'}(\mathcal{F}^{(9)}_{\rho\sigma}\mathcal{F}^{(9)}_{\delta\tau}) + c^{rz}c^w_r\,\mathrm{tr}_z(\Lambda^{(9)}\mathcal{F}^{(9)}_{\mu\nu})\,\mathrm{tr}_w(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau})\\ &+c^{rw}c^z_r\,\mathrm{tr}_w(\Lambda^{(5)}\mathcal{F}^{(5)}_{\mu\nu})\,\mathrm{tr}_z(\mathcal{F}^{(9)}_{\rho\sigma}\mathcal{F}^{(9)}_{\delta\tau}) + c^{rw}c^{w'}_r\,\mathrm{tr}_w(\Lambda^{(5)}\mathcal{F}^{(5)}_{\mu\nu})\,\mathrm{tr}_{w'}(\mathcal{F}^{(5)}_{\rho\sigma}\mathcal{F}^{(5)}_{\delta\tau})]\ ,\end{aligned} \tag{331}$$

given by the divergence of the covariant equation for $A^{(5)}_\mu$,

$$\hat{e}D_\nu[\hat{v}^r c_{rw}\mathcal{F}^{(5)}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}] + \frac{1}{6}\epsilon^{\mu\nu\rho\alpha\beta\gamma}c_{rw}\mathcal{F}^{(5)}_{\nu\rho}\tilde{H}^r_{\alpha\beta\gamma} = 0\ , \tag{332}$$

and the divergence of the covariant equation for $A^{(9)}_\mu$. Without 9-9 vectors, (332) is both geometric and gauge-covariant, while, if 9-9 vectors are present, the geometric structure is violated by gauge-covariant terms proportional again to $c^{rz}c^w_r$.

12 Conclusion

The main purpose of this publication was to investigate the covariant, consistent and gravitational anomalies in gauged supergravity. We have shown how general gauge theories with axionic shift symmetries, generalized Chern-Simons terms and quantum anomalies can be formulated in a way that is covariant with respect to electric-magnetic duality transformations. We performed our analysis first in rigid supersymmetry. Using superconformal techniques, we could then show that only one cancellation had to be checked to extend the results to supergravity. It turns out that the Chern-Simons term does not need any gravitino corrections and can thus be added as such to the matter-coupled supergravity actions. Our paper provides thus an extension to the general framework of coupled chiral and vector multiplets in $N = 1$ supergravity. We have completed the coupling of (1, 0) six-dimensional supergravity to tensor and vector multiplets. The coupling to tensor multiplets is of a more conventional nature, and parallels similar constructions in other supergravity models. Our work is here confined to the field equations, but a lagrangian formulation of the (anti)self-dual two-forms is now possible, following the proposal of Pasti, Sorokin and Tonin and indeed, results to this effect have been presented in a superspace formulation. The Yang-Mills currents are not conserved, and the consistent residual gauge anomaly is accompanied by a corresponding anomaly in the supersymmetry current. In completing these results to all orders in the fermi fields, we have come to terms with another peculiar feature of anomalies, neatly displayed by these field equations: anomalous divergences of gauge currents are typically accompanied by corresponding anomalies in current commutators. We have shown that cancellation of gravitational, gauge, and mixed anomalies gives a sufficient constraint on six-dimensional supersymmetric theories of gravity with gauge and matter fields that in some cases all models consistent with anomaly cancellation admit a realization through string theory. We have ruled out a number of infinite families of models which satisfy anomaly factorization, so that the gap is rather small between the set of known 6D models satisfying anomaly cancellation and the set of models realized through string compactification. We have conjectured that all consistent 6D supergravity theories with Lagrangian descriptions can be realized in string theory, and that this set of models can be identified from low-energy considerations. All $N = (1, 0)$ supersymmetric theories in 6D with one gravity and one tensor multiplet which are free of anomalies or other quantum inconsistencies admit a string construction. It would also be interesting to formulate the matter coupled anomaly-free supergravity theories in six dimensions such that the classically gauge invariant and supersymmetric part of the action is identified and the anomaly corrections are determined by means of the anomaly equations. We conduct a systematic search for anomaly-free six-dimensional $N = 1$ chiral supergravity theories. In six dimensions, cancellation of gauge, gravitational, and mixed anomalies strongly constrains the set of quantum field theories which can be coupled consistently to gravity. Anomaly cancellation has turned out to be a crucial guiding principle for the identification of consistent D = 6 theories for the same reason as in the D = 10 case. The D = 6 anomaly cancellation conditions are weaker than those in D = 10, they are still very stringent, especially in the case of gauged supergravity theories. To obtain the fully consistent equations of motion at the quantum level one must also take into account the non-local corrections to the one loop effective action. This raises the question of which equations of motion are to be solved in search of special solutions of the theory. Constructing a consistent quantum theory of gravity has proven to be substantially more difficult than identifying a quantum theory describing the other forces in nature. Even if it is known that consistent superstring theories can be formulated in six dimensions and that six-dimensional supergravity can arise as their low-energy limit, this is not the only reason for investigating D = 6 supergravity. In fact, while supergravities in D = 10 and D = 11 spacetime dimensions are of direct interest as backgrounds for strings, membranes and M-theory, one frequently performs compactifications down to D = 6 to clarify relations among these theories, which are hidden in their ten or eleven-dimensional formulations. Supergravity theories in diverse dimensions play nowadays an important role as low-energy effective field theories of superstring and membrane theories. Supergravity theories have been extensively studied in four dimensions, of course because of their direct physical relevance, and in ten and eleven dimensions of their fundamental features. Explicit knowledge of this set of theories gives us a powerful tool for exploring the connection between string theory and low-energy physics. The anomalies are the key to a deeper research and understanding of gauged supergravity. We hope to have conveyed the idea that anomalies play an important role in supergravity and their cancellation has been and still is a valuable guide for constructing consistent quantum supergravity theories. The treatment of anomalies makes fascinating contacts with several branches of modern theoretical physics.

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