Knot, refractive index, and scalar field

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We construct the geometric optical knot in 3-dimensional Euclidean (flat) space of the Abelian Chern-Simons integral using the variables (the Clebsch variables) of the complex scalar field, i.e. the function of amplitude and the phase, where the phase is related to the refractive index.

Keywords: *geometric optical knot, helicity, Chern-Simons integral, integer, complex scalar field, phase, refractive index, 3-dimensional Euclidean (flat) space.*

It is commonly believed there exists no topological object in the linear theory, such as the Maxwell's theory. It is because of *a topological theory must be a non-linear theory*[1]. The existence of topological object, *a knot*, in the Maxwell's linear theory so far has not been well known[2]. How could a knot exist in the Maxwell's linear theory?

In the Maxwell's theory, the electromagnetic fields (*the set* of the solutions of Maxwell equations) in *vacuum* has *a subset field with a topological structure*[1]. Any electromagnetic field is *locally equal to* a subset field i.e. any electromagnetic field can be obtained by patching together subset fields (except in a zero measure set) but *globally different*[1]. This means that *the difference* between the set of the subset fields and all the electromagnetic fields in the Maxwell's theory in a vacuum is *global* instead of local, since *the subset fields obey the topological quantum condition* that the electromagnetic helicity (consists of electric and magnetic helicities) is equal to *an integer number*[1].

The electromagnetic field satisfies a linear field equation, but a subset field satifies a non-linear field equation. Both fields, the electromagnetic field and a subset field, satisfy the linear field equation in the case of the weak field[3]. It means that *a non-linear subset field theory reduces to the Maxwell's linear theory in the case of the weak field*. The space where the weak field lives approximately represents the vacuum space. *A knot could exist in the vacuum Maxwell's theory because of the vacuum Maxwell's theory is the weak field limit*[3] *of a non-linear subset field theory*.

In this article, *we propose there exists a knot in the geometrical optics, as a solution of the eikonal equation*. The reason is, in fact, there exists a knot in the Maxwell's theory[1–3] and the geometrical optics (the eikonal equation) can be derived from the Maxwell's theory (Maxwell equations)[4–6]. We treat the geometrical optics as an Abelian $U(1)$ *local* gauge theory[7,8], the same as the Abelian $U(1)$ Maxwell's gauge theory. *To the best of our knowledge, the formulation of a knot in the geometrical optics has not been done yet*[1,2,9,10].

Let us consider *a map* of a subset field (consists of *a complex scalar field*) from a finite radius r to an infinite r which implies from the strong field to the weak field. A scalar field has, *by definition*, the property that its value for a finite r depends on the magnitude and the direction of the position vector $\vec{r}$, but for an infinite r it is *well defined*[3] (it depends on the magnitude only). In other words, for an infinite r, a scalar field is *isotropic*. Throughout this article we will work with the classical scalar field.

The property of such a scalar field can be interpreted as *a map* $S^3 \to S^2$[1] where S^3 and S^2 are 3-dimensional and 2-dimensional spheres, respectively. As maps of this kind *can be classified* in *homotopy classes*, labelled by *a topological invariant* called *the Hopf index*[1], *an integer number*. We see there exists (one) dimensional reduction in such map. We consider this dimensional reduction related to the isotropic (well defined) property of a scalar field for an infinite r. The property of a scalar field as a function of space seem likely in harmony with the property of space-time. The space-time could be locally anisotropic, but globally isotropic (the distribution of matter-energy in the universe is assumed to be homogeneous).

In Ranada works[1,3], because of the subset fields have well-defined property at infinity, so the subset fields can be interpreted as maps $S^3 \to S^2$, after identifying, via stereographic projection, $\mathbb{R}^3 \cup \{\infty\}$ with the sphere S^3 and the complete complex plane $C \cup \{\infty\}$ with the sphere S^2. These maps can be classified in *homotopy classes*, labelled by the value of the corresponding Hopf indexes, *the topological invariants*[1,3]. The other names of the topological invariant are *the topological charge, the winding number* (*the degree* of a continuous mapping)[11]. In physics, *the topological charge which is independent to the space metric tensor can be interpreted as energy*[12].

In physics, the idea of a knot, *topologically stable matter*, had been proposed in 1868 by Lord Kelvin that *the atoms could be knots or links of vorticity lines of aether*[2]. *A knot is a smooth-embedding of a circle in* $\mathbb{E}^3$[10], 3-dimensional Euclidean space[13]. Two knots are *equivalent* if one knot can be *deformed continuously* into the other *without crossing itself*[10].

In electrodynamics, *a knot could be formed by bending the electric and magnetic field lines* (the geometric concept of *magnetic lines of force* - those lines of force are today designated by the symbol $\vec{H}$, *the magnetic field* - is due to Faraday[14]) so that they could form *closed* *loops*[2]. *A set of closed loops in space forms a link*[15]. These closed loops can be *linked*[2] (although links do not actually need to be linked[16]). If two closed loops of field lines are *linked* then we have *a non-vanishing Gauss integral (Gauss linking integral). This linking could provide the topological structure*[2]. *The self-linking number (an integer number)* i.e. a non-vanishing Gauss integral describes *the knottedness*[2].

In mathematics, especially in algebraic topology, *a knot is defined by the Hopf index*[2]. The Hopf index is related to *the Hopf invariant*[1]. In turn, the Hopf invariant is related to *a non-trivial Hopf map*[17].

Suppose that we have a scalar field as a function of position vector, $\phi(\vec{r})$, with a property that, as we mentioned, can be interpreted using the non-trivial Hopf map written below[1,3]

$$\phi(\vec{r}) : S^3 \to S^2 \tag{1}$$

This non-trivial Hopf map is related to the Hopf invariant[17], $\mathcal{H}$, expressed as an integral[17–19]

$$\mathcal{H} = \int_{S^3} \omega \wedge d\omega \tag{2}$$

where ω is *a 1-form* on S^3[17].

The relation between the Hopf invariant and the Hopf index, h, can be written as[1]

$$\mathcal{H} = h\ \gamma^2 \tag{3}$$

where γ is *the total strength of the field*, that is the sum of the strengths of all *the tubes* formed by the integral lines of electric and magnetic fields[1].

The Hopf invariant have a deep relationship with *the Abelian Chern-Simons action*[17] *(the Chern-Simons integral)* and *self-helicity* in magnetohydrodynamics[17].

In the case of the 3-dimensional Euclidean (flat) space, $\mathbb{E}^3$, the Abelian Chern-Simons integral could be related to the topological object, i.e. *the geometric optical helicity* or *the geometric optical knot*, h_{go}[20], as follow

$$h_{\text{go}} = \int_{\mathbb{E}^3} \varepsilon^{\alpha\mu\nu}\ \vec{A}_\alpha\ \vec{F}_{\mu\nu}\ d^3x \tag{4}$$

where h_{go} is *integer* (.., -2, -1, 0, 1, 2, ..), $\varepsilon^{\alpha\mu\nu}$ is the Levi-Civita symbol, $\alpha, \mu, \nu = 1, 2, 3$ denote the 3-dimensional space, $\vec{A}_\alpha$ is *the* $U(1)$ *gauge potential*[21], $\vec{F}_{\mu\nu}$ is *the* $U(1)$ *gauge field tensor*[21] (the field strength tensor), $\int_{\mathbb{E}^3} d^3x$ shows that we work in 3-dimensional Euclidean space.

Using the scalar field, ϕ, the field strength can be written as[1]

$$\vec{F}_{\mu\nu} = \vec{f}_{\mu\nu} = \theta\ \frac{\partial_\mu\phi^*\ \partial_\nu\phi - \partial_\nu\phi^*\ \partial_\mu\phi}{(1+\phi^*\phi)^2} \tag{5}$$

where $\theta = 1/(2\pi i)$ and ϕ^* is the complex conjugate of the scalar field. We call eq.(5) as *the non-linear field equation* where *the nonlinearity* is shown by *the* $\phi^*\phi$ *term*.

In the case of the weak field, i.e. $\phi << 1$ so $\phi^*\phi << 1$ then the denominator in eq.(5) *can be taken* as being equal to one and $f_{\mu\nu}(\phi)$ (5) is *equivalent* to the Maxwell linear theory[1]. We interpret the Maxwell's linear theory in a vacuum is the same as the non-linear field theory in the case of weak field due to the field is taken far away from the source (electric charge or current).

Let us assume[22] that the scalar field could be written as

$$\phi = \rho\ e^{iq} \tag{6}$$

and

$$f = -1/[2\pi(1+\rho^2)] \tag{7}$$

ρ is *the amplitude*, q is *the phase*, f is *the function of amplitude*. This assumption is based on the wave point of view of the field. We could interpret that the scalar field, ϕ, as *the disturbance* where the physical disturbance is the real part of ϕ[23].

In the case of the weak field and by using the components of the scalar field, f and q, eq.(5) can be written as[22]

$$\vec{F}_{\mu\nu} = \vec{f}_{\mu\nu} = \partial_\mu(f\ \partial_\nu q) - \partial_\nu(f\ \partial_\mu q) \tag{8}$$

where f and q are known as *the Clebsch variables*[22]. The eq.(8) is equal to

$$\vec{F}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu \tag{9}$$

We call eqs.(8), (9) as *the linear field equations*.

By observing the equality of eq.(8) and (9), we see that[22]

$$\vec{A}_\nu = f\ \partial_\nu q \tag{10}$$

Eq.(10) shows that *the gauge potential (the gauge vector field) can be written using the Clebsch variables of the scalar field*.

By substituting eqs.(8), (10) into eq.(4), we obtain

$$h_{\text{go}} = \int_{\mathbb{E}^3} \varepsilon^{\alpha\mu\nu} f\ \partial_\alpha q\ \{\partial_\mu(f\ \partial_\nu q) - \partial_\nu(f\ \partial_\mu q)\}\ d^3x \tag{11}$$

where the phase can be written as[20,24]

$$q = X(\psi_1 - ct) = X\left(\int_{x_1}^{x_2} n\ d^3x - ct\right) \tag{12}$$

$X = f_\theta/c$, f_θ is the angular frequency, c is the speed of light in vacuum space, ψ_1 is also called *the phase*, t is time and n is *the refractive index*. The refractive index is *the real* scalar function of coordinates with positive values, *the slowness* at a point[7]. The refractive index is typically supplied as *known input, given*, and we seek *the solution, the phase*[7], ψ_1. The integral $\int_{x_1}^{x_2} d^3x$ shows the propagation of ray from the initial position, x_1, to the final position, x_2, in 3-dimensional space.

By substituting eq.(12) into eq.(11), we obtain

$$h_{\rm go} = \int_{\mathbb{E}^3} \varepsilon^{\alpha\mu\nu}\ f\ \partial_\alpha \left[X\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)\right] \left\{\partial_\mu \left\{f\ \partial_\nu \left[X\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)\right]\right\} - \partial_\nu \left\{f\ \partial_\mu \left[X\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)\right]\right\}\right\} d^3x \tag{13}$$

We see from eq.(13) *there exists the relation between the geometric optical knot and the refractive index.* It means that *the knot could exists in the geometrical optics.*

Mathematically, the interesting one is if the complex scalar field is *a smooth single-valued function of its variables.* The smooth single-valued function of the complex scalar field will give rise to the existence of *the singularities of the phase*[23,25–28]. This phase singularity[25] where the phase is undefined[25] or indeterminate[26,29] have been shown to have *a well-defined mathematical structure*[29].

Physically, in our case this well-defined mathematical structure is the geometric optical knot which could be obtained for the weak scalar field. The analysis of the phase singularity is given in a separated article[30].

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