Why All Perfect Number Are Even
Derek Tucker

Abstract: We prove the non-existence of odd perfect numbers, by the non-existence of natural numbers with sum of divisors equal to two.

As usual, ceil denotes the ceiling function, i.e. the least integer greater than a given value; p and n denote an arbitrary prime (2, 3, 5 …such that p is a sum but not product of smaller natural numbers) and natural number (1, 2, 3… $n_{+1}=n+1$) respectively; $\sigma_s(n)$ denotes the sum of the divisors of n raised to the s^{th} power; and $v_p(n)$ denotes the power of p in the prime factorization of n. To investigate the sum of divisors function empirically with the web based graphing calculator, Desmos™, we use,

$$\sigma_s(n) \coloneqq s_{igma}(s,n) = \sum_{k=1}^{n} k^s \left(1 - \text{ceil}\left\lceil \frac{n \bmod k}{k} \right\rceil\right).$$

The definition of a perfect number [1, 2], is a natural number with the sum of its' divisors equal to twice the number itslef. Symbolically, $\sigma_1(n) = 2n$. Our proof turns on well known properies of the sum of divisors function and a lemma. The sum of divisors of a prime number is one more than the prime, because a prime is only divisible by one and itself. The sum of divisors function is multiplicative. When a function is multiplicative, given co-prime co=factors, f and f', the product of co-factors evaluated as independent arguments, equals the function evaluated with the single argument as their product. Symbolically, this is,

$$\gcd(f, f') = 1\,, \qquad \sigma_1(f)\sigma_1(f') = \sigma_1(ff').$$

Lemma: No natural number has sum of divisors equal to two. $\nexists n\colon \sigma_1(n) = 2.$

Proof: Direct.

1. $\sigma_1(1) = 1.$
2. $\sigma_1(p) = p + 1.$
3. $\sigma_1(n) = \prod_{k=1}^{\infty} \sigma_1\left(p_k^{\alpha_k}\right),\ \alpha_k = v_{p_k}(n).$ By the fundamental theorem of arithmetic.
4. Since no primes are less than two, and, $\sigma_1(n)$ for all n greater than 1, is the product of numbers at least three, the lemma must be true by the properties of multiplication. ■.

Theorem: There are no odd perfect numbers.

Proof. By contradiction.

1. Suppose N is an odd perfect number. Then, because the sum of divisors function is multiplicative, we can find a pair of natural numbers that are co-prime factors. f and f' .of N, such that, by the definition of a perfect number,
$$\sigma_1(f)\sigma_1(f') = 2N.$$
2. We know N is odd by the premise, and since the factors are co-prime, we have, $\sigma_1(f) = N$, and, $\sigma(f') = 2$.
3. By the lemma, ,f' does not exist. contradicting the multiplicative nature of the sum of divisors function. Hence, there can be no odd perfect numbers. ■.

References

[1] Euclid, Elements, Alexandria, 300 BC.

[2] N. Sloan, "Online Encyclopedia of Integer Sewuences," in *A000203 a(n) = sigma(n), the sum of the divisors of n. Also called sigma_1(n).*, https://oeis.org/A000203, 2022.