What is the factorization of $(x^n + y^n)$ when *n* is an even positive integer?

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(4 December 2022)

Abstract: The main motivation behind this paper is the question 'What is the factorization of $(x^n + y^n)$ when n is an even positive integer?' which *was* and *is* frequently asked on the Internet by many high school and university students and, to our knowledge, even the specialized textbooks and research articles have not yet answered it, and over time the question itself is not only transformed into a problem that needs to be solved but also became an interesting subject of investigation. In the present paper, the question is positively answered and the problem is solved through the detailed study of the factorizations that leads directly to the concept of algebraic factorization with fractional degree and positive integral order, and an apparently new type of indefinite irrational and rational integrals.

Keywords: factorization, odd positive integer, even positive integer, algebraic expressions, integrals

MSC (2020): 12D05, 34A34, 44A45

1. Introduction

The title of this paper is a question repeatedly asked on the Net by many high school and university students and it seems none got any satisfactory answer to the question which is transformed into a problem that needs attention and needs to be solved.

To our surprise, until now, even the very specialized textbooks [1-10] and research articles have not tackled this problem.

Algebraically speaking, factorization simply means transforming an algebraic expression, usually a polynomial, into product of linear factors. For example, the factorization of the expression $(x^3 - 2x^2 - 5x + 6)$ is $(x-1)(x+2)(x-3)$. Without doubt, the factorization as a mathematical tool has always played an important role in algebra, analysis, number theory, numerical analysis and so on. For instance, how can we evaluate the following indefinite integral

$$\int \frac{dx}{x^3 - 2x^2 - 5x + 6}, \tag{1}$$

without performing factorization? Or how can we solve the first-order non-linear ODE

$$2zz' - \frac{1}{1+x^4} = 0, \tag{2}$$

without doing factorization? Frankly, we cannot since in these cases, the factorization is absolutely inevitable.

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It has long been known that, sometimes, it is quite impossible to factorize polynomial into linear factors employing only rational numbers, but it may be possible to factorize an algebraic expression containing terms with degree *n* into a product containing terms with degree *m* less than *n*. For example, the expression $\left(x^6 - x^4 + 2\sqrt{3}x^3 - \sqrt{3}x + 3\right)$ can, with some difficulty, be factorized as $\left(x^3 + \sqrt{3}\right)\left(x^3 - x + \sqrt{3}\right)$. The expression in each bracket cannot be further factorized using only rational numbers. In this sense, we usually say that these factors are irreducible over the rational numbers. However, since the set of irrational numbers is largest and dense than the set of rational numbers, there are a lot of important expressions that are irreducible over rational number, but which can be factorized if one, of course, allows irrational numbers or real numbers.

Since $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$ (where $\mathbb{N}$ is the set of non-negative integers, *i.e*., "natural numbers", and algebraically $\mathbb{N} = \mathbb{Z}_+$ that is why $\mathbb{N} \subset \mathbb{Z}$) hence over the real numbers every polynomial can be factorized into a product of linear factors and/or quadratic factors, and over the complex numbers every polynomial can be completely factorized into linear factors.

In passing, from the factorization of the expression $(x^n - 1)$ exclusively performed over the integer numbers occurred an interesting type of polynomials called 'cyclotomic polynomials' which were also studied by many mathematicians. For example, $(x^4 - 1) = (x - 1)(x^3 + x^2 + x + 1)$, the polynomial $(x^3 + x^2 + x + 1)$ is an illustrative example of a cyclotomic polynomial. In fact, the questions of factoring and developing methods of factoring, at a more advanced level, were an active part of an extensive work of renowned mathematicians such as Euler, Gauss, Galois, Abel, … etc.

2. Factorization of $(x^n - y^n)$

The factorization of the expression

$$f_n(x, y) = (x^n - y^n), \qquad \forall x, y \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{3}$$

We have the well-known result

$$(x^n - y^n) = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}), \tag{4}$$

or in compact form

$$(x^n - y^n) = (x - y)\sum_{j=0}^{n-1} x^{n-j-1}y^j = (x - y)\sum_{j=1}^{n} x^{n-j}y^{j-1}. \tag{5}$$

This factorization is valid when *n* is an odd or even positive integer, *i.e*., $n = 2r$ or $n = 2r + 1$ and $r \in \mathbb{N}$. Further, it is worthwhile to note that besides the usual factorization (4) or (5), there are four interesting formulae for factorizing the expression (3) whose derivation can be performed as follows. Consider the expression

$$A_n(x, y) = \left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right), \tag{6}$$

which is valid for the following cases:

i) $\forall x, y \in \mathbb{R}_+$ if $n = 2r + 1$; $r \in \mathbb{N}$,

ii) $\forall x, y \in \mathbb{R}$ if $n = 2r$; $r \in \mathbb{N}$.

Now, squaring expression (6) to get

$$A_n^2(x,y) = \left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right)^2 = x^n + 2(xy)^{\frac{n}{2}} + y^n \,. \qquad (7$$

Subtracting $2(xy)^{\frac{n}{2}} + 2y^n$ from both sides of (7) and rearranging to obtain

$$(x^n - y^n) = \left[\left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right)^2 - 2(xy)^{\frac{n}{2}}\right] - 2y^n,$$

$$= \left[\left[\left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right)^2 - 2(xy)^{\frac{n}{2}}\right]^{\frac{1}{2}} - [2y^n]^{\frac{1}{2}}\right]\left[\left[\left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right)^2 - 2(xy)^{\frac{n}{2}}\right]^{\frac{1}{2}} + [2y^n]^{\frac{1}{2}}\right],$$

$$= \left[[x^n + y^n]^{\frac{1}{2}} - [2y^n]^{\frac{1}{2}}\right]\left[[x^n + y^n]^{\frac{1}{2}} + [2y^n]^{\frac{1}{2}}\right].$$

Finally, introducing the radicals to find the first desired formula

$$(x^n - y^n) = \left(\sqrt{x^n + y^n} - \sqrt{2y^n}\right)\left(\sqrt{x^n + y^n} + \sqrt{2y^n}\right), \qquad (8)$$

which is valid for the cases (*i*) and (*ii*).

The second formula can be derived in this way

$$(x^n - y^n) = \left(x^{\frac{n}{2}} - y^{\frac{n}{2}}\right)\left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right),$$

$$\left(x^{\frac{n}{2}} - y^{\frac{n}{2}}\right) = \left(x^{\frac{n}{4}} - y^{\frac{n}{4}}\right)\left(x^{\frac{n}{4}} + y^{\frac{n}{4}}\right),$$

$$\left(x^{\frac{n}{4}} - y^{\frac{n}{4}}\right) = \left(x^{\frac{n}{8}} - y^{\frac{n}{8}}\right)\left(x^{\frac{n}{8}} + y^{\frac{n}{8}}\right),$$

$$\left(x^{\frac{n}{8}} - y^{\frac{n}{8}}\right) = \left(x^{\frac{n}{16}} - y^{\frac{n}{16}}\right)\left(x^{\frac{n}{16}} + y^{\frac{n}{16}}\right),$$

$$\left(x^{\frac{n}{16}} - y^{\frac{n}{16}}\right) = \left(x^{\frac{n}{32}} - y^{\frac{n}{32}}\right)\left(x^{\frac{n}{32}} + y^{\frac{n}{32}}\right),$$

$$\left(x^{\frac{n}{32}} - y^{\frac{n}{32}}\right) = \left(x^{\frac{n}{64}} - y^{\frac{n}{64}}\right)\left(x^{\frac{n}{64}} + y^{\frac{n}{64}}\right),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left(x^{\frac{n}{k}} - y^{\frac{n}{k}}\right) = \left(x^{\frac{n}{2k}} - y^{\frac{n}{2k}}\right)\left(x^{\frac{n}{2k}} + y^{\frac{n}{2k}}\right).$$

After performing successive substitutions and rearrangement, we get the second formula

$$f_n^{(m)}(x,y) \equiv (x^n - y^n) = \left(x^{\frac{n}{m}} - y^{\frac{n}{m}}\right)\left(x^{\frac{n}{m}} + y^{\frac{n}{m}}\right)\left(x^{\frac{2n}{m}} + y^{\frac{2n}{m}}\right)\left(x^{\frac{4n}{m}} + y^{\frac{4n}{m}}\right)\cdots\left(x^{\frac{kn}{m}} + y^{\frac{kn}{m}}\right), \quad (9)$$

or in compact form

$$f_n^{(m)}(x,y) \equiv (x^n - y^n) = \left(x^{\frac{n}{m}} - y^{\frac{n}{m}}\right)\prod_{k=1}^{\left[\frac{m}{2}\right]}\left(x^{\frac{kn}{m}} + y^{\frac{kn}{m}}\right)\cdot \qquad (10)$$

The third formula can be derived as follows. We have from (10) the factorization with degree $n = 1$ and an arbitrary order m:

$$(x-y) = \left(x^{\frac{1}{m}} - y^{\frac{1}{m}}\right) \prod_{k=1}^{\left[\frac{m}{2}\right]} \left(x^{\frac{k}{m}} + y^{\frac{k}{m}}\right).$$

Furthermore, we have according to (5)

$$(x^n - y^n) = (x-y) \sum_{j=1}^{n} x^{n-j} y^{j-1}.$$

Therefore, after substitution, we get the expected formula

$$g_n^{(m)}(x,y) \equiv (x^n - y^n) = \left[\left(x^{\frac{1}{m}} - y^{\frac{1}{m}}\right) \prod_{k=1}^{\left[\frac{m}{2}\right]} \left(x^{\frac{k}{m}} + y^{\frac{k}{m}}\right)\right] \sum_{j=1}^{n} x^{n-j} y^{j-1}. \tag{11}$$

Formulae (10) and (11) are exclusively valid for the case when $\forall x, y \in \mathbb{R}_+;\ \ n, m \in \mathbb{N}: m = 2, 4, 8, 16, 32 \cdots$

Finally, the fourth formula can be deduced from (8) in this way:

$$(x^n - y^n) = \left[(x^n + y^n)^{\frac{1}{2}} - (2y^n)^{\frac{1}{2}}\right]\left[(x^n + y^n)^{\frac{1}{2}} + (2y^n)^{\frac{1}{2}}\right]$$

$$= \left[(x^n + y^n)^{\frac{1}{4}} - (2y^n)^{\frac{1}{4}}\right]\left[(x^n + y^n)^{\frac{1}{4}} + (2y^n)^{\frac{1}{4}}\right]\left[(x^n + y^n)^{\frac{1}{2}} + (2y^n)^{\frac{1}{2}}\right]$$

$$= \left[(x^n + y^n)^{\frac{1}{8}} - (2y^n)^{\frac{1}{8}}\right]\left[(x^n + y^n)^{\frac{1}{8}} + (2y^n)^{\frac{1}{8}}\right]\left[(x^n + y^n)^{\frac{1}{4}} + (2y^n)^{\frac{1}{4}}\right]\left[(x^n + y^n)^{\frac{1}{2}} + (2y^n)^{\frac{1}{2}}\right]$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$= \left[(x^n + y^n)^{\frac{1}{m}} - (2y^n)^{\frac{1}{m}}\right]\left[(x^n + y^n)^{\frac{1}{m}} + (2y^n)^{\frac{1}{m}}\right] \cdots \left[(x^n + y^n)^{\frac{1}{2}} + (2y^n)^{\frac{1}{2}}\right].$$

From where we obtain the desired formula in compact form

$$h_n^{(m)}(x,y) \equiv (x^n - y^n) = \left[(x^n + y^n)^{\frac{1}{m}} - (2y^n)^{\frac{1}{m}}\right] \prod_{k=1}^{\left[\frac{m}{2}\right]} \left[(x^n + y^n)^{\frac{k}{m}} + (2y^n)^{\frac{k}{m}}\right], \tag{12}$$

which is exclusively valid for the cases:

1) $\forall x, y \in \mathbb{R}_+$ if $n = 2r + 1;\ r \in \mathbb{N};\ m \in \mathbb{N}\backslash\{0\}: m = 2, 4, 8, 16, 32$ ...
2) $\forall x, y \in \mathbb{R}$ if $n = 2r;\ r \in \mathbb{N};\ m \in \mathbb{N}\backslash\{0\}: m = 2, 4, 8, 16, 32$ ...

Remark, n and m in (8), (9), (10), (11) and (12) are, respectively, called the '*degree*' and '*order*' of factorization. Furthermore, the formulae (10) and (12) are also valid when the degree is a fraction of the form p/q. Hence, $\forall x, y \in \mathbb{R}_+;\ p \in \mathbb{N},\ q, m \in \mathbb{N}\backslash\{0\}: m = 2, 4, 8, 16, 32$ ... we have

$$f_{p/q}^{(m)}(x,y) \equiv \left(x^{\frac{p}{q}} - y^{\frac{p}{q}}\right) = \left(x^{\frac{p}{qm}} - y^{\frac{p}{qm}}\right) \prod_{k=1}^{\left[\frac{m}{2}\right]} \left(x^{\frac{kp}{qm}} + y^{\frac{kp}{qm}}\right), \tag{13}$$

$$h_{p/q}^{(m)}(x,y) \equiv \left(x^{\frac{p}{q}} - y^{\frac{p}{q}}\right) = \left[\left(x^{\frac{p}{q}} + y^{\frac{p}{q}}\right)^{\frac{1}{m}} - \left(2y^{\frac{p}{q}}\right)^{\frac{1}{m}}\right] \prod_{k=1}^{\left[\frac{m}{2}\right]} \left[\left(x^{\frac{p}{q}} + y^{\frac{p}{q}}\right)^{\frac{k}{m}} + \left(2y^{\frac{p}{q}}\right)^{\frac{k}{m}}\right]. \tag{14}$$

To our knowledge, the formulae (8), (10), (11) and (12) seem to have not been considered by the textbooks' authors.

From the formulae (9) or (10) we list *the first five factorizations*

$$f_1^{(2)}(x,y) = \left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right),$$

$$f_1^{(4)}(x,y) = \left(x^{\frac{1}{4}} - y^{\frac{1}{4}}\right)\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right),$$

$$f_1^{(8)}(x,y) = \left(x^{\frac{1}{8}} - y^{\frac{1}{8}}\right)\left(x^{\frac{1}{8}} + y^{\frac{1}{8}}\right)\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right),$$

$$f_1^{(16)}(x,y) = \left(x^{\frac{1}{16}} - y^{\frac{1}{16}}\right)\left(x^{\frac{1}{16}} + y^{\frac{1}{16}}\right)\left(x^{\frac{1}{8}} + y^{\frac{1}{8}}\right)\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right),$$

$$f_1^{(32)}(x,y) = \left(x^{\frac{1}{32}} - y^{\frac{1}{32}}\right)\left(x^{\frac{1}{32}} + y^{\frac{1}{32}}\right)\left(x^{\frac{1}{16}} + y^{\frac{1}{16}}\right)\left(x^{\frac{1}{8}} + y^{\frac{1}{8}}\right)\left(x^{\frac{1}{4}} + y^{\frac{1}{4}}\right)\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right).$$

Example 2.1.

Let us show the utility of the factorization (9) or (10) by evaluating the following definite integral

$$I_{n,m} = \int_0^1 [(1-t)t]^{\frac{n}{m}} \prod_{k=1}^{\left[\frac{m}{2}\right]} (1-t)^{\frac{kn}{m}} dt, \tag{15}$$

where $t \in \mathbb{R}_+;\ n, m \in \mathbb{N}\backslash\{0\}\colon m = 2, 4, 8, 16, 32 \cdots$

The integration can be performed with the help of the formula (10). Putting $x = 1 - t$ and $y = 0$ in (10) then multiplying both sides by $t^{\frac{n}{m}}$, this gives

$$t^{\frac{n}{m}}(1-t)^n = [(1-t)t]^{\frac{n}{m}} \prod_{k=1}^{\left[\frac{m}{2}\right]} (1-t)^{\frac{kn}{m}}.$$

Integrating both sides from 0 to 1 yields

$$\int_0^1 t^{\frac{n}{m}}(1-t)^n dt = \int_0^1 [(1-t)t]^{\frac{n}{m}} \prod_{k=1}^{\left[\frac{m}{2}\right]} (1-t)^{\frac{kn}{m}}\, dt. \tag{16}$$

Let us focus our attention on the LHS of (16) which has in fact the form of the well-known beta function

$$B\left(\frac{n}{m}+1, n+1\right) = \frac{\Gamma\left(\frac{n}{m}+1\right)\Gamma(n+1)}{\Gamma\left(\frac{n}{m}+n+2\right)} = \int_0^1 t^{\frac{n}{m}}(1-t)^n dt. \tag{17}$$

Therefore, the required evaluation of integral (15) is

$$I_{n,m} = \frac{\left(\frac{n}{m}\right) n!}{\left(\frac{n}{m}+n+1\right)\left(\frac{n}{m}+n\right)} \frac{\Gamma\left(\frac{n}{m}\right)}{\Gamma\left(\frac{n}{m}+n\right)}\,. \tag{18}$$

Theorem 2.2. Let $x \in \mathbb{R}_+$. If $n, m \in \mathbb{N}\backslash\{0\}$: $m = 2, 4, 8, 16, 32 \cdots$, then

$$\lim_{n\to m}\left[\frac{\sum_{j=1}^{n} x^{n-j}}{\prod_{k=1}^{\left[\frac{m}{2}\right]}\left(1+x^{k\frac{n}{m}}\right)}\right] = 1. \tag{19}$$

Proof: We have according to (5) and (10), respectively

$$(x^n - y^n) = (x - y)\sum_{j=1}^{n} x^{n-j}y^{j-1},$$

$$(x^n - y^n) = \left(x^{\frac{n}{m}} - y^{\frac{n}{m}}\right)\prod_{k=1}^{\left[\frac{m}{2}\right]}\left(x^{\frac{kn}{m}} + y^{\frac{kn}{m}}\right), \quad x \neq y$$

By combining the above expressions, we obtain

$$\frac{\sum_{j=1}^{n} x^{n-j}y^{j-1}}{\prod_{k=1}^{\left[\frac{m}{2}\right]}\left(x^{\frac{kn}{m}}+y^{\frac{kn}{m}}\right)} = \frac{x^{\frac{n}{m}}-y^{\frac{n}{m}}}{x-y}\cdot \tag{20}$$

Putting $y = 1$ in (20), rearranging and introducing the limit on both sides to get (19):

$$\lim_{n\to m}\left[\frac{\sum_{j=1}^{n} x^{n-j}}{\prod_{k=1}^{\left[\frac{m}{2}\right]}\left(1+x^{\frac{kn}{m}}\right)}\right] = \lim_{n\to m}\left[\frac{1-x^{\frac{n}{m}}}{1-x}\right] = 1. \qquad \text{QED.}$$

Theorem 2.3. Let $m, k \in \mathbb{N}\backslash\{0\}$. If $m > k$ such that $m = 2, 4, 8, 16, 32 \cdots$, then

$$\frac{\arcsin\left[m-\sum_{k=1}^{\left[\frac{m}{2}\right]} k\right]}{\arccos\left[\sum_{k=1}^{\left[\frac{m}{2}\right]} k-m\right]} = \frac{1}{2}\cdot \tag{21}$$

Proof: To prove this theorem we must use the formula (10). Thus, with this aim, putting $y = 0$ in (10), to obtain

$$x^n = \left(x^{\frac{n}{m}}\right)\prod_{k=1}^{\left[\frac{m}{2}\right]}\left(x^{\frac{kn}{m}}\right)\cdot \tag{22}$$

Now, let us factorize '*e*', that is the case when *x* = *e*, for degree *n* =1 and an arbitrary order *m* where $m = 2, 4, 8, 16, 32 \cdots$ Thus, after substitution in (22), we get

$$e = \left(e^{\frac{1}{m}}\right)\prod_{k=1}^{\left[\frac{m}{2}\right]}\left(e^{\frac{k}{m}}\right) = \left(e^{\frac{1}{m}}\right)\left(e^{\frac{1}{m}}\right)\left(e^{\frac{2}{m}}\right)\left(e^{\frac{4}{m}}\right)\left(e^{\frac{8}{m}}\right)\left(e^{\frac{16}{m}}\right)\left(e^{\frac{32}{m}}\right)\left(e^{\frac{64}{m}}\right)\cdots\ \left(e^{\frac{1}{2}}\right)$$

$$= e^{\frac{1}{m}\left(1+1+2+4+8+16+32+64\cdots\ \frac{m}{2}\right)}. \tag{23}$$

Introducing the logarithm on both sides of (23) and rearranging the terms to obtain

$$m - \left(1 + 2 + 4 + 8 + 16 + 32 + 64\ \cdots\ \frac{m}{2}\right) = 1. \tag{24}$$

Remark $$\left(1+2+4+8+16+32+64\ \cdots\ \frac{m}{2}\right)=\sum_{k=1}^{\left[\frac{m}{2}\right]}k, \tag{25}$$

therefore, by combining (24) and (25), we find

$$m-\sum_{k=1}^{\left[\frac{m}{2}\right]}k\ =1=\sin\left(\frac{\pi}{2}\right), \tag{26}$$

from where we get

$$\arcsin\left[m-\sum_{k=1}^{\left[\frac{m}{2}\right]}k\right]=\frac{\pi}{2}\cdot \tag{27}$$

Further, we have from (24) and (25)

$$\sum_{k=1}^{\left[\frac{m}{2}\right]}k-m\ =-1=\cos(\pi), \tag{28}$$

from where we obtain

$$\arccos\left[\sum_{k=1}^{\left[\frac{m}{2}\right]}k-m\right]=\pi. \tag{29}$$

Multiplying both sides of (29) by $\frac{1}{2}$ to get

$$\frac{1}{2}\arccos\left[\sum_{k=1}^{\left[\frac{m}{2}\right]}k-m\right]=\frac{\pi}{2}\cdot \tag{30}$$

Finally, combining (27) and (30) to find (21). QED.

3. Factorization of (x^n+y^n)

The well-known and commonly used factorization of the expression

$$\chi_n(x,y)=(x^n+y^n)\ ,\qquad \forall x,y\in\mathbb{R},\ \ n\in\mathbb{N}\ , \tag{31}$$

is in general of the form

$$(x^n+y^n)=(x+y)(x^{n-1}-x^{n-2}y+x^{n-3}y^2-\ \cdots\ +y^{n-1}), \tag{32}$$

or in compact form

$$(x^n+y^n)=(x+y)\sum_{j=0}^{n-1}(-1)^j x^{n-j-1}y^j=(x+y)\sum_{j=1}^{n}(-1)^{j+1}x^{n-j}y^{j-1}. \tag{33}$$

The factorization (32) or (33) is valid if and only if *n* is an odd positive integer, and for this reason many students asked the central question, the title of this paper.

4. What is the factorization of (x^n+y^n) when *n* is an even positive integer?

At present, we arrive at the main subject of this paper and our aim is not simply answering but also investigating, exploring and exploiting the result.

As we have already seen, the factorization (32) or (33) of the expression (31) is not valid when *n* is an even (exponent) positive integer. However, there are other formulae for factorization, two of which are, respectively, valid if and only if n is an even positive integer non-multiple of 4 and 8, the others are valid when n is odd or even. The two first formulae, which are actually partial answers to the principal question, can be derived as follows.

4.1. Derivation of a formula valid for even-exponent positive integer non-multiple of 4

We have in accordance with expression (32)

$$(t^m+\tau^m)=(t+\tau)(t^{m-1}-t^{m-2}\tau+t^{m-3}\tau^2-t^{m-4}\tau^3+\cdots+\tau^{m-1}), \tag{34}$$

where $t,\tau\in\mathbb{R}$; $m,r\in\mathbb{N}$: $m=2r+1$.

On putting $t=x^2$, $\tau=y^2$ in (34), we obtain

$$(x^{2m}+y^{2m})=(x^2+y^2)\left(x^{2m-2}-x^{2m-4}y^2+x^{2m-6}y^4-x^{2m-8}y^6+x^{2m-10}y^8-\cdots+y^{2m-2}\right),$$

replacing $2m$ by n to get the desired formula

$$(x^n+y^n)=(x^2+y^2)(x^{n-2}-x^{n-4}y^2+x^{n-6}y^4-x^{n-8}y^6+x^{n-10}y^8-x^{n-12}y^{10}+\cdots$$
$$+y^{n-2}). \tag{35}$$

Or in compact form

$$(x^n+y^n)=(x^2+y^2)\sum_{j=1}^{\left[\frac{n}{2}\right]}(-1)^{j+1}x^{n-2j}y^{2j-2}. \tag{36}$$

Formulae (35) and (36) are exclusively valid for every $x,y\in\mathbb{R}$ and every n even positive integer non-multiple of 4, *i.e.*, $n=2r$ with $r=1,3,5,7,9,11,13,\ldots$

The first five factorizations

$(x^2+y^2)=(x^2+y^2)$,

$(x^6+y^6)=(x^2+y^2)(x^4-x^2y^2+y^4)$,

$(x^{10}+y^{10})=(x^2+y^2)(x^8-x^6y^2+x^4y^4-x^2y^6+y^8)$,

$(x^{14}+y^{14})=(x^2+y^2)(x^{12}-x^{10}y^2+x^8y^4-x^6y^6+x^4y^8-x^2y^{10}+y^{12})$,

$(x^{18}+y^{18})=(x^2+y^2)(x^{16}-x^{14}y^2+x^{12}y^4-x^{10}y^6+x^8y^8-x^6y^{10}+x^4y^{12}-x^2y^{14}+y^{16})$.

4.2. Derivation of a formula valid for even-exponent positive integer non-multiple of 8

This formula can be derived by using the same previous procedure, that is, putting $t=x^4$, $\tau=y^4$ in (34), we get

$$(x^{4m}+y^{4m})=(x^4+y^4)(x^{4m-4}-x^{4m-8}y^4+x^{4m-12}y^8-x^{4m-16}y^{12}+x^{4m-20}y^{16}-\cdots+\ y^{4m-4}).$$

Finally, substituting $4m$ for n, to find the requested formula

$$(x^n+y^n)=(x^4+y^4)(x^{n-4}-x^{n-8}y^4+x^{n-12}y^8-x^{n-16}y^{12}+x^{n-20}y^{16}-\cdots+y^{n-4}). \quad (37)$$

Or equivalently

$$(x^n+y^n)=\left(x^2-\sqrt{2}xy+y^2\right)\left(x^2+\sqrt{2}xy+y^2\right)\sum_{j=1}^{\left[\frac{n}{4}\right]}(-1)^{j+1}x^{n-4j}y^{4j-4}. \quad (38)$$

Formulae (37) and (38) are exclusively valid for every $x, y \in \mathbb{R}$ and every *n* even positive integer non-multiple of 8, *i.e.*, $n = 4r$ with $r = 1, 3, 5, 7, 9, 11, 13, \cdots$

To our knowledge, the formulae (35), (36), (37) and (38) have never been mentioned in the textbooks otherwise the central question itself should not be frequently asked by students at all.

The first five factorizations

$$(x^4+y^4)=(x^4+y^4),$$

$$(x^{12}+y^{12})=(x^4+y^4)(x^8-x^4y^4+y^8),$$

$$(x^{20}+y^{20})=(x^4+y^4)(x^{16}-x^{12}y^4+x^8y^8-x^4y^{12}+y^{16}),$$

$$(x^{28}+y^{28})=(x^4+y^4)(x^{24}-x^{20}y^4+x^{16}y^8-x^{12}y^{12}+x^8y^{16}-x^4y^{20}+y^{24}),$$

$$(x^{36}+y^{36})=(x^4+y^4)(x^{32}-x^{28}y^4+x^{24}y^8-x^{20}y^{12}+x^{16}y^{16}-x^{12}y^{20}+x^8y^{24}-x^4y^{28}+y^{32}).$$

4.3. Derivation of some general formulae valid for odd or even-exponent positive integer

After we have derived the formulae (35), (36), (37) and (38) as partial answers, now we will give a complete answer to the principal question by means of general formulae valid for odd or even-exponent positive integer, which can be derived as follows. Consider the expression

$$B_n(x,y)=\left(x^{\frac{n}{2}}+y^{\frac{n}{2}}\right). \quad (39)$$

Expression (39) is valid for the cases (*i*) and (*ii*).

Now, squaring expression (39) to get

$$B_n^2(x,y)=\left(x^{\frac{n}{2}}+y^{\frac{n}{2}}\right)^2=x^n+2(xy)^{\frac{n}{2}}+y^n, \quad (40)$$

from where we obtain

$$(x^n + y^n) = \left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right)^2 - 2(xy)^{\frac{n}{2}} = \left[\left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right) - \sqrt{2(xy)^{\frac{n}{2}}}\right]\left[\left(x^{\frac{n}{2}} + y^{\frac{n}{2}}\right) + \sqrt{2(xy)^{\frac{n}{2}}}\right].$$

Finally, rearranging and introducing the radicals to get the desired formula

$$K_n(x,y) \equiv (x^n + y^n) = \left(\sqrt{x^n} - \sqrt{2\sqrt{(xy)^n}} + \sqrt{y^n}\right)\left(\sqrt{x^n} + \sqrt{2\sqrt{(xy)^n}} + \sqrt{y^n}\right), \qquad (41)$$

which is valid for the cases (i) and (ii), respectively.

The first ten factorizations

$$K_1(x,y) = \left(\sqrt{x} - \sqrt{2\sqrt{xy}} + \sqrt{y}\right)\left(\sqrt{x} + \sqrt{2\sqrt{xy}} + \sqrt{y}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$K_2(x,y) = \left(x - \sqrt{2xy} + y\right)\left(x + \sqrt{2xy} + y\right); \qquad \forall x,y \in \mathbb{R}_+ \text{ or } \forall x,y \in \mathbb{R}_-$$

$$K_3(x,y) = \left(\sqrt{x^3} - \sqrt{2\sqrt{(xy)^3}} + \sqrt{y^3}\right)\left(\sqrt{x^3} + \sqrt{2\sqrt{(xy)^3}} + \sqrt{y^3}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$K_4(x,y) = \left(x^2 - \sqrt{2}xy + y^2\right)\left(x^2 + \sqrt{2}xy + y^2\right); \qquad \forall x,y \in \mathbb{R}$$

$$K_5(x,y) = \left(\sqrt{x^5} - \sqrt{2\sqrt{(xy)^5}} + \sqrt{y^5}\right)\left(\sqrt{x^5} + \sqrt{2\sqrt{(xy)^5}} + \sqrt{y^5}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$K_6(x,y) = \left(x^3 - \sqrt{2(xy)^3} + y^3\right)\left(x^3 + \sqrt{2(xy)^3} + y^3\right); \qquad \forall x,y \in \mathbb{R}_+ \text{ or } \forall x,y \in \mathbb{R}_-$$

$$K_7(x,y) = \left(\sqrt{x^7} - \sqrt{2\sqrt{(xy)^7}} + \sqrt{y^7}\right)\left(\sqrt{x^7} + \sqrt{2\sqrt{(xy)^7}} + \sqrt{y^7}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$K_8(x,y) = \left(x^4 - \sqrt{2}(xy)^2 + y^4\right)\left(x^4 + \sqrt{2}(xy)^2 + y^4\right); \qquad \forall x,y \in \mathbb{R}_+ \text{ or } \forall x,y \in \mathbb{R}_-$$

$$K_9(x,y) = \left(\sqrt{x^9} - \sqrt{2\sqrt{(xy)^9}} + \sqrt{y^9}\right)\left(\sqrt{x^9} + \sqrt{2\sqrt{(xy)^9}} + \sqrt{y^9}\right); \qquad \forall x,y \in \mathbb{R}_+$$

$$K_{10}(x,y) = \left(x^5 - \sqrt{2(xy)^5} + y^5\right)\left(x^5 + \sqrt{2(xy)^5} + y^5\right). \qquad \forall x,y \in \mathbb{R}_+ \text{ or } \forall x,y \in \mathbb{R}_-$$

Actually, the factorization (41) is also valid for the following important case, that is, when the exponent itself is of the form n/m. Hence, $\forall x,y \in \mathbb{R}_+$; $n,m \in \mathbb{N}: m \neq 0$, we have

$$\left(x^{\frac{n}{m}} + y^{\frac{n}{m}}\right) = \left(\sqrt{x^{n/m}} - \sqrt{2\sqrt{(xy)^{n/m}}} + \sqrt{y^{n/m}}\right)\left(\sqrt{x^{n/m}} + \sqrt{2\sqrt{(xy)^{n/m}}} + \sqrt{y^{n/m}}\right). \quad (42)$$

In addition to the formulae (41) and (42), there are three other ones which can be deduced by using the same previous procedures. To this end, we have

$$-x^n + y^n = -(x^n - y^n),$$

then adding $2x^n$ to both sides to get

$$x^n + y^n = 2x^n - (x^n - y^n),$$

introducing the radicals to obtain the desired formula

$$(x^n + y^n) = \left[\sqrt{2x^n} - \sqrt{(x^n - y^n)}\right]\left[\sqrt{2x^n} + \sqrt{(x^n - y^n)}\right]. \quad (43)$$

The formula (43) is valid for the following cases:

$a)$ $\forall x, y \in \mathbb{R}_+ : x \geq y\,;\ \ n \in \mathbb{N}$,

$b)$ $\forall x, y \in \mathbb{R} : |x| \geq |y|\,;\ \ n, r \in \mathbb{N} : n = 2r$.

From (43) we can deduce this one

$$Q_n^{(m)}(x,y) \equiv (x^n + y^n) = \left[(2x^n)^{\frac{1}{m}} - (x^n - y^n)^{\frac{1}{m}}\right] \prod_{k=1}^{\left[\frac{m}{2}\right]} \left[(2x^n)^{\frac{k}{m}} + (x^n - y^n)^{\frac{k}{m}}\right], \quad (44)$$

where $m \in \mathbb{N}\backslash\{0\}$: $m = 2, 4, 8, 16, 32 \ldots$

Finally, the third formula is of the form

$$R_n^{(m)}(x,y) \equiv (x^n + y^n) =$$

$$= \left[\left(\sqrt{x^n} + \sqrt{y^n}\right)^{\frac{1}{m}} - \left(2\sqrt{(xy)^n}\right)^{\frac{1}{2m}}\right] \prod_{k=1}^{m} \left[\left(\sqrt{x^n} + \sqrt{y^n}\right)^{\frac{k}{m}} + \left(2\sqrt{(xy)^n}\right)^{\frac{k}{2m}}\right], \quad (45)$$

which is exclusively valid for the following cases:

I) $\forall x, y \in \mathbb{R}_+$ if $n = 2r + 1$; $r \in \mathbb{N}$; $m \in \mathbb{N}\backslash\{0\}$: $m = 1, 2, 4, 8, 16, 32 \ldots$

II) $\forall x, y \in \mathbb{R}$ if $n = 2r$; $r \in \mathbb{N}$; $m \in \mathbb{N}\backslash\{0\}$: $m = 1, 2, 4, 8, 16, 32 \ldots$

Theorem 4.4. Let $\theta \in \left[0,\ \frac{\pi}{2}\right[$. If $k, m \in \mathbb{N} : m = 2, 4, 8, 16, 32\ \cdots$ then

$$\frac{1}{\left[\left(2\cos^2(\theta)\right)^{\frac{1}{m}} - \left(\cos(2\theta)\right)^{\frac{1}{m}}\right]} = \prod_{k=1}^{\left[\frac{m}{2}\right]} \left[\left(2\cos^2(\theta)\right)^{\frac{k}{m}} + \left(\cos(2\theta)\right)^{\frac{k}{m}}\right]. \quad (46)$$

Proof: To prove this theorem we must use the formula (44) and the trigonometric identity $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$. Thus, putting $x = \cos(\theta)$, $y = \sin(\theta)$ in (44), to obtain

$$\left(\cos^{n}(\theta)+\sin^{n}(\theta)\right)=\left[\left(2\cos^{n}(\theta)\right)^{\frac{1}{m}}-\left(\cos^{n}(\theta)-\sin^{n}(\theta)\right)^{\frac{1}{m}}\right]\prod_{k=1}^{\left[\frac{m}{2}\right]}\left[\left(2\cos^{n}(\theta)\right)^{\frac{k}{m}}+\left(\cos^{n}(\theta)-\sin^{n}(\theta)\right)^{\frac{k}{m}}\right].$$

Now, let us consider the interesting special case when $n=2$, then the above expression becomes after substitution and rearrangement:

$$\frac{1}{\left[\left(2\cos^{2}(\theta)\right)^{\frac{1}{m}}-\left(\cos(2\theta)\right)^{\frac{1}{m}}\right]}=\prod_{k=1}^{\left[\frac{m}{2}\right]}\left[\left(2\cos^{2}(\theta)\right)^{\frac{k}{m}}+\left(\cos(2\theta)\right)^{\frac{k}{m}}\right]. \qquad \text{QED.}$$

5. Apparently new type of indefinite irrational integrals

As a direct consequence of the formulae (41) and (42), we have the following (apparently) new indefinite rational integrals:

I. $I_n=\int\frac{dx}{\sqrt{x^n}-\sqrt{2\sqrt{x^n}+1}}$

II. $I_n=\int\frac{dx}{\sqrt{x^n}+\sqrt{2\sqrt{x^n}+1}}$

III. $I_n=\int\frac{x^n}{\sqrt{x^n}-\sqrt{2\sqrt{x^n}+1}}\,dx$

IV. $I_n=\int\frac{x^n}{\sqrt{x^n}+\sqrt{2\sqrt{x^n}+1}}\,dx$

V. $I_n=\int\frac{x^n+1}{\sqrt{x^n}-\sqrt{2\sqrt{x^n}+1}}\,dx$

VI. $I_n=\int\frac{x^n+1}{\sqrt{x^n}+\sqrt{2\sqrt{x^n}+1}}\,dx$

VII. $I_n=\int\frac{\sqrt{x^n}+\sqrt{2\sqrt{x^n}+1}}{\sqrt{x^n}-\sqrt{2\sqrt{x^n}+1}}\,dx$

VIII. $I_n=\int\frac{\sqrt{x^n}-\sqrt{2\sqrt{x^n}+1}}{\sqrt{x^n}+\sqrt{2\sqrt{x^n}+1}}\,dx$

IX. $I_{n,m}=\int\frac{dx}{\sqrt{x^{n/m}}-\sqrt{2\sqrt{x^{n/m}}+1}}$

X. $I_{n,m}=\int\frac{dx}{\sqrt{x^{n/m}}+\sqrt{2\sqrt{x^{n/m}}+1}}$

XI. $I_{n,m}=\int\frac{x^n}{\sqrt{x^{n/m}}-\sqrt{2\sqrt{x^{n/m}}+1}}\,dx$

XII. $I_{n,m} = \int \frac{x^n}{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}} + 1}}\, dx$

XIII. $I_{n,m} = \int \frac{x^n + 1}{\sqrt{x^{n/m}} - \sqrt{2\sqrt{x^{n/m}} + 1}}\, dx$

XIV. $I_{n,m} = \int \frac{x^n + 1}{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}} + 1}}\, dx$

XV. $I_{n,m} = \int \frac{\sqrt{x^{n/m}} - \sqrt{2\sqrt{x^{n/m}} + 1}}{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}} + 1}} dx$

XVI. $I_{n,m} = \int \frac{\sqrt{x^{n/m}} + \sqrt{2\sqrt{x^{n/m}} + 1}}{\sqrt{x^{n/m}} - \sqrt{2\sqrt{x^{n/m}} + 1}} dx$

6. Mixed-factorization

Definition 6.1. Mixed-factorization is a factorization in which the exponent m is, at the same time, order and co-degree.

In what follows, we omit the details of the derivations and give only the result that is to say the two formulae for the mixed-factorization when $\forall x, y \in \mathbb{R}_+$; $n, m \in \mathbb{N}: m \neq 0$, then the first one is

$$\left(x^{\frac{n}{m}} + y^{\frac{n}{m}}\right)^m - (x^n + y^n) = \left(\sqrt{\left(x^{\frac{n}{m}} + x^{\frac{n}{m}}\right)^m} - \sqrt{x^n + y^n}\right)\left(\sqrt{\left(x^{\frac{n}{m}} + x^{\frac{n}{m}}\right)^m} + \sqrt{x^n + y^n}\right)$$

$$= \sum_{j=1}^{m-1} C_m^j \, x^{n-j\frac{n}{m}} y^{j\frac{n}{m}}, \tag{47}$$

and the second one is

$$\left(x^{\frac{n}{m}} - y^{\frac{n}{m}}\right)^m - (x^n + (-1)^m y^n) =$$

$$= \left(\sqrt{\left(x^{\frac{n}{m}} - y^{\frac{n}{m}}\right)^m} - \sqrt{x^n + (-1)^m y^n}\right)\left(\sqrt{\left(x^{\frac{n}{m}} - x^{\frac{n}{m}}\right)^m} + \sqrt{x^n + (-1)^m y^n}\right)$$

$$= \sum_{j=1}^{m-1} (-1)^j C_m^j \, x^{n-j\frac{n}{m}} y^{j\frac{n}{m}}, \qquad C_m^j = \frac{m!}{j!(m-j)!} \cdot \tag{48}$$

6.2. Apparently new type of indefinite rational integrals

In terms of utility, the formulae (47) and (48) can play an important role. For instance, with the help of formula (47), the following (apparently) new indefinite rational integrals can be evaluated

i. $I_{n,m} = \int \left(1 + x^{\frac{n}{m}}\right)^m dx,$ $x \in \mathbb{R}_+;\quad n,m \in \mathbb{N}: m \neq 0$

ii. $I_{n,m} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m}{1+x} dx,$

iii. $I_{n,m} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m}{1-x} dx,$ $x \in \mathbb{R}_+ \backslash \{1\}$

iv. $I_{n,m}^{(\alpha)} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m}{1+x^{\alpha}} dx,$ $\alpha \in \mathbb{R}$

v. $I_{n,m}^{(\alpha)} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m}{1-x^{\alpha}} dx,$ $x \in \mathbb{R}_+ \backslash \{1\}$

vi. $I_{n,m} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m - (1+x^n)}{1+x} dx,$

vii. $I_{n,m} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m - (1+x^n)}{1-x} dx,$ $x \in \mathbb{R}_+ \backslash \{1\}$

viii. $I_{n,m}^{(\alpha)} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m - (1+x^n)}{1+x^{\alpha}} dx,$

ix. $I_{n,m}^{(\alpha)} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m - (1+x^n)}{1-x^{\alpha}} dx,$ $x \in \mathbb{R}_+ \backslash \{1\}$

x. $I_{n,m} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m + (1+x^n)}{1+x} dx,$

xi. $I_{n,m} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m + (1+x^n)}{1-x} dx,$ $x \in \mathbb{R}_+ \backslash \{1\}$

xii. $I_{n,m}^{(\alpha)} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m + (1+x^n)}{1+x^{\alpha}} dx,$

xiii. $I_{n,m}^{(\alpha)} = \int \frac{\left(1+x^{\frac{n}{m}}\right)^m + (1+x^n)}{1-x^{\alpha}} dx,$ $x \in \mathbb{R}_+ \backslash \{1\}.$

Theorem 6.3. Let $x, y \in \mathbb{R}_+$ and $n, j \in \mathbb{N}$. If $x \neq 0$ and $n \geq j$ then

$$\left(1 - \sqrt{2\sqrt{\left(\frac{y}{x}\right)^n} + \sqrt{\left(\frac{y}{x}\right)^n}}\right)\left(1 + \sqrt{2\sqrt{\left(\frac{y}{x}\right)^n} + \sqrt{\left(\frac{y}{x}\right)^n}}\right) = \left[\sum_{j=0}^{n} C_n^j \left(\frac{y}{x}\right)^j - \sum_{j=1}^{n-1} C_n^j \left(\frac{y}{x}\right)^j\right]. \quad (49)$$

Proof: To prove this theorem we must use the well-known binomial formula and the formula (41). To this end, we have for the usual case when $n, j \in \mathbb{N}$ such that $n \geq j$:

$$(x+y)^n = (x^n + y^n) + \sum_{j=1}^{n-1} C_n^j \, x^{n-j} y^j = \sum_{j=0}^{n} C_n^j \, x^{n-j} y^j,$$

from where we get

$$(x^n + y^n) = \left[\sum_{j=0}^{n} C_n^j \, x^{n-j} y^j - \sum_{j=1}^{n-1} C_n^j \, x^{n-j} y^j\right]. \tag{50}$$

Expression (50) is generally valid for $x, y \in \mathbb{R}$, but for the purpose of the proof, we can reduce the set $\mathbb{R}$ of validity to $\mathbb{R}_+$ with the additional condition $x \neq 0$. Hence, we can rewrite (50) in the form

$$\left[1 + \left(\frac{y}{x}\right)^n\right] = \left[\sum_{j=0}^{n} C_n^j \left(\frac{y}{x}\right)^j - \sum_{j=1}^{n-1} C_n^j \left(\frac{y}{x}\right)^j\right]. \tag{51}$$

Moreover, let $t, \tau \in \mathbb{R}_+$ and $n \in \mathbb{N}$, thus we have according to the factorization (41)

$$(t^n + \tau^n) = \left(\sqrt{t^n} - \sqrt{2\sqrt{(t\tau)^n}} + \sqrt{\tau^n}\right)\left(\sqrt{t^n} + \sqrt{2\sqrt{(t\tau)^n}} + \sqrt{\tau^n}\right). \tag{52}$$

Putting $t = 1$ and $\tau = \frac{y}{x}$ with $x \neq 0$ in (52) to get

$$\left[1 + \left(\frac{y}{x}\right)^n\right] = \left(1 - \sqrt{2\sqrt{\left(\frac{y}{x}\right)^n}} + \sqrt{\left(\frac{y}{x}\right)^n}\right)\left(1 + \sqrt{2\sqrt{\left(\frac{y}{x}\right)^n}} + \sqrt{\left(\frac{y}{x}\right)^n}\right). \tag{53}$$

Finally, by combining (51) and (53), we obtain (49). QED

Theorem 6.4. Let $x \in \mathbb{R}_+$. If $n, m \in \mathbb{N}: m \neq 0$, then

$$\int_0^1 \left(1 + x^{\frac{n}{m}}\right)^m dx = \frac{n+2}{n+1} + \sum_{j=1}^{m-1} C_m^j \, \frac{m}{m(n+1) - jn}. \tag{54}$$

Proof: To prove this theorem we must use the formula (47). Hence, to this end, putting $y = 1$ in (47), rearranging the terms and integrating from 0 to 1 to obtain

$$\int_0^1 \left(1 + x^{\frac{n}{m}}\right)^m dx = \int_0^1 (1 + x^n) dx + \int_0^1 \sum_{j=1}^{m-1} C_m^j \, x^{n - j\frac{n}{m}} \, dx. \tag{55}$$

Let us focus our attention on the RHS of (55). A direct integration leads to

$$\int_0^1 (1 + x^n) dx + \int_0^1 \sum_{j=1}^{m-1} C_m^j \, x^{n - j\frac{n}{m}} \, dx = \left[x + \frac{x^{n+1}}{n+1}\right]_0^1 + \left[\sum_{j=1}^{m-1} C_m^j \, \frac{x^{n - j\frac{n}{m} + 1}}{n - j\frac{n}{m} + 1}\right]_0^1,$$

$$= \frac{n+2}{n+1} + \sum_{j=1}^{m-1} C_m^j \, \frac{m}{m(n+1) - jn}. \tag{56}$$

Finally, from (56) and (55), we find (54). QED.

Theorem 6.5. Let $x \in \mathbb{R}_+\backslash\{0\}$. If $n, m \in \mathbb{N}\backslash\{0\}: n < m$, then

$$\int_0^\infty \left[1 + \left[(-1)^m + x^n \left[1 + \sum_{j=1}^{m-1} (-1)^j C_m^j \, x^{-j\frac{n}{m}}\right]\right]^{\frac{1}{m}}\right] \frac{dx}{x(1+x)} = \frac{\pi}{\sin\left(\frac{n}{m}\right)\pi} \cdot \qquad (57)$$

Proof: To prove this theorem we must use the formula (48). Thus, putting $y = 1$ in (48) and rearranging the terms to obtain

$$\left(x^{\frac{n}{m}} - 1\right)^m = (x^n + (-1)^m) + \sum_{j=1}^{m-1} (-1)^j C_m^j \, x^{n-j\frac{n}{m}} y^{j\frac{n}{m}} .$$

Multiplying both sides by $(x + x^2)^m$ to get

$$\left(\frac{x^{\frac{n}{m}} - 1}{x + x^2}\right)^m = \left[(-1)^m + x^n \left[1 + \sum_{j=1}^{m-1} (-1)^j C_m^j \, x^{-j\frac{n}{m}}\right]\right] \frac{1}{(x+x^2)^m}.$$

From where we find

$$\frac{x^{\frac{n}{m}} - 1}{x + x^2} = \left[(-1)^m + x^n \left[1 + \sum_{j=1}^{m-1} (-1)^j C_m^j \, x^{-j\frac{n}{m}}\right]\right]^{\frac{1}{m}} \frac{1}{x(1+x)}.$$

Finally, rearranging the terms and integrating from 0 to ∞:

$$\int_0^\infty \left[1 + \left[(-1)^m + x^n \left[1 + \sum_{j=1}^{m-1} (-1)^j C_m^j \, x^{-j\frac{n}{m}}\right]\right]^{\frac{1}{m}}\right] \frac{dx}{x(1+x)} = \int_0^\infty \frac{x^{\frac{n}{m}-1}}{(1+x)} dx = \frac{\pi}{\sin\left(\frac{n}{m}\right)\pi} \cdot \qquad \text{QED.}$$

7. The concept of algebraic factorization with fractional degree and positive integral order

For notational convenience, we adopt the notation $[x]_{\frac{1}{n},k}$ where $x = (a - b) \in \mathbb{R}_+; n, k \in \mathbb{N}$: $n = 2, 4, 8, 16, 32, \dots$, to denote the '*algebraic factorization with fractional degree* $\frac{1}{n}$ *and positive integral order* k *of positive real number* x'. The above-mentioned concept is a direct natural consequence of the formulae derived in this paper, which allow us to factorize any given positive real number. For instance, the usual arithmetic factorization of 15 is 3×5 but 11 cannot be arithmetically factorized because it is a prime number. However, according to the aforesaid concept, 3, 5, 11 and 15 are algebraically factorable. For instance, since $3 = (5 - 2) = (5 + 0) - (2 + 0)$ thus '3' has an algebraic factorization with fractional degree ½ and positive integral order 0: $[3]_{\frac{1}{2},0} = \left(5^{\frac{1}{2}} - 2^{\frac{1}{2}}\right)\left(5^{\frac{1}{2}} + 2^{\frac{1}{2}}\right)$, and the factorization with fractional degree $\frac{1}{n}$ and positive integral order k of '3' is:

$$[3\,]_{\frac{1}{n},k} = \left[(5 + k)^{\frac{1}{n}} - (2 + k)^{\frac{1}{n}}\right] \prod_{r=1}^{\left[\frac{n}{2}\right]} \left[(5 + k)^{\frac{r}{n}} + (2 + k)^{\frac{r}{n}}\right],$$

where $n, k \in \mathbb{N}$: $n = 2, 4, 8, 16, 32 \ldots$

From all that, we arrive at the following theorem.

Theorem 1.7. Let $c, a, b \in \mathbb{R}_+$. If $c = a - b$, then c admitting an algebraic factorization with fractional degree $\frac{1}{n}$ and positive integral order k of the form

$$[c]_{\frac{1}{n},k} = \left[(a+k)^{\frac{1}{n}} - (b+k)^{\frac{1}{n}}\right] \prod_{r=1}^{\left[\frac{n}{2}\right]} \left[(a+k)^{\frac{r}{n}} + (b+k)^{\frac{r}{n}}\right]. \tag{58}$$

where $n, k \in \mathbb{N}$: $n = 2, 4, 8, 16, 32 \ldots$

Proof: Since $c, a, b \in \mathbb{R}_+$ this implies $c = a - b \geq 0$. Consequently

$$\begin{aligned} c &= (a+0) - (b+0) \\ &= (a+1) - (b+1) \\ &= (a+2) - (b+2) \\ &= (a+3) - (b+3) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= (a+k) - (b+k) \end{aligned}$$

Or equivalently

$$\begin{aligned} c &= \left[(a+0)^{\frac{1}{2}} - (b+0)^{\frac{1}{2}}\right]\left[(a+0)^{\frac{1}{2}} + (b+0)^{\frac{1}{2}}\right] \\ &= \left[(a+1)^{\frac{1}{4}} - (b+1)^{\frac{1}{4}}\right]\left[(a+1)^{\frac{1}{4}} + (b+1)^{\frac{1}{4}}\right]\left[(a+1)^{\frac{1}{2}} + (b+1)^{\frac{1}{2}}\right] \\ &= \left[(a+2)^{\frac{1}{8}} - (b+2)^{\frac{1}{8}}\right]\left[(a+2)^{\frac{1}{8}} + (b+2)^{\frac{1}{8}}\right]\left[(a+2)^{\frac{1}{4}} + (b+2)^{\frac{1}{4}}\right]\left[(a+2)^{\frac{1}{2}} + (b+2)^{\frac{1}{2}}\right] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= \left[(a+k)^{\frac{1}{n}} - (b+k)^{\frac{1}{n}}\right]\left[(a+k)^{\frac{1}{n}} + (b+k)^{\frac{1}{n}}\right]\cdots\left[(a+k)^{\frac{1}{4}} + (b+k)^{\frac{1}{4}}\right]\left[(a+k)^{\frac{1}{2}} + (b+k)^{\frac{1}{2}}\right] \end{aligned}$$

Finally, the last expression can be put in compact form to get (58). QED.

8. Conclusion

In this paper, the frequently asked question "What is the factorization of $(x^n + y^n)$ when n is an even positive integer?" by high school and university students on the Net has been answered, generalized and extended to the fractional exponents. The detailed study of the factorizations led directly to the concept of algebraic factorization with fractional degree and positive integral order, and an apparently new type of indefinite irrational and rational integrals.

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