On the sequence $\left\{x_N \in (0,1) \wedge 3x_N^N + 3x_N = 1, N = 1, 2, 3, ...\right\}$

Edgar Valdebenito

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Abstract

We study the sequence :

$$\left\{x_N \in (0,1) \wedge 3x_N^N + 3x_N = 1, N = 1, 2, 3, ...\right\}$$

Introduction

Let $\{x_N\}$ be the sequence given by

$$0 < x_N < 1 \,,\, 3x_N^N + 3x_N = 1 \,,\, N \in \{1, 2, 3, ...\}$$

In this note we give some formulas related to $\{x_N\}$.

The number Pi is defined by

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - ...\right) = 3.1415926535\,...$$

Formulas

Entry 1.

$$x_1 = \frac{1}{6} \,,\, x_2 = \frac{\sqrt{21} - 3}{6} \,,\, x_\infty = \frac{1}{3}$$

Entry 2. for $N = 3, 4, 5, ...$ we have

$$x_N = \sum_{k=0}^{\infty} \frac{(Nk)!\,(-1)^k\, 3^{-(N-1)k-1}}{((N-1)k+1)!\,k!}$$

Entry 3. for $N = 1, 2, 3, ...$ we have

$$\pi = 2\sqrt{3} \sum_{n=0}^{\infty} (-x_N)^n \sum_{k=0}^{\left[\frac{n}{N}\right]} \binom{n-(N-1)k}{k} \frac{(-1)^{(N-1)k}}{2n - 2(N-1)k + 1}$$

Entry 4. for $N = 1, 2, 3, ...$ we have

$$\pi = 3 \sum_{n=0}^{\infty} (x_N)^n \sum_{k=0}^{\left[\frac{n}{N}\right]} \binom{2n - 2(N-1)k}{n-(N-1)k} \binom{n-(N-1)k}{k} \frac{\left(\frac{3}{16}\right)^{n-(N-1)k}}{2n - 2(N-1)k + 1}$$

Entry 5. for $N = 1, 2, 3,...$ we have

$$\pi = 6\sqrt{3} \sum_{n=0}^{\infty} (-1)^n \, x_N^{\;n+1} \sum_{k=0}^{\left[\frac{n}{N}\right]} \binom{n-(N-1)\,k}{k} \frac{(-1)^{Nk} 3^k}{2n - 2Nk + 1}$$

Entry 6. for $N >> 1$ we have

$$x_N \approx \frac{1}{3} - \frac{3^{-N}}{1 + N\,3^{-N+1}} + \frac{(N-1)N\,3^{-3N+2}}{2\left(1 + N\,3^{-N+1}\right)^3} - ...$$

Entry 7. for $N = 1, 2, 3,...$ we have

$$y_{N,\,0} = 0\;,\; y_{N,\,k+1} = \frac{\left(1 - 3\,y_{N,\,k}\right)^N 3^{-N} + N\,3^{-N+1}\,y_{N,\,k}}{1 + N\,3^{-N+1}}\;,\; k = 0, 1, 2,... \Rightarrow \lim_{k\to\infty} y_{N,\,k} = \frac{1}{3} - x_N$$

Entry 8. for $N = 2, 3, 4,...$ we have

$$x_N = z_N^{\;-\frac{1}{N-1}}$$

where

$$z_N = \left(3 + 3\left(3 + 3(3 + ...)^{\frac{N-1}{N}}\right)^{\frac{N-1}{N}}\right)^{\frac{N-1}{N}}$$

Entry 9. for $N = 2, 3, 4,...$ we have

$$x_N = \frac{1}{3} - \left\{\left(\frac{1}{3} - \left(\frac{1}{3} - \left(\frac{1}{3} - ...\right)^N\right)^N\right)^N\right\}$$

Entry 10.

$$x_N < x_{N+1}\;,\; N = 1, 2, 3,...$$
$$\left(x_N\right)^N > \left(x_{N+1}\right)^{N+1}\;,\; N = 1, 2, 3,...$$

Entry 11. for $N = 2, 3, 4,...$ we have

$$c(N,k) = 3\,c(N,k-1) + 3\,c(N,k-N)\;,\; c(N,0) = 1,\; c(N,-k) = 0,\; k = 1, 2, 3,...$$

$$\lim_{k\to\infty} \frac{c(N,k)}{c(N,k+1)} = x_N$$

Examples:

$$c(2,k) = \{1, 3, 12, 45, 171, 648, 2457, 9315,...\}$$
$$c(3,k) = \{1, 3, 9, 30, 99, 324, 1062, 3483,...\}$$
$$c(4,k) = \{1, 3, 9, 27, 84, 261, 810, 2511,...\}$$

Entry 12. for $N = 2, 3, 4,...$ we have

$$f(N,z) = \frac{1}{1 - 3z - 3z^N}$$

$$g(N)=\int_0^{2\pi} f\left(N, \frac{1}{4}+\frac{e^{ix}}{6}\right)e^{ix}\,dx \quad, i=\sqrt{-1}$$

$$x_N=\left(\frac{1}{N}\left(-1-\frac{4\pi}{g(N)}\right)\right)^{\frac{1}{N-1}}$$

Entry 13.

$$\frac{\pi}{\sqrt{3}}=24\sum_{n=1}^{\infty}\frac{3^{-n}\cdot(n+1)}{4n^2-1}\sum_{k=1}^{n}(-1)^{k-1}x_n+18\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left(\frac{x_n}{3}\right)^n$$

$$\frac{\pi}{\sqrt{3}}=12\sum_{n=1}^{\infty}3^{-n}\sum_{k=1}^{n}\frac{(-1)^{k-1}x_k}{2k-1}+18\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1}\left(\frac{x_n}{3}\right)^n$$

Entry 14. for $N=2, 3, 4,...$ we have

$$y_{N,0}=0 \;,\; y_{N,k+1}=\frac{1}{3+3\left(y_{N,k}\right)^{N-1}} \;, k=0,1,2,3,... \Rightarrow \lim_{k\to\infty} y_{N,k}=x_N$$

$$y_{N,2k}<x_N<y_{N,2k+1} \;, k=0,1,2,3,...$$

Entry 15. for $N=1, 2, 3,...$ we have

$$\frac{1}{x_N}=3+3\left(3+3\left(3+3\left(3+3(3+...)^{1-N}\right)^{1-N}\right)^{1-N}\right)^{1-N}$$

Entry 16. for $N=1, 2, 3,...$ we have

$$\pi=6\sqrt{x_N}\sum_{n=0}^{\infty}(-1)^n(x_N)^n\sum_{k=0}^{\left[\frac{n}{N}\right]}\binom{2n-2(N-1)k}{n-(N-1)k}\binom{n-(N-1)k}{k}\frac{(-1)^{Nk}\left(\frac{3}{4}\right)^k}{(2n-2Nk+1)\binom{2n-2Nk}{n-Nk}}$$

References

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