A Formula of the Dirichlet Character Sum

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Abstract. In this paper, We use the Fourier series expansion of real variables function, We give a formula to calculate the Dirichlet character sum, and four special examples are given.

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The calculation of the Dirichlet character sum is very important in the number theory. This paper uses the Fourier series expansion of the functions, we give a general formula for calculating the Dirichlet character sum, Then, four examples are given to illustrate.

In this paper, χ_q denote the Dirichlet primitive character of mod q, If $f(x)$ is a real function, we write

$$f^*(x) = \frac{f(x+0) + f(x-0)}{2}$$

and

$$G(n, \chi_q) = \sum_{k=1}^{q-1} \chi_q(k) e\left(\frac{kn}{q}\right) \qquad \tau(\chi_q) = \sum_{k=1}^{q-1} \chi_q(k) e\left(\frac{k}{q}\right)$$

where $e(x) = e^{2\pi i x}$

First, let's give some lemmas.

Lemma 1. If χ_q is the primitive character of module q, then we have

$$G(n,\chi_q)=\sum_{k=1}^{q-1}\chi_q(k)e\left(\frac{kn}{q}\right)=\overline{\chi}_q(n)\tau(\chi_q)$$

see page 287 of references[1].

Lemma 2. If χ_q is the primitive real character of module q, then we have

$$\tau(\chi_q)=\begin{cases}\sqrt{q} & if\ \ \chi_q(-1)=1\\ i\sqrt{q} & if\ \ \chi_q(-1)=-1\end{cases}$$

see page 167 of references[2].

Lemma 3. If the function $f(x)$ is defined in the interval $[0,1]$ and satisfies the Dirichlet condition, then we have

$$f^*(x)=\int_0^1 f(t)dt+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\exp(2\pi inx)\int_0^1 f(t)\exp(-2\pi int)dt$$

see page 421 of references[3].

Now, we give the theorem of this paper.

Theorem. If the function $f(x)$ is defined in the interval $[0,1]$ and satisfies the Dirichlet condition, then, when $\chi_q(-1)=1$, we have

$$\sum_{k=1}^{q-1}\chi_q(k)f^*\left(\frac{k}{q}\right)=2\tau(\chi_q)\sum_{n=1}^{\infty}\overline{\chi}_q(n)\int_0^1 f(t)\cos(2\pi nt)dt$$

when $\chi_q(-1)=-1$, we have

$$\sum_{k=1}^{q-1} \chi_q(k) f^* \left(\frac{k}{q}\right) = -2i\tau(\chi_q) \sum_{n=1}^{\infty} \overline{\chi}_q(n) \int_0^1 f(t) sin(2\pi nt) dt$$

Proof. By lemma 3, we have

$$f^*(x) = \int_0^1 f(t)dt + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \exp(2\pi inx) \int_0^1 f(t) \exp(-2\pi int) dt$$

we take $x = \frac{k}{q}$, $1 \leq k \leq q-1$, then

$$f^* \left(\frac{k}{q}\right) = \int_0^1 f(t)dt + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \exp\left(2\pi i \frac{nk}{q}\right) \int_0^1 f(t) \exp(-2\pi int) dt$$

Multiply the above formula by $\chi_q(k)$, then sum over k, we have

$$\sum_{k=1}^{q-1} \chi_q(k) f^* \left(\frac{k}{q}\right) = \sum_{k=1}^{q-1} \chi_q(k) \int_0^1 f(t)dt$$

$$+ \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{k=1}^{q-1} \chi_q(k) \exp\left(2\pi i \frac{nk}{q}\right) \int_0^1 f(t) \exp(-2\pi int) dt$$

By Lemma 1,

$$\sum_{k=1}^{q-1} \chi_q(k) \exp\left(2\pi i \frac{nk}{q}\right) = \overline{\chi}_q(n)\tau(\chi_q) \quad and \quad \sum_{k=1}^{q-1} \chi_q(k) = 0$$

We have

$$\sum_{k=1}^{q-1} \chi_q(k) f^*\left(\frac{k}{q}\right) = \tau(\chi_q) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \overline{\chi}_q(n) \int_0^1 f(t)\exp(-2\pi i n t)dt$$

$$= \tau(\chi_q) \sum_{n=1}^{\infty} \overline{\chi}_q(n) \int_0^1 f(t)\exp(-2\pi i n t)dt$$

$$+\tau(\chi_q) \sum_{n=1}^{\infty} \overline{\chi}_q(-n) \int_0^1 f(t)\exp(2\pi i n t)dt$$

$$= \tau(\chi_q) \sum_{n=1}^{\infty} \overline{\chi}_q(n) \Bigg(\int_0^1 f(t)\exp(-2\pi i n t)dt$$

$$+\overline{\chi}_q(-1) \int_0^1 f(t)\exp(2\pi i n t)dt \Bigg)$$

therefore, when $\chi_q(-1) = 1$, we have

$$\sum_{k=1}^{q-1} \chi_q(k) f^*\left(\frac{k}{q}\right) = 2\tau(\chi_q) \sum_{n=1}^{\infty} \overline{\chi}_q(n) \int_0^1 f(t)\cos(2\pi n t)dt$$

when $\chi_q(-1) = -1$, we have

$$\sum_{k=1}^{q-1} \chi_q(k) f^*\left(\frac{k}{q}\right) = -2i\tau(\chi_q) \sum_{n=1}^{\infty} \overline{\chi}_q(n) \int_0^1 f(t)\sin(2\pi n t)dt$$

This completes the proof of the theorem.

From this theorem, we can see that the calculation of the character sum becomes the calculation of integrals.

Below, we give a few special examples.

The first example.

Let χ_q be the primitive real character and $\chi_q(-1) = -1$, then

$$\sum_{k=1}^{q-1} \chi_q(k) \left(\frac{k}{q}\right)^2 = -\frac{\sqrt{q}}{\pi} L(1, \chi_q)$$

By Theorem and Lemma 2, easily seen

$$\sum_{k=1}^{q-1} \chi_q(k) \left(\frac{k}{q}\right)^2 = 2\sqrt{q} \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 t^2 \sin(2\pi n t)dt$$

We compute the integral as follows

$$\int_0^1 t^2 \sin(2\pi n t)dt = -\frac{1}{2\pi n} \int_0^1 t^2 \, d\cos(2\pi n t)$$

$$= -\frac{1}{2\pi n} + \frac{2}{2\pi n} \int_0^1 t \cos(2\pi n t)dt = -\frac{1}{2\pi n} + \frac{2}{(2\pi n)^2} \int_0^1 t \, d\sin(2\pi n t)$$

$$= -\frac{1}{2\pi n} - \frac{2}{(2\pi n)^2} \int_0^1 \sin(2\pi nt)dt = -\frac{1}{2\pi n}$$

therefore

$$\sum_{k=1}^{q-1} \chi_q(k) \left(\frac{k}{q}\right)^2 = -\frac{\sqrt{q}}{\pi} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{n} = -\frac{\sqrt{q}}{\pi} L(1, \chi_q)$$

The second example.

Let χ_q be the primitive real character and $\chi_q(-1) = 1$, then

$$\sum_{k=1}^{q-1} \chi_q(k) \log k = -\frac{\sqrt{q}}{2} L(1, \chi_q) + c\sqrt{q}$$

where c is a absolute constant.

By Theorem and Lemma 2, we have

$$\sum_{k=1}^{q-1} \chi_q(k) \log \frac{k}{q} = 2\sqrt{q} \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 \log t \cos(2\pi nt)dt$$

Now, let's compute the integral

$$\int_0^1 \log t \cos(2\pi nt)dt = \frac{1}{2\pi n} \int_0^1 \log t \, d \sin(2\pi nt)$$

$$= -\frac{1}{2\pi n}\int_0^1 \frac{1}{t}\sin(2\pi nt)dt = -\frac{1}{2\pi n}\int_0^n \frac{1}{t}\sin(2\pi t)dt$$

$$= -\frac{1}{2\pi n}\int_0^\infty \frac{1}{t}\sin(2\pi t)dt + \frac{1}{2\pi n}\int_n^\infty \frac{1}{t}\sin(2\pi t)dt$$

easily seen

$$-\frac{1}{2\pi n}\int_0^\infty \frac{1}{t}\sin(2\pi t)dt = -\frac{1}{2\pi n}\int_0^\infty \frac{1}{t}\sin t dt = -\frac{1}{4n}$$

because

$$\frac{1}{2\pi n}\int_n^\infty \frac{1}{t}\ \sin(2\pi t)dt = -\frac{1}{(2\pi)^2 n}\int_n^\infty \frac{1}{t}\,d\cos(2\pi t)$$

$$= \frac{1}{(2\pi n)^2} - \frac{1}{(2\pi)^2 n}\int_n^\infty \frac{1}{t^2}\ \cos(2\pi t)dt \ll \frac{1}{n^2} + \frac{1}{n}\int_n^\infty \frac{1}{t^2}dt \ll \frac{1}{n^2}$$

as well as

$$\sum_{k=1}^{q-1}\chi_q(k)\log\frac{k}{q} = \sum_{k=1}^{q-1}\chi_q(k)\log k - \log q\sum_{k=1}^{q-1}\chi_q(k) = \sum_{k=1}^{q-1}\chi_q(k)\log k$$

This completes the proof.

The third example.

Let χ_q be the primitive real character and $\chi_q(-1) = 1$, then

$$\sum_{k=1}^{q-1} \chi_q(k)\, e^{\frac{k}{q}} = 2(e-1)\sqrt{q} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{1 + 4\pi^2 n^2}$$

Let χ_q be the primitive real character and $\chi_q(-1) = -1$, then

$$\sum_{k=1}^{q-1} \chi_q(k)\, e^{\frac{k}{q}} = -4\pi(e-1)\sqrt{q} \sum_{n=1}^{\infty} \frac{\chi_q(n)\, n}{1 + 4\pi^2 n^2}$$

Proof. When $\chi_q(-1) = 1$, by Theorem and Lemma 2, we have

$$\sum_{k=1}^{q-1} \chi_q(k)\, e^{\frac{k}{q}} = 2\sqrt{q} \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 e^t \, \cos(2\pi n t) dt$$

Because, we know

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$$

Therefore

$$\int_0^1 e^t \cos(2\pi n t)\, dt = \frac{e-1}{1 + 4\pi^2 n^2}$$

Therefore

$$\sum_{k=1}^{q-1} \chi_q(k)\, e^{\frac{k}{q}} = 2(e-1)\sqrt{q} \sum_{n=1}^{\infty} \frac{\chi_q(n)}{1+4\pi^2 n^2}$$

When $\chi_q(-1) = -1$, by Theorem and Lemma 2, we have

$$\sum_{k=1}^{q-1} \chi_q(k)\, e^{\frac{k}{q}} = 2\sqrt{q} \sum_{n=1}^{\infty} \chi_q(n) \int_0^1 e^t \, \sin(2\pi n t) dt$$

Because, we know

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

Therefore

$$\int_0^1 e^t \sin(2\pi n t)\, dt = -(e-1) \frac{2\pi n}{1+4\pi^2 n^2}$$

Therefore

$$\sum_{k=1}^{q-1} \chi_q(k)\, e^{\frac{k}{q}} = -4\pi(e-1)\sqrt{q} \sum_{n=1}^{\infty} \frac{\chi_q(n)\, n}{1+4\pi^2 n^2}$$

This completes the proof.

The fourth example.

This is a well-known formula.

We write

$$F(y)=\sum_{1\le k\le qy}\chi_q(k) \qquad and \qquad F^*(y)=\frac{F(y+0)+F(y-0)}{2}$$

When $\chi_q(-1)=1$, we have

$$F^*(y)=\frac{\tau(\chi_q)}{\pi}\sum_{n=1}^{\infty}\frac{\overline{\chi}_q(n)}{n}\sin(2\pi ny)$$

When $\chi_q(-1)=-1$, we have

$$F^*(y)=\frac{\tau(\chi_q)}{i\pi}L(1,\overline{\chi}_q)-\frac{\tau(\chi_q)}{i\pi}\sum_{n=1}^{\infty}\frac{\overline{\chi}_q(n)}{n}\cos(2\pi ny)$$

Proof. Let $0<y<1$, we define the function $f(x)$ as follow

$$f(x)=\begin{cases} 1 & if\ \ 0\le x\le y\\ 0 & if\ \ y<x<1\end{cases}$$

By Theorem, when $\chi_q(-1)=1$, we have

$$\sum_{k=1}^{q-1}\chi_q(k)f^*\left(\frac{k}{q}\right)=2\tau(\chi_q)\sum_{n=1}^{\infty}\overline{\chi}_q(n)\int_0^y\cos(2\pi nt)dt$$

Because

$$\int_0^y \cos(2\pi nt)dt = \frac{1}{2\pi n}\int_0^y d\sin(2\pi nt) = \frac{\sin(2\pi ny)}{2\pi n}$$

Therefore

$$F^*(y) = \frac{\tau(\chi_q)}{\pi}\sum_{n=1}^{\infty}\frac{\overline{\chi}_q(n)}{n}\sin(2\pi ny)$$

By Theorem, when $\chi_q(-1) = -1$, we have

$$\sum_{k=1}^{q-1}\chi_q(k)f^*\left(\frac{k}{q}\right) = -2i\tau(\chi_q)\sum_{n=1}^{\infty}\overline{\chi}_q(n)\int_0^y \sin(2\pi nt)dt$$

Because

$$\int_0^y \sin(2\pi nt)dt = -\frac{1}{2\pi n}\int_0^y d\cos(2\pi nt) = -\frac{1}{2\pi n}(\cos(2\pi ny) - 1)$$

Therefore

$$F^*(y) = \frac{\tau(\chi_q)}{i\pi}L(1,\overline{\chi}_q) - \frac{\tau(\chi_q)}{i\pi}\sum_{n=1}^{\infty}\frac{\overline{\chi}_q(n)}{n}\cos(2\pi ny)$$

This completes the proof.

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