ON PROBLEMS AND THEIR SOLUTION SPACES

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Abstract. We introduce and develop the logic of existence of solution to problems. We use this theory to answer the question of Florentin Smarandache in logic. We answer this question in the negative.

1. Introduction

The endeavour of finding solutions to problems or at least knowing that a problem is solvable appears to be very compelling. It has various related class of problems that remains unsolved till date. Perhaps the best known of all is the P versus NP problem in computer science. In [1] Florentin Smarandache ask the deceptively simple question

Question 1.1. Is it true that for any question there is at least an answer? Reciprocally, is any assertion the result of at least a question?

We develop a much more consolidated theory of problems and their solution spaces to study the structure and the inner workings of problems, whose solutions may or may not exist. By studying this structure into details, we obtain a negative answer to the question posed

Theorem 1.2. *There exists a problem with no solution.*

It turns out that this result holds for irreducible problems, a certain class of problems we will study in the sequel. This result is obtained via a certain infinite argument under the assumption of a positive answer to the major question, to obtain a certain infinite sub-covers of problem spaces whose indices becomes infinitesimally small and never running into extinction.

2. Problems and solution spaces

In this section we introduce and develop the notion of problem and their corresponding solution spaces.

Definition 2.1. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all problems to be solved to provide solution X to problem Y the problem space induced by providing solution X to problem Y. We denote this space with $\mathcal{P}_Y(X)$. If K is any subspace of the space $\mathcal{P}_Y(X)$, then we denote this relation with $K \subseteq \mathcal{P}_Y(X)$. If the space K is a subspace of the space $\mathcal{P}_Y(X)$ with $K \neq \mathcal{P}_Y(X)$, then we write $K \subset \mathcal{P}_Y(X)$. We say problem V is a sub-problem of problem Y if a solution to problem Y furnishes a solution to

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problem V. If V is a sub-problem of the problem Y, then we write $V \leq Y$. If V is a sub-problem of the problem Y and $V \neq Y$, then we write $V < Y$ and we call V a proper sub-problem of Y.

Definition 2.2. Let $\mathcal{P}_Y(X)$ be the problem space induced by providing the solution X to problem Y. Then we call the number of problems in the space (size) the **complexity** of the space and denote by $\mathbb{C}[\mathcal{P}_Y(X)]$ the complexity of the space. We make the assignment $Z \in \mathcal{P}_Y(X)$ if problem Z is also a problem in this space.

Definition 2.3. Let X denotes a solution (resp. answer) to problem Y (resp. question). Then we call the collection of all solutions to problems obtained as a result of providing the solution X to problem Y the solution space induced by providing solution X to problem Y. We denote this space with $\mathcal{S}_Y(X)$. If K is any subspace of the space $\mathcal{S}_Y(X)$, then we denote this relation with $K \subset \mathcal{S}_Y(X)$. We make the assignment $T \in \mathcal{S}_Y(X)$ if solution T is also a solution in this space.

Proposition 2.1. *Let $\mathcal{S}_Y(X)$ be the solution space induced by providing solution X to problem Y. Then $X \in \mathcal{S}_Y(X)$.*

Proof. This follows by virtue of Definition 2.3. $\square$

Definition 2.4. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to problem Y. Then we call the number of solutions in the space (size) the **index** of the space and denote by $\mathbb{I}[\mathcal{S}_Y(X)]$ the index of this space.

Definition 2.5. Let $\mathcal{S}_Y(X)$ be the solution space induced by providing the solution X to problem Y. Then by the **entropy** of the space, we mean the expression

$$\mathcal{E}[S] = \frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]}.$$

In the sequel we formalize the notion that the problem space induced by providing a solution to a problem should - by necessity - contain this solution. The argument is an iteration of a never diminishing entropy of larger and larger solution spaces. We launch formally the following arguments.

Theorem 2.6. *Let $\mathcal{P}_Y(X)$ be the induced problem space of providing solution X to problem Y. Then $Y \in \mathcal{P}_Y(X)$.*

Proof. Let us suppose to the contrary that for any problem space $Y \notin \mathcal{P}_Y(X)$. Since Y is a solved problem, it must belong to some problem space, say $\mathcal{P}_V(U)$. In particular we have the containment

$$Y \in \mathcal{P}_V(U).$$

Since X is a solution to problem Y and V has solution U, it follows that X is a solution obtained as a result of providing solution U to problem V. It follows that $X \in \mathcal{S}_V(U)$ so that the embedding

$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U)$$

holds, since $X \in \mathcal{S}_Y(X)$. Again $V \notin \mathcal{P}_V(U)$ under the assumption, so that V belongs to some problem space, say $\mathcal{P}_K(L)$. That is, $V \in \mathcal{P}_K(L)$, a problem space induced by providing solution L to problem K. Since U is a solution to problem V

and K has solution L, it must be a problem solved as a result of providing solution L to problem K. It follows that $U \in \mathcal{S}_K(L)$ and the embedding holds

$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U) \subset \mathcal{S}_K(L)$$

since $U \in \mathcal{S}_V(U)$. By iterating the argument in this manner under the assumption that $G \notin \mathcal{P}_G(F)$ for an arbitrary problem space, we obtain the infinite embedding

$$\mathcal{S}_Y(X) \subset \mathcal{S}_V(U) \subset \mathcal{S}_K(L) \subset \cdots \subset \cdots .$$

It follows from this the following infinite decreasing sequence of the entropy of solution spaces towards zero

$$\frac{1}{\mathbb{I}[\mathcal{S}_Y(X)]} > \frac{1}{\mathbb{I}[\mathcal{S}_V(U)]} > \frac{1}{\mathbb{I}[\mathcal{S}_K(L)]} > \cdots > \cdots$$

which is not possible. This completes the proof of the theorem. □

Definition 2.7. Let Y and V be any two problems. Then we say problem Y is equivalent to problem V if providing solution to problem Y also provides a solution to problem V and conversely providing a solution to problem V also provides a solution to problem Y. We denote the equivalence with $V \equiv Y$.

Next we expose a simple criterion for creating a subspace of a problem space.

Proposition 2.2. *Let $X \in \mathcal{S}_V(U)$ and $Y \in \mathcal{P}_V(U)$. If X is a solution to problem Y, then*

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Proof. Under the requirement $Y \in \mathcal{P}_V(U)$, then Y is a sub-problem to be solved to provide solution U to problem V. Since $X \in \mathcal{S}_V(U)$, it follows that X is a solution obtained by providing solution U to problem V. Since X solves Y and $Y \in \mathcal{P}_Y(X)$, it follows that

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

□

We use the following criterion to determine the solubility of a problem.

Proposition 2.3. *Let V be a problem with solution U. If $Y \in \mathcal{P}_V(U)$, then Y must have a solution.*

Proof. Clearly problem V is solved by U with an induced problem space $\mathcal{P}_V(U)$. Since this space consist of all sub-problems to be solved in order to provide solution U to problem V and $Y \in \mathcal{P}_V(U)$, then Y has a solution. □

3. Reducible and irreducible problems

In this section, we classify problems in a problem space into two main categories. We study the notion of irreducibility and reducibility of a problem.

Definition 3.1. Let V be a problem. Then we say V is reducible if there exists a proper sub-problem of V with no proper sub-problem. On the other hand, we say problem V is irreducible if every proper sub-problem of V has a proper sub-problem.

It is a well-known problem to determine if every problem has a solution. Using this classification, we can deduce that there must exist a problem with no solution. It turns out that irreducible problems satisfies this property.

Theorem 3.2. *There exists a problem with no solution.*

Proof. Suppose to the contrary that every problem has a solution. It suffices to argue with only irreducible problems. Now, let V be an irreducible problem with solution U. Consider the induced problem space $\mathcal{P}_V(U)$. Then from Theorem 2.6 $V \in \mathcal{P}_V(U)$. Since V is irreducible, we choose a proper sub-problem Y of V with solution X and construct the problem space $\mathcal{P}_Y(X)$ and solution spaces $\mathcal{S}_Y(X)$. Then $Y \in \mathcal{P}_V(U)$ and $X \in \mathcal{S}_V(U)$ so that

$$\mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

Again V is irreducible so that we can choose a proper sub-problem Z of Y with solution R. Then under the same arguments, we have the chain of sub-covers of problem spaces

$$\mathcal{P}_Z(R) \subset \mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

By iterating the argument under the same assumption that every problem has a solution, we obtain the infinite chain of sub-covers of smaller problem spaces

$$\cdots \subset \cdots \subset \mathcal{P}_Z(R) \subset \mathcal{P}_Y(X) \subset \mathcal{P}_V(U).$$

This is impossible and this completes the proof. $\square$

We can now state another important criterion for determining the solubility of a problem, provided we can put it on par with some category of problems.

Proposition 3.1. *Let V and Y be any two problems such that $V \equiv Y$. If V is irreducible, then Y cannot be solved.*

Proof. Let $V \equiv Y$ and suppose Y has a solution. Then it follows that V must also have a solution, contradicting the requirement that V is irreducible. $\square$

4. **Regular and irregular problems**

In this section we classify problems according to the structure of their sub-problems. We study the notion of regular and irregular problem.

Definition 4.1. Let V be a problem and $\{Y_i\}_{i\geq 1}$ be the sequence of all the sub-problems of V. Then we say V is regular if

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq V.$$

We say it is irregular if there exists sub-problems Y_j and Y_k of V such that $Y_j \not\leq Y_k$ and $Y_k \not\leq Y_j$.

De facto, regular problem can easily be solved as opposed to irregular problems, where a solution to one sub-problem cannot in anyway be modified and advanced to obtain a solution to other sub-problems. This makes the theory much more tractable with reducible problems.

4.1. **Maximal and minimal sub-problems.**

Definition 4.2. Let V be a problem and Y a proper sub-problem of V. Then we say Y is the maximal sub-problem of V if all other proper sub-problems of V are sub-problems of Y. We say it is the minimal sub-problem of V if it is a sub-problem of all other sub-problems of V.

Next we relate the notion of minimal sub-problem to the notion of reducibility.

Proposition 4.1. *Let V be a problem. If there exists a minimal sub-problem of V, then V must be reducible.*

Proof. Let Y be the minimal sub-problem of problem V. Then Y has no proper sub-problem. This implies that V must be reducible. □

In a similar fashion we relate the notion of maximal sub-problem with the notion of regularity.

Theorem 4.3. *Let V be a problem. If every sub-problem of V has a maximal proper sub-problem, then V must be regular.*

Proof. Let Y be the maximal proper sub-problem of V, since $V \leq V$. Then we have the relation $Y < V$ and every other proper sub-problem of V must be a sub-problem of Y. Since every sub-problem of V has a maximal sub-problem, we let Z be the maximal proper sub-problem of Y then $Z < Y$ and every other proper sub-problems of Y are sub-problems of Z. Since the proper sub-problems of V excluding Y are proper sub-problems of Y and the remaining excluding Z are sub-problems of Z, we obtain the chain of sub-problems

$$\cdots < Z < Y < V$$

and thus chain contains all the sub-problems of V. This proves that V must be a regular problem. □

5. **Connected and disconnected problem spaces**

In this section we study the existence of solutions to problems by deriving an information about the status of related and analogous problems.

Definition 5.1. Let V be a problem with solution U and Y a problem with solution X. Then we say the induced problem spaces $\mathcal{P}_V(U)$ and $\mathcal{P}_Y(X)$ are connected if and only if

$$\mathcal{P}_V(U) \cap \mathcal{P}_Y(X) \neq \emptyset.$$

We say the connection is high if

$$\frac{|\mathcal{P}_V(U) \cap \mathcal{P}_Y(X)|}{|\mathcal{P}_V(U)|} \geq \frac{1}{2} \quad \textbf{and} \quad \frac{|\mathcal{P}_V(U) \cap \mathcal{P}_Y(X)|}{|\mathcal{P}_Y(X)|} \geq \frac{1}{2}.$$

Otherwise, we say the connection is low. On the other hand, we say the problem spaces are disconnected if and only if

$$\mathcal{P}_V(U) \cap \mathcal{P}_Y(X) = \emptyset.$$

Proposition 5.1. *Let Y be a problem with solution X. If V is also a problem with a maximal proper sub-problem Z such that $Z \in \mathcal{P}_Y(X)$ and V is regular, then V must be solvable and the induced problem space must be connected to $\mathcal{P}_Y(X)$.*

Proof. Since problem Y has solution X, each problem in the space $\mathcal{P}_Y(X)$ has also been solved. The requirement $Z \in \mathcal{P}_Y(X)$ implies that problem Z has been solved. Since V is regular, we have the chain of all sub-problems of V as

$$\cdots \leq Y_3 \leq Y_2 \leq Y_1 \leq Z$$

since Z is the maximal sub-problem of V. Since Z is solved, it follows that all the sub-problems of V is solved and V must have a solution, say T, with induced problem space $\mathcal{P}_V(T)$. The latter claim follows by noting that $Z \in \mathcal{P}_V(T) \cap \mathcal{P}_Y(X)$. $\square$

6. **Further Remarks**

The theory as developed is just the preliminary and the first phase of the theory to study problems and their generative solutions. The notion of the time complexity of problems and their sub-problems is a notion to be explored in our next phase of this project, motivated in part by the P versus NP problem. We suspect the following assertions to be true

Conjecture 6.1. Let V be a problem. If V has a minimal and a maximal sub-problem, then V must be a regular problem.

Conjecture 6.2. Let V be a problem with solution U and Y a problem with solution X. If V be regular and the spaces $\mathcal{P}_V(U)$ and $\mathcal{P}_Y(X)$ are highly connected, then Y must also be regular.

[1].

References

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